ON TOTAL MASSES OF BALAYAGED MEASURES

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Introduction

Beurling and Deny [1], [2] introduced the notion of Dirichlet spaces. They [2] showed the existence of balayaged measures and equilibrium measures in the theory of Dirichlet spaces. In this paper, we shall show that the following equivalence is valid for a Dirichlet space on a locally compact Hausdorff space X.

(1) *For a pure potential u_μ such that S_μ, the support of μ, is compact and for a compact neighborhood ω of S_μ, let μ' be the balayaged measure of μ to $\mathscr{C}\omega$. Then*

$$\int d\mu = \int d\mu' .$$

(2) *For an increasing net $(\omega_\alpha)_{\alpha\in I}$ of relatively compact open sets satisfying $\omega_\alpha \nearrow X$, let ν_α be the equilibrium measure of ω_α. Then the net $(\nu_\alpha)_{\alpha\in I}$ converges vaguely to* 0.

Furthermore we shall examine anologous equivalences for a special Dirichlet space on a locally compact abelian group X.

1. Preliminaries on Dirichlet spaces

According to Beurling and Deny [2], we define a normal contraction of the complex plane $\mathfrak{C}$.

Definition 1. A transformation T of $\mathfrak{C}$ into itself is called a normal contraction if it satisfies the following conditions:

$$T(0)=0 \text{ and } |Tz_1 - Tz_2| \leq |z_1 - z_2|$$

for any couple of z_1 and z_2 in $\mathfrak{C}$.

Let X be a locally compact Hausdorff space and let $C_k = C_k(X)$ be the space of complex valued continuous functions with compact support provided with the topology of uniform convergence.

Definition 2.[1] Let ξ be a positive Radon measure in X which is every-

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[1] Cf. [2], p. 209.

where dense (i.e., $\xi(\omega)>0$ for each non-empty open set ω in X). A Hilbert space $D=D(X,\xi)$ is called a ξ-Dirichlet space (simply, a Dirichlet space) if each element in D is a complex valued function $u(x)$ which is locally summable for ξ and the following three conditions are satisfied.

a) For each compact subset K in X, there exists a positive constant $A(K)$ such that, for any u in D

$$\int_K |u(x)|\,d\xi(x)\leq A(K)\,\|u\|.$$

b) $C_k\cap D$ is dense both in C_k and in D.

c) For any normal contraction T and any u in D,

$$Tu\in D \quad \text{and} \quad \|Tu\|\leq\|u\|.$$

More precisely, two functions which are equal to *p.p.* in X[2] represent the same element in D. The norm in D is denoted by $\|\cdot\|$, the associated scalar product by $(\cdot,\ \cdot)$.

DEFINITION 3.[3] An element u in D is called a potential if there exists a Radon measure μ such that

$$(u,v)=\int\bar{v}\,d\mu$$

for any v in $C_k\cap D$. Such an element u is denoted by u_μ. Especially if μ is positive, u_μ is called a pure potential.

It is evident that the subspace of linear combinations of pure potentials is dense in D.

DEFINITION 4.[4] We say that a property holds *p.p.p.* on a subset E in X if the property holds μ-*p.p.* for any pure potential u_μ such that $S_\mu\subset E$.[5]

It is evident that a property holds *p.p.* on a subset E in X if the property holds *p.p.p.* on E, because for any complex valued bounded measurable function f with compact support, there exists the potential u_f generated by f.

In order to prove our main theorem, we need the following lemmas. Let D be a Dirichlet space on X. For each element u in D, the refinement of u is

2) A property is said to hold *p.p.* in a subset E in X if the property holds in E except a set which is locally of ξ-measure 0.

3) Cf. [2], p. 209.

4) Cf. [7].

5) S_μ is the support of μ.

denoted by u^*.[6]

LEMMA 1. *For elements u and v in D, suppose that* $u(x)\geqq v(x)$ *p.p. in an open set G. Then* $u^*(x)\geqq v^*(x)$ *p.p.p. in G.*

Proof. For any pure potential u_μ such that $S_\mu\subset G$, it is sufficient to prove that

$$(u, u_\mu)\geqq(v, u_\mu),$$

because

$$(u, u_\mu)=\int u^*d\mu \quad\text{and}\quad (v, u_\mu)=\int v^*d\mu\,.$$

Similarly as in the proof of Lemma 3 in [7], there exist positive bounded measurable functions f_n with compact support such that $f_n(x)=0$ *p.p.* in $\mathscr{C}G$ and the sequence (u_{f_n}) converges weakly u_μ in D. By our assumption,

$$(u, u_{f_n})=\int u(x)f_n(x)d\xi\geqq\int v(x)f_n(x)d\xi=(v, u_{f_n})\,.$$

Hence

$$(u, u_\mu)=\lim_{n\to\infty}(u, u_{f_n})\geqq\lim_{n\to\infty}(v, u_{f_n})=(v, u_\mu)\,.$$

This completes the proof.

By Lemma 1, we obtain the following domination theorem.

LEMMA 2. *For pure potentials* u_{μ_1} *and* u_{μ_2} *in D, suppose that*

$$u_{\mu_1}(x)\geqq u_{\mu_2}(x)$$

p.p. in some open neighborhood ω *of* S_{μ_2}. *Then*

$$u_{\mu_1}\geqq u_{\mu_2}$$

Proof. By Lemma 1,

$$u^*_{\mu_1}(x)\geqq u^*_{\mu_2}(x)$$

p.p.p. in ω. It is known that there exists a pure potential u_ν such that[7]

$$u_\nu=\inf(u_{\mu_1}, u_{\mu_2})\,.$$

6) Cf. [2], p. 210.

7) Cf. [4], Lemma 2, p. 5.

By above, it holds that

$$u_\nu^*(x)=\mu_{\mu_2}^*(x) \quad p.p.p. \text{ in } \omega,$$
$$u_\nu^*(x)\leq u_{\mu_2}^*(x) \quad p.p.p. \text{ in } X.$$

Then we have

$$\|u_{\mu_2}-u_\nu\|^2=\|u_{\mu_2}\|^2-2(u_{\mu_2}, u_\nu)+\|u_\nu\|^2$$
$$=\int u_{\mu_2}^* d\mu_2-2\int u_\nu^* d\mu_2+\int u_\nu^* d\nu$$
$$=\int u_\nu^* d\nu-\int u_\nu^* d\mu_2$$
$$=\int (u_\nu^*-u_{\mu_2}^*) d\nu\leq 0 .$$

Hence

$$u_{\mu_2}=u_\nu, \text{ i.e., } u_{\mu_1}\geq u_{\mu_2}.$$

This completes the proof.

By the above lemma, we obtain the following unicity theorem.

Corollary. *Let u_{μ_1} and u_{μ_2} be two potentials in D. If*

$$u_{\mu_1}(x)=u_{\mu_2}(x) \quad p.p.$$

in some neighborhood of $S_{\mu_1}\cup S_{\mu_2}$, then $\mu_1=\mu_2$.

This is evident by Lemma 2.

Lemma 3. *For elements u and v in D, the following equalities hold.*

(1) $(\alpha u+\beta v)^*(x)=\alpha u^*(x)+\beta v^*(x) \quad p.p.p. \text{ in } X,$

(2) $(\inf(u, v))^*(x)=\inf(u^*(x), v^*(x)) \quad p.p.p. \text{ in } X.$

The proof is evident by Lemma 1 and the fact that $(u^*)^*(x)=u^*(x)$ $p.p.p.$ in X for any u in D.

Lemma 4. *For any pure potential u_μ in D with $\int d\mu<+\infty$ and any positive number M, there exists a pure potential u_{μ_M} such that*

$$u_{\mu_M}(x)=\inf(u_\mu(x), M) \text{ and } \int d\mu_M\leq\int d\mu .$$

Proof. The existence of a pure potential u_{μ_M} is followed from a result of

Deny.[8] For a relatively compact open set ω, let u_ν be the equilibrium potential of ω.[9] Then

$$\int_\omega d\mu_M \leq \int u_\nu^* d\mu_M = (u_\nu, u_{\mu_M}) = \int u_{\mu_M}^* d\nu$$

$$= \int \inf(u_\mu^*(x), M) d\nu \leq \int u_\mu^* d\nu$$

$$= \int u_\nu^* d\mu \leq \int d\mu .$$

ω being arbitrary, we obtain

$$\int d\mu_M \leq \int d\mu .$$

This completes the proof.

Now we define the spectrum of an element in D. Given an element u in D, there exists the greatest open set ω having the following property: $(u, v)=0$ for any v in $C_k \cap D$ with support in ω.

Definition 5.[9] The complementary set of such an open set ω is called the spectrum of u, denoted by $s(u)$.

Evidently for any potential u_μ in D, $s(u_\mu)=S_\mu$.

We put, for an open set ω,

$$D_\omega^{(1)} = \overline{\{u \in D;\ s(u) \subset \omega\}},$$

$$D_\omega^{(2)} = \overline{\{f \in C_k \cap D;\ S_f \subset \omega\}},$$

and for a closed set F in X,

$$D_F^{(1)} = \{u \in D;\ s(u) \subset F\},$$

$$D_F^{(3)} = \{u \in D;\ u^*(x)=0 \quad p.p.p. \text{ on } F\}.$$

Lemma 5. *Let u_μ be a pure potential in D and let F be a closed set in X. Then there exists a pure potential $u_{\mu'}$ in D such that*

(1) μ' is supported by F and $\int d\mu' \leq \int d\mu$,

(2) $u_\mu^*(x) = u_{\mu'}^*(x)$ $p.p.p.$ on F,

(3) $u_\mu(x) \geq u_{\mu'}(x)$ $p.p.$ in X,

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8) Cf. [4], p. 6.

9) Cf. [2], p. 215.

(4) $u_{\mu'}$ is equal to the projection of u_μ to $D_F^{(1)}$.

Proof. We put

$$E_{u_{\mu'}F}=\{u \varepsilon D; u^*(x) \geq u_\mu^*(x) \quad p.p.p. \text{ on } F\}.$$

Then $E_{u_\mu, F}$ is a non-empty closed convex set in D. Let u' be the unique element which minimizes the norm in $E_{u_\mu, F}$. Similarly as the proof of Beurling and Deny's Balayage Theorem,[10] we can prove that u' is a pure potential in D and the conditions (1)-(3) are satisfied. Furthermore similarly as the proof of Lemma 3 in [7], we can prove that the condition (4) is satisfied.

We remark that for a pure potential u_μ in D, the element which satisfied the conditions (1)-(3) is uniquely determined in D by Lemma 2. We call such a pure potential $u_{\mu'}$ the balayaged potential of u_μ to F and the positive measure μ' the balayaged measure of μ to F.

LEMMA 6. For an open set ω in X, $D_\omega^{(2)}$ is a Dirichlet space on ω with the norm induced from the norm in D. Let u'_μ be a pure potential in $D_\omega^{(2)}$ such that S_μ is compact in ω. Then there exists a potential u_μ in D such that

$$u'_\mu(x)=u_\mu(x)-u_{\mu'}(x),$$

where $u_{\mu'}$ is the balayaged potential of u_μ to $\mathscr{C}\omega$.

Proof. It is evident that $D_\omega^{(2)}$ become a Dirichlet space on ω by the norm induced from the norm in D. We may assume that

$$D_\omega^{(2)}=\{u-u_1; u \varepsilon D_\omega^{(1)}\},$$

where u_1 is the projection of u to $D_{\mathscr{C}\omega}^{(1)}$. Hence there exists an element v in $D_\omega^{(1)}$ such that

$$u'_\mu=v-v_1$$

Obviously

$$s(v-v_1) \subset s_\mu \cup \mathscr{C}\omega.$$

S_μ being compact in ω, $s(v)=S_\mu$ and for any φ in $C_k \cap D_\omega^{(2)}$,

$$(v, \varphi)=\int \bar{\varphi}(x) d\mu,$$

that is, for any φ in $C_k \cap D$,

$$(v, \varphi)=\int \bar{\varphi}(x) d\mu.$$

10) Cf. [2], p. 210 and [7].

Therefore $v=u_\mu$ and by Lemma 5, $v_1=u_{\mu'}$. This completes the proof.

2. Main theorems

By the above lemmas, we obtain the following main theorems.

THEOREM I. *Let D be a Dirichlet space on X. Then the following two conditions are equivalent.*

(I. 1) *For a pure potential u_μ in D such that S_μ is compact in X and a compact neighborhood ω of S_μ, let μ' be the balayaged measure of μ to $\mathscr{C}\omega$.*[11] *Then*

$$\int d\mu=\int d\mu' .$$

(I. 2) *For an increasing net (ω_α) of relatively compact open sets with $\omega_\alpha\nearrow X$, the net of the equilibrium measures ν_α of ω_α*[12] *converges vaguely to* 0.

Proof. First we prove the implication (I. 1) $\Rightarrow$ (I. 2). Since the net $(u^*_{\nu_\alpha})$ is increasing and converges to 1 *p.p.p.* in X,

$$\lim_{\alpha\in I}\int u^*_{\nu_\alpha}d\mu=\lim_{\alpha\in I}\int u^*_{\nu_\alpha}d\mu'$$

for any u_μ in D such that S_μ is compact and any compact neighborhood ω. S_μ being compact,

$$0=\lim_{\alpha\in I}\int u^*_{\nu_\alpha}d(\mu-\mu')=\lim_{\alpha\in I}\int(u^*_\mu-\mu^*_{\mu'})d\nu_\alpha .$$

We take a fixed function φ in C_k. Next we take a relatively compact open sets ω and ω_0 such that

$$S_\varphi\subset\omega_0\subset\bar{\omega}_0\subset\omega .$$

Let u'_ν be the equilibrium potential of ω_0 in the Dirichlet space $D^{(2)}_\omega$. By Lemma 6, there exists a pure potential u_ν in D such that

$$u'_\nu=u_\nu-u_{\nu'},$$

where $u_{\nu'}$ is the balayaged potential of u_ν to $C\omega$. Furtheremore we take a relatively compact open set ω' such that $\bar{\omega}\subset\omega'$. Let ν'' be the balayaged measure of ν to $\mathscr{C}\bar{\omega}'$. Then

$$u_{\nu'}(x)\geqq u_{\nu''}(x)$$

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11) Cf. [2], p. 210 and [7].

12) Cf. [2], p. 210 and [7].

$p.p.$ in $\mathscr{C}\omega$. By Lemma 2,

$$u_{\nu'}(x) \geq u_{\nu''}(x)$$

$p.p.$ in X. That is,

$$u_\nu^*(x) - u_{\nu'}^*(x) \leq u_\nu^*(x) - u_{\nu''}^*(x)$$

$p.p.p.$ in X. Hence there exists a positive number M such that

$$\overline{\lim_{\alpha \in I}} \int |\varphi| d\nu_\alpha \leq \overline{\lim_{\alpha \in I}} M \int (u_\nu^* - u_{\nu'}^*) d\nu_\alpha$$

$$\leq \overline{\lim_{\alpha \in I}} M \int (u_\nu^* - u_{\nu''}^*) d\nu_\alpha = \lim_{\alpha \in I} M \int (u_\nu^* - u_{\nu''}^*) d\nu_\alpha = 0 .$$

Therefore the net $(\nu_\alpha)_{\alpha \in I}$ converges vaguely to 0.

Next we prove the implication (I. 2) ⇨ (I. 1). Let u_μ and $u_{\mu'}$ be the elements in our theorem. By Lemma 4, for each positive number M, there exists positive measures μ_M and μ'_M such that

$$u_{\mu_M} = \inf(u_\mu, M) \text{ and } u_{\mu'_M} = \inf(u_{\mu'}, M) .$$

Since we have

$$\int d\mu_M \leq \int d\mu \text{ and } \int d\mu'_M \leq \int d\mu' ,$$

we may assume that there exist bounded linear functionals T and T' on C, where C is the Banach space of bounded continuous functions in X with norm

$$\| f \|_C = \sup_{x \in X} | f(x) | .$$

Then

$$\int f \, d\mu_M \longrightarrow T(f) \text{ and } \int f \, d\mu'_M \longrightarrow T'(f)$$

as $M \to \infty$ for any f in C. On the other hand

$$\| u_{\mu_M} \|^2 = \int u_{\mu_M}^* d\mu_M \leq \int u_\mu^* d\mu_M = \int u_{\mu_M}^* d\mu \leq \int u_\mu^* d\mu = \| u_\mu \|^2 .$$

Similarly we have

$$\| u_{\mu'_M} \|^2 \leq \| u_{\mu'} \|^2 .$$

For any bounded measurable function f with compact support, we have the following convergences

$$\lim_{M\to\infty}\int u_{\mu_M}(x)f(x)d\xi(x)=\int u_\mu(x)f(x)d\xi(x)\,,$$

$$\lim_{M\to\infty}\int u_{\mu'_M}(x)f(x)d\xi(x)=\int u_{\mu'}(x)f(x)d\xi(x)\,.$$

Hence (u_{μ_M}) and $(u_{\mu'_M})$ converges weakly to u_μ and $u_{\mu'}$, respectively, because the totality of potentials generated by such functions f is dense in D. Now for any φ in $C_k\cap D$,

$$T(\varphi)=\int\varphi d\mu \text{ and } T'(\varphi)=\int\varphi d\mu'\,.$$

By the denseness of $C_k\cap D$ in C_k,

$$T(f)\geq\int f\,d\mu \quad\text{and}\quad T'(f)\geq\int f\,d\mu'$$

for any f in C. On the othr hand by lemma 4,

$$T(1)\leq\int d\mu \quad\text{and}\quad T'(1)\leq\int d\mu'\,.$$

That is,

$$T(1)=\int d\mu \quad\text{and}\quad T'(1)=\int d\mu'\,.$$

By the above equality, it is sufficient to prove that for any $M>0$,

$$\int d\mu_M=\int d\mu'_M\,.$$

Since we have

$$0\leq u^*_{\mu_M}-u_{\mu'_M}\leq M \quad p.p.p. \text{ in } X\,,$$

$$u^*_{\mu_M}-u^*_{\mu'_M}=0 \quad p.p.p. \text{ in } \mathscr{C}\omega\,,$$

there exists a function φ in C_k such that

$$u^*_{\mu_M}(x)-u^*_{\mu'_M}(x)\leq\varphi(x)$$

$p.p.p.$ in X. Hence by our assumption,

$$\overline{\lim_{\alpha\in I}}\int(u^*_{\mu_M}-u^*_{\mu'_M})d\nu_\alpha\leq\lim_{\alpha\in I}\int\varphi d\nu_\alpha=0\,.$$

Therefore

$$\lim_{\alpha\in I}\int(u^*_{\mu_M}-u^*_{\mu'_M})d\nu_\alpha=\lim_{\alpha\in I}\int u^*_{\nu_\alpha}d(\mu_M-\mu'_M)=0\,.$$

That is,

$$\int d\mu_M = \int d\mu'_M .$$

This completes the proof.

Remark 1. *In the above theorem, a sufficient condition for the condition* (I. 1) *to be satisfied is the followingq*(I.2′).

(I. 2′) *There exists an increasing net* $(\omega_\alpha)_{\alpha\in I}$ *of relatively compact open sets such that* $\omega_\alpha \nearrow X$ *and the net of the equilibrium measures* ν_α *of* ω_α *converges vaguely to* 0.

Remark 2. *Let X be a bounded domain in the n-dimensional Euclidean space* $R^n (n \geq 2)$ *with sufficiently smooth boundary and let* $0 < \alpha < 2$. *In the Dirichlet space* D^0_α *on X introduced by Elliott* [6], *the condition* (I. 1) *in Theorem* I *is not satisfied, because for any sequence* (ω_m) *of relatively compact open sets tending to X, the sequence* (ν_m) *of the equilibrium measures of* ω_m *converges vaguely to* $m(x)$, *where*

$$m(x) = J_\alpha \int_{\mathscr{C}X} |x-y|^{-\alpha-n} dy$$

and J_α *is a positive constant.*

By Beurling and Deny's Representation Theorem[13] and our Theorem I, we obtain the following

THEOREM II. *Let D be a Dirichlet space on X. Then the following two conditions are equivalent.*

(II. 1) *For a pure potential* u_μ *in D such that* S_μ *is compact and a compact neighborhood* ω *of* $S_{\mu'}$, *let* μ' *be the balayaged measure of* μ *to* $\mathscr{C}\omega$. *Then*

$$\int d\mu = \int d\mu' .$$

(II. 2) *There exist a positive Hermitian form* $N(f, g)$ *on* $C_K \cap D$ *with a local character*[14] *and a positiveve symmetric measure* σ *in* $X \times X - \delta$ (δ *is the diagonal set of* $X \times X$) *such that*

$$(f, g) = N(f, g) + \iint (f(x) - f(y))(\bar{g}(x) - \bar{g}(y)) d\sigma(x, y) .$$

By Beurling and Deny's Representation Theorem and the remark with respect to it in [7], it is evident that the conditions (I. 1) and (II. 2) are equivalent.

13) Cf. [2], pp. 211 and [7].

14) This means that $N(f, g) = 0$ if g is constant in some neighborhood of S_f.

3. Special Dirichlet spaces

According to Beurling and Deny [2], we define a negative definite function in a locally compact abelian group X and a special Dirichlet space on X.

DEFINITION 6.[15] A complex valued continuous function $\lambda(x)$ defined in X is said to be negative definite if the following Hermitian form

$$\sum_{i,j=1}^{n}(\lambda(x_i)+\overline{\lambda(x_j)}-\lambda(x_i-x_j))\rho_i\bar{\rho}_j$$

is positive for each set of n points $x_1, x_2, \ldots\ldots, x_n$ in X and each n complex number $\rho_1, \rho_2, \ldots\ldots, \rho_n$ $(n=1, 2, \ldots\ldots)$.

DEFINITION 7.[16] A Dirichlet space $D=D(X, \xi)$ is said to be special if X is a locally compact abelian group and ξ is the Haar measure on X, the following condition being satisfied.

d) If $U_x u$ is the function obtained from u in D by the translation $x \varepsilon X$(i.e., $U_x u(y)=u(y-x)$), then

$$U_x u \varepsilon D \quad \text{and} \quad \| U_x u \| = \| u \|.$$

Buerling and Deny [2] showed the following important result.

To a special Dirichlet space D on X corresponds a real valued negative definite function $\lambda(\hat{x})$ on the dual group $\hat{X}$ of X such that λ^{-1} is locally summable and the following equality holds:

$$\| u \|^2 = \int \lambda(\hat{x}) | \hat{u}(\hat{x}) |^2 d\hat{x} \tag{1}$$

for any u in $C_k \cap D$, where û is the Fourier transform of u.

Conversely, such a negative definite function $\lambda(\hat{x})$ on $\hat{X}$ defines, by means of (1), a special Dirichlet space on X.

Furthermore for a special Dirichlet space D, there exists a positive measure κ having λ^{-1} as the Fourier transform. We call this measure κ the kernel of D. We [7] proved the following proposition.

PROPOSITION. *Let D be a special Dirichlet space on X and let κ be the kernal of D. For a point x in X and a compact neighborhood ω of x, there exists a positive measure ε'_x such that*

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15) Cf. [2], p. 214 and [4], p. 8.
16) Cf. [2], p. 215 and [4], p. 9.

(1) ε'_x *supported by* $\overline{\mathscr{C}\omega}$ *and* $\int d\varepsilon'_x \leq 1$,

(2) $\kappa * \varepsilon_x = \kappa * \varepsilon'_x$ *as a measure in* $\mathscr{C}\omega$,

(3) $\kappa * \varepsilon_x \geq \kappa * \varepsilon'_x$ *in* X .

This meaure ε'_x is called the balayaged measure of the unit measure ε_x at x to $\mathscr{C}\omega$.

To prove the second main theorem, we need the following lemmas.

LEMMA 7. *Let D be a special Dirichlet space on X. For each increaesing net* $(\omega_\alpha)_{\alpha\in I}$ *of relatively compact open sets with* $\omega_\alpha \nearrow X$, *the net* $(\nu_\alpha)_{\alpha\in I}$ *converges vaguely to* $\lambda(\hat{o})$ *if* $\lambda(\hat{o}) \neq 0$, *where* ν_α *is the equilibrium measure of* ω_α .

Cf. [7], Lamma 12.

LEMMA 8. *Let* D_1 *and* D_2 *be special Dirichlet spaces on X and* $\lambda_1(\hat{x})$ *and* $\lambda_2(\hat{x})$ *be the negative definite functions of* D_1 *and* D_2, *respectively. If* $\lambda_1(\hat{x}) \geq \lambda_2(\hat{x})$, *then* $D_1 \subset D_2$.

Proof. For any u in $C_k \cap D_1$,

$$\| u \|_2^2 = \int |\hat{u}(\hat{x})|^2 \lambda_2(\hat{x}) d\hat{x} \leq \int |\hat{u}(\hat{x})|^2 \lambda_1(\hat{x}) d\hat{x} = \| u \|_1^2 ,$$

where $\| \cdot \|_i$ is the norm in D_i. Then u is in D_2 and $\| u \|_2 \leq \| u \|_1$.

Therefore $D_1 \subset D_2$, because $C_k \cap D_1$ is dense in D_1. This completes the proof.

By Lemma 7 and Lemma 8, we obtain the following

LEMMA 9. *Let D be a special Dirichlet space on X and let* $\lambda(\hat{x})$ *be the negative definite function of D. For each increasing net* $(\omega_\alpha)_{\alpha\in I}$ *of relatively compact open sets with* $\omega_\alpha \nearrow X$, *the net* $(\nu_\alpha)_{\alpha\in I}$ *converges vaguely to* $\lambda(\hat{o})$, *where* ν_α *is the equilibrium measure of* ω_α .

Proof. For a fixed positive number c, we put

$$\lambda'(\hat{x}) = \lambda(\hat{x}) + c .$$

Then λ' is negative definite and λ'^{-1} is finite continuous in X. Let D' be the special Dirichlet space on X such that $\lambda'(\hat{x})$ is the negative definite function of D'. For our net $(\omega_\alpha)_{\alpha\in I}$, we take another net $(\omega'_\alpha)_{\alpha\in I}$ of relatively compact open sets such that

$$\overline{\omega_\alpha} \subset \omega'_\alpha$$

for any α. Let u'_{μ_α} be the condensor potential with respect to ω_α and $\overline{\mathscr{C}\omega'_\alpha}$ in

D'. We take a fixed function φ in C_k such that

$$\varphi(x) \geq 0 \quad \text{and} \quad \int \varphi(x) dx = 1 .$$

Then $u_{\mu_\alpha} * \varphi$ is in $C_k \cap D'$. Put

$$u_\alpha(x) = u_{\mu_\alpha} * \varphi(x) .$$

Then $u_\alpha(x)$ tends to 1. For any u in $C_k \cap D'$, there exists a number α_0 such that $u_\alpha(x) = 1$ in some neighborhood of S_u for each $\alpha \geq \alpha_0$. By Beurling and Deny's Representation Theorem and Lemma 7,

$$(u, u_\alpha)' = (\lambda(\delta) + c) \int u(x) dx + 2 \iint u(x)(1 - u_\alpha(y)) d\sigma(x, y) ,$$

where $(\cdot , \cdot)'$ is the scalar product in D' and σ is a positive symmetric measure in $X \times X - \delta$ (δ is the diagonal set of $X \times X$). Hence we obtain that

$$\begin{aligned} \lim_{\alpha \in I} (u, u_\alpha)' &= \lim_{\alpha \in I} \int u(x) \mu_\alpha * \varphi(x) dx \\ &= (\lambda(\delta) + c) \int u(x) dx . \end{aligned}$$

On the other hand by Beurling and Deny's Representation theorem and Lemma 8, The net (ν_α) converges vaguely to some positive measure ν and we obtain the following equality,

$$\lim_{\alpha \in I} (u, u_\alpha) = \int u(x) d\nu(x) .$$

Since we have the equality

$$(u, u_\alpha)' - (u, u_\alpha) = c \int u(x) \overline{u_\alpha}(x) dx = c \int u(x) u_\alpha(x) dx ,$$

$$\lim_{\alpha \in I} (u, u_\alpha) = \int u(x) d\nu(x) = \lambda(\delta) \int u(x) dx .$$

By the denseness of $C_k \cap D'$ in C_k, we have the equality $\nu = \lambda(\delta)$ as a measure in X. This completes the proof.

By Theorem I and the above lemma, we obtain the following theorem.

THEOREM III. *Let D be a special Dirichlet space on X. Then the following three condition are equivalent.*

(1) *There exist a point x in X and a compact neighborhood ω of x such that*

$$\int d\varepsilon'_x = 1\,,$$

where ε'_x *is the balayaged measure of* ε_x *to* $\mathscr{C}\omega$.

(2) $$\lambda(\hat{o}) = 0\,.$$

(3) *For any point* x *in* X *and any compact neighborhood* ω *of* x, *the total mass of the balayaged measure of* ε_x *to* $\mathscr{C}\omega$ *is equal to* 1.

Proof. First we shall prove the implication (1) ⇨ (2). Assume that $\lambda(\hat{o}) \neq 0$. Then λ^{-1} is finite continuous in X, because

$$\lambda(\hat{x}) \geq \lambda(\hat{o})$$

for all $\hat{x}$ in $\hat{X}$. By Bochner's theorem, the total mass of the kernel κ of D is finite. By the unicity theorem with respect to special Dirichlet spaces (Cf. [7]),

$$\int d\kappa > \int d(\kappa * \varepsilon'_x) = \int d\kappa \cdot \int d\varepsilon'_x$$

for each x and each compact neighborhood ω of x. That is, the total mass of ε'_x is less than 1. This contradicts our assumption.

The implication (2) ⇨ (3) is evidently followed from Theorem I and Lemma 9.

The implication (3) ⇨ (1) is evident.

This completes the proof.

Moreover we obtain the following

THEOREM IV. *Let* D *be a special Dirichlet space on* X *and let* $\lambda(\hat{x})$ *be the negative definite function of* D. *Assume that* $\lambda(\hat{o}) \neq 0$. *Then for any increasing net* $(\omega_\alpha)_{\alpha \in I}$ *of compact neighborhoods of* x *in* X *with* $\omega_\alpha \nearrow X$, *we obtain the following convergence*

$$\lim_{\alpha \in I} \int d\varepsilon'_\alpha = 0\,,$$

where ε'_α *is the balayaged measrure of* ε_x *to* $\mathscr{C}\omega_\alpha$.

Proof. By our assumption, $\lambda(\hat{x}) > 0$ for any $\hat{x}$ in X. Hence the total mass of the kernel κ of D is finite. Since

$$\int d\varepsilon'_\alpha < 1 \quad \text{and} \quad S_{\varepsilon'_\alpha} \subset \overline{\mathscr{C}\omega_\alpha}\,,$$

we obtain the convergence

$$\lim_{\alpha \in I}\int f(x)d\varepsilon'_\alpha=0\,,$$

for any finite continuous function f vanishing at infinity. On the other hand

$$u_{\mu'_\alpha}(x)\geq u_{\mu'_\beta}(x)$$

if $\alpha\leq\beta$ for any pure potential u_μ, where $u_{\mu'_\alpha}$ and $u_{\mu'_\beta}$ are the balayaged potentials of u_μ to $\mathscr{C}\omega_\alpha$ and $\mathscr{C}\omega_\beta$, respectively. Hence

$$\kappa*\varepsilon'_\alpha\geq\kappa*\varepsilon'_\beta$$

if $\alpha\leq\beta$. Since the total mass of κ is finite, there exists a positive measure η such that

$$\lim_{\alpha \in I}\int f(x)d(\kappa*\varepsilon'_\alpha)=\int f(x)d\eta$$

for any bounded continouus function f in X. For each φ in C_k, $\kappa*\varphi(x)$ is a finite continuous function vanishing at infinity, and hence

$$\lim_{\alpha \in I}\int\varphi(x)d(\kappa*\varepsilon'_\alpha)=\lim_{\alpha \in I}\int\kappa*\varphi(x)d\varepsilon'_\alpha=0.$$

Therefore $\eta=0$. Now since the total mass of κ is not zero, there exists a bounded measurable function f in X such that

$$\kappa*f(x)\geq 1$$

in X. Then

$$\begin{aligned}\overline{\lim_{\alpha \in I}}\int d\varepsilon'_\alpha &\leq \overline{\lim_{\alpha \in I}}\int\kappa*f(x)d\varepsilon'_\alpha\\ &=\overline{\lim_{\alpha \in I}}\int f(x)d(\kappa*\varepsilon'_\alpha)=\lim_{\alpha\in I}\int f(x)d(\kappa*\varepsilon'_\alpha)=0\,.\end{aligned}$$

This completes the proof.

Remark. Let D be a Dirichlet space on a locally compact Hausdorff space X. It is an open question if the same result with Theorem IV exists when ν_α, the equilibrium measure of ω_α, tends vaguely to a non-zero measure for an increasing net (ω_α) of relatively compact open sets with $\omega_\alpha\nearrow X$.

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