ON POSITIVE SOLUTIONS OF THE HEAT EQUATION

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday

1. Consider the positive and twice continuously differentiable solutions u of the heat equation

$$(1) \qquad \left(\Delta - \frac{\partial}{\partial t}\right)u = 0 \ , \qquad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

in an open t-strip $\Omega = R_n \times (0, T)$ for some $T > 0$, where R_n is the n-dimensional Euclidean space.

In this note, we prove a theorem of Fatou type on u and, as its application, the uniqueness theorem for the Cauchy problem of (1).

2. The following theorem corresponds to Fatou's theorem on harmonic functions.

THEOREM 1. *Let u be any positive solution of* (1) *in Ω. Then $\lim_{t\to 0+} u(x, t)$ exists for almost every $x = (x_1, x_2, \ldots\ldots, x_n) \in R_n$.*

Proof. We begin with the Poisson-Stieltjes integral representation of $u(x, t)$ which is valid at least near the hyperplane $t = 0$. The representation is classical when $n = 1$ (cf. [5]).

Let t_0 be any fixed value such that $0 < t_0 < T$ and ε be any number such that $0 < \varepsilon < T - t_0$. Then we can represent $u(x, t + \varepsilon)$ as follows:

$$(2) \qquad u(x, t+\varepsilon) = \int_{R_n} k(y - x, t) u(y, \varepsilon) dy \qquad \text{in } \Omega_\varepsilon \ ,$$

where $k(y, t) = (4\pi t)^{-n/2} \exp(-|y|^2/4t)$, $|y|^2 = \sum_{i=1}^{n} y_i^2$

and $\Omega_\varepsilon = R_n \times (0, T - \varepsilon)$ (e.g. [2] p. 42–48).

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By choosing $x=0, t=t_0$ in (2), we see that the Borel measures

$$d\mu_\varepsilon = k(y, t_0)u(y, \varepsilon)dy$$

are uniformly bounded for sufficiently small $\varepsilon > 0$. Hence, by Frostman's selection theorem, there exists a sequence$\{\mu_{\varepsilon_j}\}_{j=1}^\infty$ of Borel measures converging to some Borel measure μ as $\varepsilon_1 > \varepsilon_2 > \ldots\ldots > \varepsilon_j > \ldots\ldots \to 0$. This means that $\lim \mu_{\varepsilon_j}(X) = \mu(X)$ for each Borel set X with the boundary of μ-measure zero. Hence, for any point (x, t) in $\Omega_0 = R_n \times (0, t_0)$, we have

$$\begin{aligned} u(x, t+\varepsilon_j) &= \int_{R_n} k(y-x, t)u(y, \varepsilon_j)dy \\ &= \int_{R_n} \frac{k(y-x, t)}{k(y, t_0)} d\mu_{\varepsilon_j} \to \int_{R_n} \frac{k(y-x, t)}{k(y, t_0)} d\mu \qquad \text{as } j \to \infty , \end{aligned}$$

where the passage to the limit is justified by noting the choice of μ_{ε_j} and by the obvious estimate

$$k(y-x, t)/k(y, t_0) = O(e^{-\text{const.}|y|^2}) \text{ as } |y| \to \infty .$$

By setting $d\sigma = d\mu/k(y, t_0)$, we obtain the desired representation

$$u(x, t) = \int_{R_n} k(y-x, t)d\sigma \qquad \text{in } \Omega_0 , \tag{3}$$

where σ is obviously finite for the bounded Borel sets in R_n.

Now, we consider the Lebesgue decomposition of $d\sigma$:

$$d\sigma = \varphi(y)dy + ds ,$$

where the density $\varphi \geqq 0$ is locally summable on R_n and $s \geqq 0$ is singular. By the strong version of Lebesgue's theorem and by the fact that the *symmetric derivative* $D_{\text{sym}}s(x)$ of s vanishes at almost every $x \in R_n$, we have

$$a^{-n}\int_{|y-x|\leqq a} \{|\varphi(y)-\varphi(x)|dy + ds(y)\} \to 0 \qquad \text{as } a \to 0 \tag{4}$$

for almost every $x \in R_n$.

We may assume that (4) holds for $x=0$. Then, for any $\varepsilon > 0$, there exists $a_0 > 0$ such that the left hand side of (4) is less than ε whenever $0 < a \leqq 2a_0$. Moreover, for any t such that $0 < t < \text{Min}\,(a_0^2, t_0)$ we choose a positive integer N such that

$$2^{N-1}b \leqq a_0 < 2^N b , \quad b = t^{1/2} .$$

By (3), we see

$$|u(0, t)-\varphi(0)|\leqq$$
$$\left(\int_{|y|<b}+\sum_{j=1}^{N}\int_{2^{j-1}b\leqq|y|<2^{j}b}+\int_{|y|>a_0}\right)k(y, t)\{|\varphi(y)-\varphi(0)|dy+ds(y)\} .$$

Estimating each integral, we see easily that

$$|u(0, t)-\varphi(0)|\leqq \text{Const. } \varepsilon+\text{Const. } \varepsilon\sum_{j=1}^{\infty}2^{-nj}$$
$$+\varphi(0)\int_{|y|>a_0}k(y, t)dy+\mu(R_n)t_0^{n/2}e^{a_0^2/4t_0}t^{-n/2}e^{-a_0^2/4t}$$
$$\leqq \text{Const. } \varepsilon \text{ as } t\to 0+ ,$$

which proves our assertion.

Remark. P.C. Rosenbloom ([3], p. 191–200) remarked without proof the validity of Fatou's theorem under a somewhat strong growth condition about $u(x, t)$.

3. Here, with the aid of Theorem 1, we prove the uniqueness theorem for positive solutions under some weak conditions (cf. [5] and [3]). For the purpose, we prepare the following lemma.

LEMMA. *If $\sigma\geqq 0$ is a Borel measure on R_n and if the upper symmetric derivative $\bar{D}_{\text{sym}}\sigma(x)$ is finite at each point $x\in R_n$, then σ is absolutely continuous with respect to the n–dimensional Lebesgue measure.*

Although we could deduce its proof from Ward's decomposition theorem ([4], p. 151–152), we state here a direct proof along the way suggested by Prof. S. Ito.

Proof. Assume that there exists a compact set K such that $\sigma(K)>0$ and such that the Lebesgue measure $|K|$ of K equals zero. Then, for a sufficiently large $M>0$, there exists a compact subset K_0 of K such that $\sigma(K_0)>0$ and $\sigma(S)/r^n\leqq M$ whenever S is a closed sphere with center in K_0 and of radius r less than M^{-1}. On the other hand, on account of $|K_0|=0$, there exists a sequence of open cubes $\{I_j\}_{j=1}^{\infty}$ such that

(a) the diameter a_j of I_j is less than M^{-1} ,

(b) $\bigcup_{j=1}^{\infty} I_j\supset K_0$ and

(c) $\sum_{j=1}^{\infty}|I_j|<(n^{n/2}M)^{-1}\sigma(K_0)$.

By (b) and (c), we have, for at least one I_{j_0} ,

$$|I_{j_0}|<(n^{n/2}M)^{-1}\sigma(K_0\cap I_{j_0}) .$$

Considering a closed sphere S_0 with center in $K_0 \cap I_{j_0} \neq \phi$ and of radius a_{j_0}, we have a contradiction.

Now we can prove the following.

THEOREM 2. *If* $\lim\sup_{t\to 0+} u(x, t) < +\infty$ *at each point* $x \in R_n$ *and if* $\lim_{t\to 0+} u(x, t) = 0$ *for almost every* $x \in R_n$, *then* $u=0$ *in* Ω.

Proof. Assume that, for some point, e.g., $x=0$, we have $\bar{D}_{\text{sym}}\sigma(0)=+\infty$. Then, there exists a sequence of radii $\{r_j\}_{j=1}^{\infty}$ converging to zero such that

$$\lim \sigma(S_j)/r_j^n = +\infty ,$$

where S_j denotes the closed sphere with center $x=0$ and of radius r_j. Thus, for a sufficiently large j, we have by (3)

$$u(0, r_j^2) \geqq \int_{|y|\leqq r_j} k(y, r_j^2)d\sigma \geqq \text{Const. } \sigma(S_j)/r_j^n ,$$

which, by letting $j \to +\infty$, leads us to a contradiction.

Hence, by the above lemma, σ is absolutely continuous, that is, $s=0$. On the other hand, by our assumption and Theorem 1, we see that $\varphi(x)=0$ for almost every $x \in R_n$. Thus, we have $\sigma=0$, that is, $u(x, t)=0$ in Ω_0. Since t_0 is arbitrary in $(0, T)$, we conclude that $u(x, t)=0$ in Ω.

4. By replacing $k(y, t)$ by the fundamental solution given in [1], we can replace Laplacian Δ in our theorems by an elliptic differential operator A of the following form:

$$A = \sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i} + c(x) ,$$

where the matrix $(a_{ij}(x))$ is symmetric and strictly positive definite for any $x \in R_n$ and

$$D^2 a_{ij}(x),\ Da_{ij}(x),\ Db_i(x),\ c(x) \text{ and } \det (a_{ij}(x))^{-1}$$

are bounded and Hölder continuous on R_n.[2)]

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