ON A CONJECTURE OF J.S. FRAME

NOBORU ITO*

To Professor Kiyoshi Noshiro on His Sixtieth Birthday

Let $\mathfrak{G}$ be a transitive group of degree n, and let $\mathfrak{G}_1$ be the stabilizer of a symbol in $\mathfrak{G}$. Then we owe to J.S. Frame the following remarkable relations between the lengths n_i of the orbits of $\mathfrak{G}_1$ and the degrees f_i of the absolutely irreducible components of the permutation matrix representation $\mathfrak{G}^*$ of $\mathfrak{G}$:

(A) If the irreducible constituents of $\mathfrak{G}^*$ are all different, then the rational number

$$F = n^{k-2} \prod_{i=1}^{k} n_i / f_i$$

is an integer, where k is the number of the orbits of $\mathfrak{G}_1$.

(C) If the irreducible constituents of $\mathfrak{G}^*$ all have rational characters, then F is a square.

Further J.S. Frame made the following conjecture ([1]):

(B) If the k numbers n_i are all different, then F is a square.

(B) is true for $k \leqq 3$ ([3], §30).

Now the purpose of this short note is to show that (B) is not true in general for $k=4$.

Let $LF_r(q)$ be the r-dimensional projective special linear group over the field of q elements such that $p=\frac{q^r-1}{q-1}$ is a prime and r is odd. Let $V_r(q)$ and $W_r(q)$ be the r-dimensional spaces of column and row vectors over the field of q elements, respectively. Let V and W be the set of one-dimensional subspaces of $V_r(q)$ and $W_r(q)$, respectively. $\left\langle\begin{matrix} x_1 \\ \vdots \\ x_r \end{matrix}\right\rangle \in V$ and $\langle y_1, \ldots\ldots, y_r \rangle \in W$ denote the one-dimensional subspaces generated by $\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in V_r(q)$ and $(y_1, \ldots\ldots, y_r)$

* This research was partially supported by National Science Foundation Grand GP-5276

$\in W_r(q)$, respectively.

$LF_r(q)$ can be considered in a natural manner as permutation groups on V and also on W. The number of elements in V and W are equal to $p=(q^r-1)/(q-1)$, and two conjugacy classes of subgroups of $LF_r(q)$ of index p correspond to the stabilizers of the symbols of V and W respectively. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the stabilizers of $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ and $<1, 0, \ldots\ldots, 0>$ in $LF_r(q)$ respectively. Let $\mathfrak{P}$ be a Sylow p-subgroup of $LF_r(q)$ and let P be a generating element of $\mathfrak{P}$. Then $P^{-i}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ $(i=0, \ldots\ldots, p-1)$ and $<1, 0, \ldots\ldots, 0>P^j$ $(j=0, \ldots\ldots, p-1)$ are all different. Making $P^{-i}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ and $<1, 0, \ldots\ldots, 0>P^j$ correspond to $\mathfrak{A}P^i$ and $P^{-j}\mathfrak{B}$, respectively, let us denote the permutation representations of $LF_r(q)$ over V and W by $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively.

$\mathfrak{A}$ consists of the matrices of the form $\begin{pmatrix} & 0 \\ * & \vdots \\ & 0 \\ & \rho \end{pmatrix}$, $\rho \in GF(q)$, whose determinants are equal to 1. $\boldsymbol{B}(\mathfrak{A})$ has two orbits, namely, $D=\{<\ldots\ldots, 0>\}$ and $W-D$, whose lengths are equal to $k=\dfrac{q^{r-1}-1}{q-1}$ and $p-k$, respectively. Thus the conjugacy class of B in G is divided into two A-classes, each of which contain k and $p-k$ subgroups, respectively.

$LF_r(q)$ admits an involutory automorphism τ such that $X^\tau=(X^t)^{-1}$ for every element X of $LF_r(q)$, where t denotes the transpose operation.

It is easy to see that $\mathfrak{A}^\tau$ is conjugate to $\mathfrak{B}$. In fact, $\mathfrak{A}^\tau$ is the stabilizer of $<0, \ldots\ldots, 0, 1>$ in $LF_r(q)$. Let $\mathfrak{G}$ be the split extension of $LF_r(q)$ by τ. We notice that $A\tau X=AX^\tau\tau$ for every element X of $LF_r(q)$.

Let us consider the following permutation representation of $\mathfrak{G}$ by the subgroup $\mathfrak{A}$:

$$X \to \begin{pmatrix} \mathfrak{A}, & \mathfrak{A}P, & \ldots\ldots, \mathfrak{A}\tau, & \mathfrak{A}\tau P, & \ldots\ldots \\ \mathfrak{A}X, & \mathfrak{A}PX, & \ldots\ldots, \mathfrak{A}\tau X, & \mathfrak{A}\tau PX, & \ldots\ldots \end{pmatrix}$$

Then it is easy to see that the lengths of the orbits of $\mathfrak{A}$ are equal to 1, $p-1$, k and $p-k$. Thus the permutation representation decomposes into four absolutely irreducible components. Then it is quite easy to see that their degrees are equal to 1, 1, $p-1$ and $p-1$. Now $F=(2p)^4(p-1)k(p-k)/(p-1)^2 =(2p)^4q^{r-2}$ is not a square, since q is a non-square and r is odd.

References

[1] J.S. Frame, The degrees of the irreducible components of simply transitive permutation groups, Duke Math. J. 3, 8–17 (1937).
[2] J.S. Frame, The double cosets of a finite group, Bull. Amer. Math. Soc. **47**, 458–467 (1941).
[3] H. Wielandt, Finite Permutation Groups, Academic Press (1964).

Nagoya University and University of Illinois Chicago, Illinois