ON LOCAL MAXIMALITY FOR THE COEFFICIENT a_6

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Dedicated to Professor K. Noshiro on his 60th birthday

1. Recently a number of authors have studied the application of Grunsky's coefficient inequalities to the study of the Bieberbach conjecture for the class of normalized regular univalent functions $f(z)$ in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n.$$

Charzynski and Schiffer [2] applied this result to give an elementary proof of the inequality $|a_4|\leqq 4$. One of the present authors [8] proved that if a_2 is real non-negative then $\Re\, a_6\leqq 6$. A natural first step in the study of the inequality for a coefficient is to prove local maximality for a_2 near to 2. Bombieri [1] announced that he had proved

$$\Re\, a_6\leqq 6-A(2-\Re\, a_2)$$

for $A>0$, $\Re\, a_2$ sufficiently near to 2. As yet to our knowledge no complete account of his result has appeared. One of the present authors has shown [7] that in many cases the Area Principle is more effective than Grunsky's method. In the present instance the Area Principle takes the form of an inequality due to Golusin [4]. In this paper we use this inequality to prove the local maximality of $\Re\, a_6$ at the Koebe function. Our theorem implies the result of Bombieri.

During the preparation of this work there appeared a paper by Garabedian, Ross and Schiffer [3] which asserts the local maximality of $\Re\, a_{2n}$, $n=2,3,\ldots\ldots$ at the Koebe function. Further consideration is required to determine its status. In any case it does not appear to include Bombieri's result.

2. Golusin's inequality and Grunsky's inequality.

Let $f(z)$ be a normalized regular function univalent in the unit disc $|z|<1$, whose expansion around $z=0$ is

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$$z+\sum_{\nu=2}^{\infty} a_\nu z^\nu .$$

Let $G_\mu(w)$ be the μ^{th} Faber polynomial which is defined by

$$g_\mu(z)=G_\mu(g(z))=z^\mu+\sum_{\nu=1}^{\infty}\frac{b_{\mu\nu}}{z^\nu} ,$$
$$g(z)=f(1/z^2)^{-1/2} .$$

Then it is known that $\nu b_{\mu\nu}=\mu b_{\nu\mu}$. Let

$$Q_m(g(z))=\sum_{\mu=1}^{m} x_\mu g_\mu(z) \quad ,$$

then Golusin's inequality has the form

$$\sum_{\nu=1}^{\infty}\nu|\sum_{\mu=1}^{m} x_\mu b_{\mu\nu}|^2\leqq\sum_{\nu=1}^{m}\nu|x_\nu|^2 , \tag{1}$$

and Grunsky's inequality has the form

$$|\sum_{\mu,\nu=1}^{m}\nu b_{\mu\nu}x_\mu x_\nu|\leqq\sum_{\nu=1}^{m}\nu|x_\nu|^2 . \tag{2}$$

One of the authors [7] pointed out that Grunsky's inequality is a direct consequence of Golusin's.

By a simple calculation we have

$$b_{11}=-\frac{1}{2}a_2,\ b_{13}=-\frac{1}{2}\left(a_3-\frac{3}{4}a_2^2\right),\ b_{15}=-\frac{1}{2}\left(a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3\right),$$

$$b_{17}=-\frac{1}{2}\left(a_5-\frac{3}{2}a_2a_4-\frac{3}{4}a_3^2+\frac{15}{8}a_3a_2^2-\frac{35}{64}a_2^4\right),$$

$$b_{22}=-a_3+a_2^2,\ b_{24}=-a_4+2a_2a_3-a_2^3 ,$$

$$b_{44}=-2a_5+4a_2a_4-8a_2^2a_3+3a_3^2+3a_2^4 ,$$

$$b_{31}=-\frac{3}{2}\left(a_3-\frac{3}{4}a_2^2\right)=3b_{13},\ b_{33}=-\frac{3}{2}\left(a_4-2a_2a_3+\frac{13}{12}a_2^3\right),$$

$$b_{35}=-\frac{3}{2}\left(a_5-2a_2a_4-\frac{5}{4}a_3^2+\frac{29}{8}a_3a_2^2-\frac{85}{64}a_2^4\right)=\frac{3}{5}b_{53} ,$$

$$b_{51}=5b_{15},\ b_{55}=-\frac{5}{2}\left(a_6-2a_2a_5-3a_3a_4+4a_2^2a_4+\frac{21}{4}a_2a_3^2\right.$$
$$\left.-\frac{59}{8}a_3a_2^3+\frac{689}{320}a_2^5\right).$$

From now on we shall use the following notations:

$$2-x+ix'=a_2\ ,$$

$$y+iy'=a_3-\frac{3}{4}a_2^2,$$

$$\eta+i\eta'=a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3,$$

$$\xi+i\xi'=a_5-\frac{3}{2}a_2a_4-\frac{3}{4}a_3^2+\frac{15}{8}a_3a_2^2-\frac{35}{64}a_2^4\,.$$

3. Lemmas.

LEMMA 1. $7(\xi^2+\xi'^2)+5(\eta^2+\eta'^2)+3(y^2+y'^2)\leqq 4x-x^2-x'^2\,.$

Proof. This is a simple consequence of the area theorem for $f(1/z^2)^{-1/2}$.

LEMMA 2. $$y\leqq 3x-\frac{15}{4}x^2+\frac{10}{3}x^3-\frac{1}{4}x'^2\ .$$

Proof. One of the authors [6] proved the following result:

$$\Re\left\{e^{-2i\Phi}\left(a_3-\frac{3}{4}a_2^2\right)\right\}\leqq 1+\frac{3}{8}\tau^2-\frac{\tau^2}{4}\log\frac{\tau}{4}+\frac{1}{4}\Re\left\{e^{-2i\Phi}a_2^2\right\}+\tau\,\Re\{e^{-i\Phi}a_2\}$$

holds for every real Φ and for every real τ satisfying $0\leqq\tau\leqq 4$.

Putting $\Phi=\pi$ and $\tau=4e^{-s}$, we have

$$y\leqq 2-8e^{-s}+6e^{-2s}+4se^{-2s}-x+\frac{x^2}{4}+4xe^{-s}-\frac{1}{4}x'^2\ .$$

By a similar discussion in [8] we have the desired result.

LEMMA 3. $$-x+\frac{x^2}{4}-\frac{x'^2}{4}\leqq y\ .$$

Proof. It is well-known that

$$\Re\,(a_2^2-a_3)\leqq 1\ .$$

This implies the desired result.

LEMMA 4. $$\eta\leqq\frac{5}{4}x-\frac{3}{4}x^2+\frac{7}{48}x^3-\frac{1}{2}x'y'+\frac{x'^2}{2}-\frac{xx'^2}{4}.$$

Proof. In (2) we select $m=3$, $x_1=\beta$, $x_2=0$, $x_3=1/3$. Then

$$|a_4-2a_2a_3+\frac{13}{12}a_2^3+2\beta\left(a_3-\frac{3}{4}a_2^2\right)+\beta^2a_2|\leqq\frac{2}{3}+2|\beta|^2\ .$$

Put $\beta=(2-x)/4$ and take the real part. Then we have

$$\eta+\frac{1}{2}x'y'+\frac{1}{12}\Big((2-x)^3-3(2-x)x'^2\Big)+\frac{1}{16}(2-x)^3$$
$$\leqq\frac{2}{3}+\frac{1}{8}(2-x)^2\ ,$$

which is nothing but the desired result.

LEMMA 5. $\quad -\eta\leqq\frac{1}{2}(2-x)y+2x-\frac{3}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}(2-x)x'^2+\frac{1}{2}x'y'\ .$

Proof. One of the authors [6] proved the following fact:

$$\Re\{e^{-3i\Phi}(-a_4+2a_2a_3-a_2^3)\}\leqq\frac{2}{3}+2\sigma^2+\frac{2}{3}\sigma^3+2\sigma\,\Re\{e^{-2i\Phi}(a_2^2-a_3)\}$$

for every real Φ and for every real σ satisfying $-1\leqq\sigma\leqq 1/3$.
Put $\Phi=0$ and $\sigma=-1+x/2$. Then we have the desired result.

LEMMA 6.

$$(2-x)\eta-x'\eta'+\frac{3}{2}(y^2-y'^2)-\frac{1}{2}\Big((2-x)^2-x'^2\Big)y+(2-x)x'y'$$
$$-2x+\frac{3}{2}x^2-\frac{1}{2}x^3+\frac{1}{16}x^4-\frac{3}{8}(2+x)^2x'^2+\frac{1}{16}x'^4\leqq 2\xi\,.$$

Proof. By Grunsky's inequality we have

$$|-2a_5+4a_2a_4-8a_2^2a_3+3a_3^2+3a_2^4|\leqq 1\ .$$

This turns out to the desired inequality taking the real part.

LEMMA 7. $\quad \xi\leqq\frac{1}{2}(2-x)\eta-\frac{1}{2}x'\eta'+\frac{1}{4}(y^2-y'^2)+\frac{1}{2}+3e^{-4x}$

$$+4xe^{-4x}+4e^{-2x}\Big(y-\frac{1}{4}(2-x)^2+\frac{1}{4}x'^2\Big)$$
$$+\frac{1}{2}\Big(-y+\frac{1}{4}(2-x)^2-\frac{1}{4}x'^2\Big)^2-\frac{1}{2}\Big(-y'+\frac{1}{2}(2-x)x'\Big)^2$$

for $0\leqq x\leqq 2$.

Proof. One of the authors [6] proved the following result:

$$-\Re\{e^{-4i\Psi}(-a_5+2a_2a_4+a_3^2-3a_2^2a_3+a_2^4)\}$$
$$\leqq\frac{1}{2}+\frac{3}{16}\sigma^4-\frac{1}{8}\sigma^4\log\frac{\sigma^2}{4}+\frac{1}{2}\Re\{e^{-4i\Psi}(a_2^4-a_3)^2\}-\sigma^2\Re\{e^{-2i\Psi}(a_2^2-a_3)\}$$

for Ψ real and $0\leqq\sigma\leqq 2$. Put $\Psi=0$ and $\sigma=2e^{-x}$, $0\leqq x\leqq 2$.

Then a simple calculation leads to the desired result.

It should be remarked that for $x \to 0$

$$y=O(x)\ ,\quad \eta=O(x)\ ,\quad \xi=O(x)\ ,$$

and

$$x'=O(x^{1/2})\ ,\quad y'=O(x^{1/2})\ ,\quad \eta'=O(x^{1/2})\ ,\quad \xi'=O(x^{1/2})\ .$$

So far as local maximality is concerned we can consider only terms of order $O(x)$. Hence we shall omit terms of higher order in the sequel.

4. By the Golusin inequality we have

$$5|x_5b_{55}+x_3b_{35}+x_1b_{15}|^2+3|x_5b_{53}+x_3b_{33}+x_1b_{13}|^2+|x_5b_{51}+x_3b_{31}+x_1b_{11}|^2 \leqq |x_1|^2+3|x_3|^2+5|x_5|^2\ .$$

Put $x_5=1$, $x_3=5\beta/6$, $x_1=5\delta$. Then we have

$$\begin{aligned}(3)\quad &\Big|a_6-2a_2a_5-3a_3a_4+4a_2^2a_4+\frac{21}{4}a_2a_3^2-\frac{59}{8}a_3a_2^3+\frac{689}{320}a_2^5\\ &+\frac{1}{2}\Big(a_5-2a_2a_4-\frac{5}{4}a_3^2+\frac{29}{8}a_3a_2^2-\frac{85}{64}a_2^4\Big)\beta+\Big(a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3\Big)\delta\Big|^2\\ &+\frac{3}{5}\Big|a_5-2a_2a_4-\frac{5}{4}a_3^2+\frac{29}{8}a_3a_2^2-\frac{85}{64}a_2^4+\frac{1}{2}\Big(a_4-2a_2a_3+\frac{13}{12}a_2^3\Big)\beta\\ &+\Big(a_3-\frac{3}{4}a_2^2\Big)\delta\Big|^2+\frac{1}{5}\Big|a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3+\frac{1}{2}\Big(a_3-\frac{3}{4}a_2^2\Big)\beta+a_2\delta\Big|^2\\ &\leqq\frac{4}{25}+\frac{1}{15}|\beta|^2+\frac{4}{5}|\delta|^2\ .\end{aligned}$$

Put $x_5=0$, $x_3=2/3$, $x_1=2\beta$. Then we have

$$\begin{aligned}&5\Big|a_5-2a_2a_4-\frac{5}{4}a_3^2+\frac{29}{8}a_3a_2^2-\frac{85}{64}a_2^4+\Big(a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3\Big)\beta\Big|^2\\ (4)\quad &+3\Big|a_4-2a_2a_3+\frac{13}{12}a_2^3+\Big(a_3-\frac{3}{4}a_2^2\Big)\beta\Big|^2+\Big|a_3-\frac{3}{4}a_2^2+a_2\beta\Big|^2\\ &\leqq\frac{4}{3}+4|\beta|^2\ .\end{aligned}$$

From (4) we have, omitting higher order terms,

$$\begin{aligned}\eta+y(2\beta-1)\leqq &(1+\beta^2)x+\frac{1}{2}x'^2-\frac{1}{2}x'y'-\frac{1}{4}(y'+\beta x')^2\\ &-\frac{3}{4}\Big(\eta'+(\beta-1)y'+x'\Big)^2-\frac{5}{4}\Big(\xi'+(\beta-1)\eta'+y'\Big)^2\end{aligned}$$

with real β. Put $\beta=5/2$. Then

$$
\begin{aligned}
\eta+4y \leqq \frac{29}{4}x+\frac{1}{2}x'^2-\frac{1}{2}x'y'-\frac{1}{4}\left(y'+\frac{5}{2}x'\right)^2 \\
-\frac{3}{4}\left(\eta'+\frac{3}{2}y'+x'\right)^2-\frac{5}{4}\left(\xi'+\frac{3}{2}\eta'+y'\right)^2 .
\end{aligned} \tag{5}
$$

From (3) putting $\beta=4$ and $\delta=2.25$ and omitting higher order terms, we have

$$
\begin{aligned}
\Re\, a_6 \leqq\ & 6+0.5\eta+2y-(10-2.25^2)x-12x'^2-14.5x'y'-3.5y'^2 \\
& -7x'\eta'-3y'\eta'-2x'\xi' \\
& -\frac{3}{4}(\xi'+\eta'+1.25y'+2x')^2-\frac{1}{4}(\eta'+2y'+2.25x')^2 .
\end{aligned} \tag{6}
$$

By (5) we have

$$
\begin{aligned}
\Re\, a_6 \leqq\ & 6-1.3125x-11.75x'^2-14.75x'y'-3.5y'^2-7x'\eta'-3y'\eta' \\
& -2x'\xi'-\frac{3}{4}(\xi'+\eta'+1.25y'+2x')^2 \\
& -\frac{1}{4}(\eta'+2y'+2.25x')^2-\frac{1}{8}\left(y'+\frac{5}{2}x'\right)^2-\frac{3}{8}\left(\eta'+\frac{3}{2}y'+x'\right)^2 \\
& -\frac{5}{8}\left(\xi'+\frac{3}{2}\eta'+y'\right)^2 .
\end{aligned}
$$

Since $x'^2+3y'^2+5\eta'^2+7\xi'^2 \leqq 4x$ omitting higher order terms in Lemma 1, we have

$$
\begin{aligned}
& \Re\, a_6 \leqq 6-F(x', y', \eta', \xi') , \\
& F(x', y', \eta', \xi')=\left(11.75+\frac{1.3125}{4}\right)x'^2+14.75x'y'+\left(3.5+\frac{3\times 1.3125}{4}\right)y'^2 \\
& \quad +7x'\eta'+3y'\eta'+\frac{5\times 1.3125}{4}\eta'^2+2x'\xi'+\frac{7}{4}1.3125\xi'^2 \\
& \quad +\frac{3}{4}(\xi'+\eta'+1.25y'+2x')^2+\frac{1}{4}(\eta'+2y'+2.25x')^2+\frac{1}{8}\left(y'+\frac{5}{2}x'\right)^2 \\
& \quad +\frac{3}{8}\left(\eta'+\frac{3}{2}y'+x'\right)^2+\frac{5}{8}\left(\xi'+\frac{3}{2}\eta'+y'\right)^2 .
\end{aligned} \tag{7}
$$

Now we shall prove the positive definiteness of $F(x', y', \eta', \xi')$. Consider $64\,F(x', y', \eta', \xi')$. This is equal to

$$
\begin{aligned}
1120x'^2+1440x'y'+528y'^2+760x'\eta'+568y'\eta'+283\eta'^2 \\
+320x'\xi'+200y'\xi'+216\eta'\xi'+235\xi'^2 .
\end{aligned}
$$

Consider the principal diagonal minor determinants

$$235, \quad \begin{vmatrix} 235 & 108 \\ 108 & 283 \end{vmatrix}, \quad \begin{vmatrix} 235 & 108 & 100 \\ 108 & 283 & 284 \\ 100 & 284 & 528 \end{vmatrix},$$

$$\begin{vmatrix} 235 & 108 & 100 & 160 \\ 108 & 283 & 284 & 380 \\ 100 & 284 & 528 & 720 \\ 160 & 380 & 720 & 1120 \end{vmatrix}.$$

Then these are positive. Hence $64F(x', y', \eta', \xi')$ is positive definite. By continuity we have

$$\Re\, a_6 \leqq 6 - Ax - Q(x', y', \eta', \xi')$$

with a suitable positive A and a suitable positive definite quadratic form $Q(x', y', \eta', \xi')$. This implies the following theorem.

THEOREM. *Let $f(z)$ be a normalized regular function univalent in $|z| < 1$, whose local expansion is*

$$z + \sum_{\nu=2}^{\infty} a_\nu z^\nu .$$

Then

$$\Re\, a_6 \leqq 6 - Ax , \quad A > 0$$

holds for $0 \leqq x < \varepsilon$. If $\Re\, a_6 = 6$ in $0 \leqq x < \varepsilon$, then $f(z)$ reduces to the Koebe function

$$\frac{z}{(1-z)^2} .$$

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