ON VARIOUS DEFINITIONS OF CAPACITY AND RELATED NOTIONS

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday

Introduction. The electric capacity of a conductor in the 3-dimensional euclidean space is defined as the ratio of a positive charge given to the conductor and the potential on its surface. The notion of capacity was defined mathematically first by N. Wiener [7] and developed by C. de la Vallée Poussin, O. Frostman and others. For the history we refer to Frostman's thesis [2]. Recently studies were made on different definitions of capacity and related notions. We refer to M. Ohtsuka [4] and G. Choquet [1], for instance. In the present paper we shall investigate further some relations among various kinds of capacity and related notions. A part of the results was announced in a lecture of the author in 1962.[1]

1. Let E and F be locally compact Hausdorff spaces and $\Phi(x, y)$ be a lower semicontinuous function on $E\times F$, satisfying $-\infty < \Phi(x, y) \leqq \infty$. This function is called a kernel. As measures we shall consider only non-negative Radon measures with compact support in E or in F. The potential $\int\Phi(x, y)d\mu(y)$ $\left(\int\Phi(x, y)d\nu(x)\text{ resp.}\right)$ of a measure μ (ν resp.) will be denoted by $\Phi(x, \mu)$($\Phi(\nu, y)$ resp.) and the double integral $\iint\Phi(x, y)d\mu(y)d\nu(x)=\int\Phi(x, \mu)d\nu(x)$ by $\Phi(\nu, \mu)$.

Let X be any non-empty set in E and μ be a measure in F. We set

$$V(X, \mu)=\sup_{x\in X}\Phi(x, \mu) \quad \text{and} \quad U(X, \mu)=\inf_{x\in X}\Phi(x, \mu).$$

Let Y be any non-empty set in F, and denote by $\mathscr{U}_Y$ the class of unit measures with compact support in Y. We put

Received August 11, 1966.

1) Capacity, Symposium on potential theory, Hakone, 1962.

$$V_X(Y)=\inf_{\mu\in\mathcal{U}_Y} V(X,\mu) \quad \text{and} \quad U_X(Y)=\sup_{\mu\in\mathcal{U}_Y} U(X,\mu)\,.$$

Similarly we define $\check{V}_Y(X)$ and $\check{U}_Y(X)$ by $\inf_{\nu\in\mathcal{U}_X}\sup_{y\in Y}\Phi(\nu,y)$ and $\sup_{\nu\in\mathcal{U}_X}\inf_{y\in Y}\Phi(\nu,y)$ respectively. B. Fuglede [3] proved the identity $V_E(K)=\check{U}_K(E)$, where K is a non-empty compact subset of F.

In the special case $E=F$ we set

$$W_i(X)=\inf_{\mu\in\mathcal{U}_X}\Phi(\mu,\mu),\ V(X)=\inf_{\mu\in\mathcal{U}_X} V(S_\mu,\mu) \quad \text{and} \quad U(X)=\sup_{\mu\in\mathcal{U}_X} U(S_\mu,\mu).$$

If the adjoint kernel $\check{\Phi}(x,y)=\Phi(y,x)$ is considered, the corresponding quantities will be denoted by $\check{W}_i(X)$, $\check{V}(X)$ and $\check{U}(X)$. We shall establish

THEOREM 1. *Suppose $E=F$ and let K be a non-empty compact set in E. Then*

$$W_i(K)=\check{W}_i(K)\leqq V(K)=\check{V}(K)\leqq\begin{Bmatrix} V_K(K)=\check{U}_K(K) \\ \\ \check{V}_K(K)=U_K(K)\end{Bmatrix}\leqq\begin{cases} V_E(K)=\check{U}_K(E) \\ \check{V}_E(K)=U_K(E) \\ U(K)=\check{U}(K)\,, \end{cases}$$

and these relations can not be improved in general.

Proof. The equalities $V_E(K)=\check{U}_K(E)$ and $\check{V}_E(K)=U_K(E)$ are special cases of the above quoted identity due to Fuglede. The equalities $V_K(K)=\check{U}_K(K)$ and $\check{V}_K(K)=U_K(K)$ are further special cases. The equalities $V(K)=\check{V}(K)$ and $U(K)=\check{U}(K)$ were found by Ohtsuka [5]; cf. [6] too. It is evident that $W_i(K)=\check{W}_i(K)$. Thus all equalities are justified.

The inequality $W_i(K)\leqq V(K)$ follows from

$$W_i(K)\leqq\int\Phi(x,\mu)d\mu(x)\leqq\sup_{x\in S_\mu}\Phi(x,\mu)$$

which is valid for any $\mu\in\mathcal{U}_K$. The inequalities $V(K)\leqq V_K(K)\leqq V_E(K)$ and $U_K(K)\leqq U(K)$ are clear.

We shall give examples in which the inequalities are strict. Consider first the space E consisting of two points x_1 and x_2. If the kernel Φ is given by the matrix $\begin{pmatrix}1 & 1\\ \frac{1}{2} & 1\end{pmatrix}$, then $W_i(K)=7/8$ and $V(K)=1$ for $K=E$. If we consider the symmetric kernel given by $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$, $V(K)=1$ but $V_K(K)=3/2$. If K consists of one point x_1 and Φ is given by $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$, then $U(K)=V_K(K)=1$ but $V_E(K)=2$.

If K consists of two points and Φ is given by $\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$, then $V_E(K)=V_K(K)=1$ but $U(K)=2$. Our proof will be completed if we can find a kernel for which $V_K(K)<\check{V}_K(K)$. This is possible, because $V_K(K)=1$ but $\check{V}_K(K)=2$ for K consisting of two points and $\Phi=\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$.

2. Suppose still $E=F$. We define $D_n(X)$ by

$$\frac{1}{n(n-1)}\inf_{x_1,\ldots\ldots,x_n\in X}\sum_{i\neq j}\Phi(x_i, x_j)\ .$$

This increases as $n\to\infty$. In fact, if we exclude the terms containing x_k and denote the remaining sum by $\sum_{i\neq j}^{(k)}$, then

$$\sum_{i\neq j}\Phi(x_i, x_j)=\frac{1}{n-2}\sum_{k=1}^{n}\sum_{i\neq j}^{(k)}\Phi(x_i, x_j)\geqq\frac{1}{n-2}\sum_{k=1}^{n}(n-1)(n-2)D_{n-1}(X)$$
$$=n(n-1)D_{n-1}(X)\ .$$

We set

$$\lim_{n\to\infty}D_n(X)=D(X)\ .$$

It is a known result that $D(K)=W_i(K)$; see, for instance, Choquet [1]. In case K is a compact set in E_3 and Φ is Newtonian, $1/D(K)$ is called the transfinite diameter of K.

We come back to the general case where E and F may not be the same. Consider two non-empty sets X and Y in E and F respectively. We set

$$nR_n(X,Y)=\sup_{y_1,\ldots\ldots,y_n\in Y}\ \inf_{x\in X}\sum_{i=1}^{n}\Phi(x, y_i)\ .$$

We shall assume $R_1(X,Y)>-\infty$ and show that $\lim_{n\to\infty}R_n(X,Y)$ exists. Choose $y_1\in Y$ such that $\inf_{x\in X}\Phi(x, y_1)>-\infty$. Then $nR_n(X,Y)\geqq\inf_{x\in X}n\Phi(x, y_1)>-\infty$. If $y_1,\ \ldots\ldots,\ y_n,\ \eta_1,\ \ldots\ldots\ ,\ \eta_m\in Y$, then

$$(n+m)R_{n+m}(X,Y)\geqq\inf_{x\in X}\left\{\sum_{i=1}^{n}\Phi(x, y_i)+\sum_{j=1}^{m}\Phi(x, \eta_j)\right\}$$
$$\geqq\inf_{x\in X}\sum_{i=1}^{n}\Phi(x, y_i)+\inf_{x\in X}\sum_{j=1}^{m}\Phi(x, \eta_j)\ ,$$

from which it follows that

$$(n+m)R_{n+m}(X,Y)\geqq nR_n(X,Y)+mR_m(X,Y)\ .$$

On the other hand,

$$nkR_n(X,Y)=k\sup_{y_1,\cdots\cdots,y_n\in Y}\inf_{x\in X}\sum_{i=1}^{n}\Phi(x,y_i)$$
$$\leqq \sup_{\eta_1,\cdots\cdots,\eta_{nk}\in Y}\inf_{x\in X}\sum_{j=1}^{nk}\Phi(x,\eta_j)=nkR_{nk}(X,Y).$$

Therefore

$$(nk+m)R_{nk+m}(X,Y)\geqq nkR_{nk}(X,Y)+mR_m(X,Y)\geqq nkR_n(X,Y)+mR_m(X,Y)$$

and hence

$$(1)\qquad R_{nk+m}(X,Y)\geqq\frac{nk}{nk+m}R_n(X,Y)+\frac{m}{nk+m}R_m(X,Y).$$

Given ε, $0<\varepsilon<1$, we choose n_0 such that

$$R_{n_0}(X,Y)\geqq\begin{cases}\overline{\lim\limits_{n\to\infty}}\,R_n(X,Y)-\varepsilon & \text{if}\quad \overline{\lim\limits_{n\to\infty}}\,R_n(X,Y)<\infty,\\ 1/\varepsilon & \text{if}\quad \overline{\lim\limits_{n\to\infty}}\,R_n(X,Y)=\infty\,.\end{cases}$$

Next we choose k_0 such that, for any $k\geqq k_0$ and every m $(0\leqq m\leqq n_0-1)$, it holds that

$$\frac{n_0k}{n_0k+m}>1-\varepsilon\quad\text{and}\quad\frac{m}{n_0k+m}R_m(X,Y)>-\varepsilon.$$

In case $\overline{\lim\limits_{n\to\infty}}\,R_n(X,Y)=\infty$, (1) yields

$$R_{n_0k+m}(X,Y)\geqq\frac{1-\varepsilon}{\varepsilon}-\varepsilon$$

for any $k\geqq k_0$ and every m, $0\leqq m\leqq n_0-1$. It follows that $\underline{\lim\limits_{n\to\infty}}\,R_n(X,Y)=\infty$.

In case $\overline{\lim\limits_{n\to\infty}}\,R_n(X,Y)<\infty$, we choose m_k $(0\leqq m_k\leqq n_0-1)$ such that

$$(2)\qquad R_{n_0k+m_k}(X,Y)\leqq\begin{cases}\underline{\lim\limits_{n\to\infty}}\,R_n(X,Y)+\varepsilon & \text{if}\quad \underline{\lim\limits_{n\to\infty}}\,R_n(X,Y)>-\infty\,,\\ -1/\varepsilon & \text{if}\quad \underline{\lim\limits_{n\to\infty}}\,R_n(X,Y)=-\infty\,.\end{cases}$$

It holds on account of (1) that

$$\underline{\lim_{k\to\infty}}\,R_{n_0k+m_k}(X,Y)\geqq R_{n_0}(X,Y)\geqq\overline{\lim_{n\to\infty}}\,R_n(X,Y)-\varepsilon\,.$$

This and (2) yield $\underline{\lim}_{n\to\infty} R_n(X,Y) \geqq \overline{\lim}_{n\to\infty} R_n(X,Y)$. Thus $\lim_{n\to\infty} R_n(X,Y)$ exists. We shall denote this limit by $R(X,Y)$.

Remark. There is an example in which $\lim_{n\to\infty} R_n(X,Y)$ does not exist. Take the x-axis as $X=E$ and $\{1, 2, \ldots\ldots\}$ as $Y=F$. We define $\Phi(x,n)$ by $(-1)^n x$. Then $R_n(X,Y)=-\infty$ if n is odd and $R_n(X,Y)=0$ if n is even.

Let us establish

THEOREM 2. *Let K be a non-empty compact set in E, and Y be any non-empty set in F. Then $R(K,Y)$ exists and*

$$R(K,Y)=U_K(Y) .$$

Proof. First we note that $R_1(K,Y)=\sup_{y\in Y}\inf_{x\in K}\Phi(x,y)>-\infty$, whence $R(K,Y)=\lim_{n\to\infty} R_n(K,Y)$ exists. For each n

$$R_n(K,Y)=\frac{1}{n}\sup_{y_1,\ldots\ldots,y_n\in Y}\ \inf_{x\in K}\sum_{i=1}^{n}\Phi(x,y_i)\leqq U_K(Y) ,$$

so that $R(K,Y)\leqq U_K(Y)$. To prove the inverse inequality take $\mu\in\mathscr{U}_Y$. Given $\varepsilon>0$, we can find a continuous function $\Phi_\varepsilon(x,y)$ on $K\times S_\mu$ such that $\Phi_\varepsilon(x,y)\leqq\Phi(x,y)$ on $K\times S_\mu$ and

$$\min_{x\in K}\Phi_\varepsilon(x,\mu)\geqq\min_{x\in K}\Phi(x,\mu)-\varepsilon .$$

There exist a finite subdivision $S_\mu=\bigcup_{i=1}^{k} Y_i$ into mutually disjoint Borel sets $Y_1,\ldots\ldots,$ Y_k and points $y_1\in Y_1,\ldots\ldots, y_k\in Y_k$ such that

$$|\Phi_\varepsilon(x,y)-\Phi_\varepsilon(x,y_i)|<\varepsilon$$

whenever $x\in K$ and $y\in Y_i$ for each i. We have

$$\left|\sum_i\Phi_\varepsilon(x,y_i)\mu(Y_i)-\Phi_\varepsilon(x,\mu)\right|\leqq\sum_i\int_{Y_i}\left|\Phi_\varepsilon(x,y_i)-\Phi_\varepsilon(x,y)\right|d\mu(y)\leqq\varepsilon$$

on K and hence

$$\min_{x\in K}\sum_i\Phi_\varepsilon(x,y_i)\mu(Y_i)\geqq\min_{x\in K}\Phi_\varepsilon(x,\mu)-\varepsilon\geqq\min_{x\in K}\Phi(x,\mu)-2\varepsilon .$$

We approximate each $\mu(Y_i)$ by a non-negative rational number r_i such that $\sum_i r_i=1$ and

$$\min_{x\in K}\sum_i \Phi_\varepsilon(x, y_i)\mu(Y_i) \leqq \min_{x\in K}\sum_i \Phi_\varepsilon(x, y_i)r_i+\varepsilon \leqq \min_{x\in K}\sum_i \Phi(x, y_i)r_i+\varepsilon .$$

Set $r_i=p_i/q$ with integers $p_i \geqq 0$ and $q>0$, and consider

$$\frac{1}{q}\left\{p_1\Phi(x, y_1)+p_2\Phi(x, y_2)+\dots\dots+p_k\Phi(x, y_k)\right\}$$

Its minimum on K is not greater than $R_q(K, Y)$. Thus

$$\min_{x\in K}\Phi(x, \mu) \leqq R_q(K, Y)+3\varepsilon.$$

Since we can take q arbitrarily large, $\min_{x\in K}\Phi(x, \mu) \leqq R(K, Y)+3\varepsilon$, whence $\min_{x\in K}\Phi(x, \mu) \leqq R(K, Y)$. Because of the arbitrariness of $\mu\in\mathscr{U}_Y$, we have $U_K(Y) \leqq R(K, Y)$, which gives the equality.

3. Finally we prove

THEOREM 3. *Let X be a non-empty set in E and L be a non-empty compact set in F. In order that there be $\mu\in\mathscr{U}_L$ such that $\Phi(x, \mu)=\infty$ for every $x\in X$, it is necessary and sufficient that $U_X(L)=\infty$.*

Proof. Suppose that there is a measure $\mu\in\mathscr{U}_L$ such that $\Phi(x, \mu)=\infty$ for every $x\in X$. Then

$$U_X(L)=\sup_{\mu\in\mathscr{U}_L}\ \inf_{x\in X}\Phi(x, \mu)=\infty .$$

Conversely assume $U_X(L)=\infty$. For each k there is $\mu_k\in\mathscr{U}_L$ such that $\Phi(x, \mu_k)>2^k$ on X. Naturally $\sum_{k=1}^{\infty}2^{-k}\mu_k\in\mathscr{U}_L$ and

$$\Phi\left(x, \sum_{k=1}^{\infty}2^{-k}\mu_k\right)=\infty \qquad \text{for every } x\in X.$$

Using Theorem 2 we obtain the following generalization of the so-called Evans-Selberg's theorem.

COROLLARY. *Let K and L be non-empty compact sets in E and F respectively. In order that there be $\mu\in\mathscr{U}_L$ such that $\Phi(x, \mu)=\infty$ for every $x\in K$, it is necessary and sufficient that $R(K, L)=\infty$.*

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