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ON A THEOREM OF RAMANAN

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Let G be a simply connected Lie group and P a parabolic subgroup without simple factor. A finite dimensional irreducible representation of P defines a homogeneous vector bundle E over the homogeneous space G/P. Ramanan [2] proved that, if the second Betti number b_2 of G/P is 1, the inequality in Definition (2.3) holds provided F is locally free. Since the notion of the H-stability was not established at that time, it was inevitable to assume that $b_2 = 1$ and F is locally free. In this paper, pushing Ramanan's idea through, we prove that E is H-stable for any ample line bundle H. Our proof as well as Ramanan's depends on the Borel-Weil theorem. If we recall that the Borel-Weil theorem fails in characteristic $p > 0$, it is interesting to ask whether our theorem remains true in characteristic $p > 0$.

§1. The Borel-Weil theorem

Let us review the Borel-Weil theorem on which the proof of our theorem heavily depends. We use the notation of Kostant [1] with slight modifications. For example, we shall denote by $\mathfrak{p}$ a parabolic Lie subalgebra which Kostant denotes by $\mathfrak{u}$. In this section all the results are stated without proofs. The details are found in the paper of Kostant cited above.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and let $(\mathfrak{g})$ be the Cartan-Killing form on $\mathfrak{g}$ namely $(x, y) = \mathrm{tr}\,(adx \circ ady)$ for $x, y \in \mathfrak{g}$.

A compact form of $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ satisfying the following conditions:

(i) $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ is the direct sum of real Lie algebra.

(ii) the Cartan-Killing form is negative definite on $\mathfrak{k}$. We fix a compact form once and for all. Let $\mathfrak{q} = i\mathfrak{k}$ so that the restriction of the Cartan-Killing form to $\mathfrak{q}$ is positive definite. Evidently we have a real decom-

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position $\mathfrak{g} = \mathfrak{q} + i\mathfrak{q}$. The star operator is defined by the formula

$$(u + iv)^* = u - iv \qquad \text{for any } u + iv \in \mathfrak{g} = \mathfrak{q} + i\mathfrak{q} .$$

Now let V be a vector space. We denote by V' the dual of V.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let ℓ be its dimension i.e., ℓ is the rank of the semi-simple Lie algebra $\mathfrak{g}$. We know that the restriction $(\mathfrak{h})$ of $(\mathfrak{g})$ to $\mathfrak{h}$ is non-singular and hence we can define a map $\mu \to x_\mu$ of $\mathfrak{h}'$ onto $\mathfrak{h}$ by the relation

$$(y, x_\mu) = \langle y, \mu \rangle \qquad \text{for all } y \in \mathfrak{h} .$$

If we define $(\mu, \lambda) = \langle x_\mu, \lambda \rangle$, we get a non-singular bilinear form $(\mathfrak{h}')$ on $\mathfrak{h}'$. If we consider $\mathfrak{g}$ as an $\mathfrak{h}$-module through the adjoint representation, then we get the decomposition of $\mathfrak{g}$ into $\mathfrak{h}$-invariant spaces:

$$\mathfrak{g} = \mathfrak{h} + \sum_{\varphi \in \mathfrak{h}'} \mathfrak{g}^\varphi$$

where $\mathfrak{h}$ acts on $\mathfrak{g}^\varphi$ through the character φ. Let $e_\varphi \in \mathfrak{g}$ denote an eigenvector corresponding to a character φ, hence $[x, e_\varphi] = \langle x, \varphi \rangle e_\varphi$ for any $x \in \mathfrak{h}$ and, by the structure theorem of semi-simple Lie algebra, $\mathfrak{g}^\varphi = \boldsymbol{C} e_\varphi$. Let $\varDelta$ be the set of characters of $\mathfrak{h}$ such that $\mathfrak{g}^\varphi \neq 0$. $\varDelta$ is the set of roots of $\mathfrak{g}$ and an eigenvector corresponding to a root is called a root vector. We know that the root vector e_φ can be chosen so that

$$\begin{aligned} (e_\varphi, e_\psi) &= 0 \qquad \text{if } \psi \neq -\varphi \\ &= 1 \qquad \text{if } \psi = -\varphi . \end{aligned}$$

Then moreover we have $[e_\varphi, e_{-\varphi}] = x_\varphi$.

Let $\mathfrak{h}^\#$ be the R-linear subspace of $\mathfrak{h}'$ generated by the set $\varDelta$. We know that the restriction of the Cartan-Killing form on $\mathfrak{h}^\#$ is positive definite. Let $\mathfrak{r}$ be a subspace of $\mathfrak{g}$ invariant under the adjoint representation of $\mathfrak{h}$. Then $\varDelta(\mathfrak{r})$ is by definition the subset of $\varDelta$ consisting of all the roots φ such that the eigenspace $\mathfrak{g}^\varphi$ is contained in $\mathfrak{r}$ and $\mathfrak{r}^0$ is the set of all the elements $z \in \mathfrak{g}$ snch that $(z, y) = 0$ for any $y \in r$.

Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. We fix $\mathfrak{b}$ once and for all. Let now consider a simply connected complex Lie group G whose Lie algebra is isomorphic to $\mathfrak{g}$. Let B be the Borel subgroup of G corresponding to the Borel subalgebra $\mathfrak{b}$. Let $\mathfrak{P}$ be the set of all the parabolic Lie subalgebra $\mathfrak{p}$ containing the Borel subalgebra $\mathfrak{b}$. Let $\mathfrak{b} \subset \mathfrak{p}$ be a parabolic Lie subalgebra and $B \subset P$ be the parabolic subgroup corresponding to $\mathfrak{p}$.

It is well-known that the quotient space $X = G/P$ is a projective algebraic variety. We assume that $\mathfrak{p}$ does not contain a simple factor. We denote by n the dimension of X. If we put $\mathfrak{g}_1 = \mathfrak{p} \cap \mathfrak{p}^*$ and $\mathfrak{m} = \mathfrak{b}^0$, then $\mathfrak{g}_1$ is reductive in $\mathfrak{g}$ and $\mathfrak{m}$ is a maximal nilpotent Lie subalgebra and $\mathfrak{m}$ is the set of all nilpotent elements in $\mathfrak{b}$. We know that if $\mathfrak{n} = \mathfrak{g}^0$, then $\mathfrak{n}$ is the maximal nilpotent ideal in $\mathfrak{p}$ and that $\mathfrak{p} = \mathfrak{g}_1 + \mathfrak{n}$. If we put $\Delta_+ = \Delta(\mathfrak{m})$ and $\Delta_- = -\Delta_+$, then $\Delta = \Delta_+ \cup \Delta_-$ is a disjoint union and there exists a subset $\Pi \subset \Delta$ such that for any element $\varphi \in \Delta$, $\varphi = \sum_{\alpha \in \Pi} n_\alpha(\varphi)\alpha$ where the n_α are non-negative or non-positive integers according as $\varphi \in \Delta_+$ or $\varphi \in \Delta_-$. The set Π is called the set of simple roots.

Let G_1 be the subgroup of G corresponding to the subalgebra $\mathfrak{g}_1$ and $Z \subset \mathfrak{h}^\# \subset \mathfrak{h}'$ be the set of all integral linear forms on $\mathfrak{h}$. Then the elements of Z are the weights of all the finite dimensional representation of G_1. Let ν_1 be a finite dimensional irreducible representation of G_1. An extremal weight of ν_1 is a weight appearing in ν_1 that becomes highest for a some lexiographical ordering of Z. We denote by W_1 the Weyl group of $\mathfrak{g}_1$. If ξ is an extremal weight of ν_1, then the collection $\{\sigma\xi\}$, $\sigma \in W_1$ is the set of all the extremal weights. Let $\xi \in Z$. We denote by ν_1^ξ the unique irreducible representation of $\mathfrak{g}_1$ having ξ as an extremal weight. Let ξ_1, ξ_2 be two elements of Z. Then the representations $\nu_1^{\xi_1}, \nu_2^{\xi_2}$ are isomorphic if and only if there exists an element $\sigma \in W_1$ such that $\sigma\xi_1 = \xi_2$. Let $\mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{g}_1$ and $D_1 = \{\mu \in Z \,|\, (\mu, \varphi) \geq 0$ for all $\varphi \in \Delta(\mathfrak{m}_1)\}$. The elements of D_1 will be called dominant. One knows that D_1 is a fundamental domain for the action of W_1 on Z. Hence every irreducible representation of G_1 is equivalent to ν_1^ξ for one and only one $\xi \in D_1$. The weight ξ is called the highest weight of the representation ν_1^ξ. Similarly $-D_1$ is a fundamental domain for the action of W_1 on Z. Hence every irreducible representation of G_1 is equivalent to ν_1^ξ for one and only one $\xi \in -D_1$. The weight ξ is called the lowest weight of the representation. An irreducible representation of G_1 is determined by its lowest weight as well as by its highest weight. When we take $\mathfrak{g}$ itself as a parabolic subgroup, we denote W, D for W_1 and D_1. Note that $D \subset D_1$ and W_1 is a subgroup of W.

Furthermore if we put

$$g_1 = \frac{1}{2} \sum_{\varphi \in \Delta(\mathfrak{m}_1)} \varphi$$

$$g_2 = \frac{1}{2} \sum_{\varphi \in \Delta(\mathfrak{m}_2)} \varphi$$

and

$$g = g_1 + g_2 ,$$

then $g_1 \in D_1$ and $g \in D$.

Define a subset $W^1 \subset W$ by putting $W^1 = \{\sigma \in W \mid \sigma\Delta_- \cap \Delta_+ \subset \Delta(\mathfrak{n})\}$. Let D_1^0 be a subset of D_1 defined by putting $D_1^0 = \{\xi \in D \mid g + \xi \text{ is regular}\}$. Recall that an element $\mu \in Z$ is said to be regular if $(\mu, \varphi) \neq 0$ for all $\varphi \in \Delta$.

LEMMA (1.1) (Kostant [1]). *The mapping $D \times W^1 \to Z$ given by $(\lambda, \sigma) \mapsto \sigma(g + \lambda) - g$ maps $D \times W^1$ bijectively onto D_1^0.*

Now for any $\sigma \in W$, we denote by $n(\sigma)$ the number of roots in $\sigma\Delta_- \cap \Delta_+$.

Let N be the subgroup of G corresponding to the subalgebra $\mathfrak{n}$ of $\mathfrak{g}$. Then $P = G_1N$ is a semi-direct product. It is not difficult to see that any irreducible representation of P is trivial on N and hence is equivalent to $\nu_1^{-\xi}$ for some $\xi \in D_1$ on G_1. Conversely if $\nu_1^{-\xi}$, $\xi \in D_1$ is an irreducible representation of G_1, we can extend it to an irreducible representation of P by giving the action of N trivial. Hereafter we will regard $\nu_1^{-\xi}$ as so extended. Thus, up to equivalence, all irreducible representation of P are of the form $\nu_1^{-\xi}$ for some $\xi \in D_1$.

Let now $\xi \in D_1$ and consider the product $G \times V_1^{-\xi}$. If we set

$$(au, s) \equiv (a, \nu_1^{-\xi}(u)s)$$

for any $a \in G, u \in P$ and $s \in V_1^{-\xi}$, then $\equiv$ is an equivalence relation, and $E^{-\xi} = G \times V_1^{-\xi} / \equiv \to G/P$ is a vector bundle with fibre $V_1^{-\xi}$. Let $a, b \in G$. If $x = bP \in X$, let $ax \in X$ denote the coset abP. Similarly if $v \in E^{-\xi}$ is the equivalence class containing $(b, s) \in G \times V_1^{-\xi}$, let $av \in E^{-\xi}$ denote the equivalence class containing (ab, s). It is clear then that if $X_0 \subset X$ is an open set in X and ψ is a local holomorphic section of $E^{-\xi}$ defined on $a^{-1}X_0$, given by $a(\psi)(x) = a\psi(a^{-1}x)$ where $x \in X_0$ is a local holomorphic section of $E^{-\xi}$ defined on X_0. Now the mapping $\psi \mapsto a\psi$ defines an operator $\rho^{-\xi}(a)$ on $H^i(X, E^{-\xi})$. Since G/P is projective $H^i(X, E^{-\xi})$ is an finite dimensional representation of G. The Borel-Weil theorem teaches us that $H^i(X, E^{-\xi})$ is irreducible and gives us its lowest weight.

Borel-Weil Theorem (1.2). Let $\xi \in D_1$. Then if $\xi \notin D_1^0$, one has $H^j(X, E) = 0$ for any j. If $\xi \in D_1^0$, then upon writing $\xi = \xi(\lambda, \sigma)$ one has $H^j(X, E^{-\xi}) = 0$ for all $j \neq n(\sigma)$ and for $j = n(\sigma)$, the representation on $H^j(X, E^{-\xi})$ is isomorphic to the irreducible representation $\nu^{-\lambda}$ of G.

We shall use the theorem in the following weak form.

COROLLARY (1.3). *Using the notation of the theorem, if* $\xi \notin D$, *then* $H^0(X, E^{-\xi}) = 0$.

For if $H^0(X, E^{-\xi}) \neq 0$, $\xi = \xi(\lambda, \sigma)$ with $n(\sigma) = 0$. $\sigma(\Delta_-) = \Delta_-$ hence $\sigma = \mathrm{id}$. Therefore $\xi \in D$.

§2. Stability of homogeneous vector bundles

LEMMA (2.1). *Let E be an irreducible homogeneous vector bundle of* rank r. *Let $s \leq r$ be an integer and E' be an irreducible component of $\Lambda^s E$. Then the first Chern class $c_1(E')$ of E' is equal to* $\dfrac{s}{r}$ rank $E' c_1(E)$.

Proof. Since E corresponds to the irreducible representation of the reductive Lie algebra, $\Lambda^s E$ is the direct sum of indecomposable homogeneous vector bundles. Since $\mathfrak{g}_1$ is reductive, $\mathfrak{g}_1$ is isomorphic to the direct sum $\mathfrak{c} \oplus \mathscr{D}\mathfrak{g}_1$ where $\mathfrak{c}$ is the center of $\mathfrak{g}_1$ and $\mathscr{D}\mathfrak{g}_1 = [\mathfrak{g}_1, \mathfrak{g}_1]$. We know that $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}_1$. Hence, denoting by $\mathfrak{k}$ a Cartan subalgebra of $\mathscr{D}\mathfrak{g}_1$, $\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{k}$. Therefore $\mathfrak{h}' = \mathfrak{c}' \oplus \mathfrak{k}'$. $\mathfrak{c}'$ is generated by the weights of representations of degree 1 of G_1 and $\mathfrak{k}'$ is generated by the root system of $\mathscr{D}\mathfrak{g}_1$. Now let $\omega^1 = (\omega^1_{(1)}, \omega^1_{(2)}) \in \mathfrak{c}' \oplus \mathfrak{k}' = \mathfrak{h}'$ be the highest weight of the irreducible representation of G_1 yielding the vector bundle E. Then other weights appearing in the representation are of the form $(\omega^1_{(1)}, \omega'_i)$ with $\omega' \in \mathfrak{k}'$ $1 \leq i \leq r$. $\det E$ is given by the representation of degree 1 of G_1 with its weight $\sum_{i=1}^r (\omega^1_{(1)}, \omega'_i) =$ trace of the representation ω^1. But $\sum_{i=1}^r (\omega^1_{(1)}, \omega'_i)$ should be in $\mathfrak{c}' \oplus 0$. Hence $\sum_{i=1}^r (\omega^1_{(1)}, \omega'_i) = r(\omega^1_{(1)}, 0)$. The weights appearing in $\Lambda^s V^{\omega^1}$ are of the form $(s\omega^1_{(1)}, \omega')$ with $\omega' \in \mathfrak{k}'$. $\det E'$ is given by the representation of degree 1 of G_1 with its weight $\sum_{j=1}^t (s\omega^1_{(1)}, \omega'_j)$ where t is the rank of E'. By the same argument as above we conclude $\sum_{j=1}^t (s\omega^1_{(1)}, \omega'_j) = ts(\omega^1_{(1)}, 0)$. This proves Lemma (2.1).

LEMMA (2.2). *Let E' be an irreducible homogeneous vector bundle over X. Assume that there exists an ample line bundle H such that* $(c_1(E') \cdot H^{n-1}) \leq 0$. *Then* $H^0(X, E') = 0$ *if* rank $E' \geq 2$.

Proof. Let $-\xi_1$ be the lowest weight of the representation of G_1 giving the vector bundle E. Let $-\xi_2$ be the weight of the representation of G_1 of degree 1 giving the line bundle $\det E'$. Hence $-\xi_2 = \operatorname{tr} \nu_1^{-\xi_1}$. If $\xi_2 \in D$, then $H^0(X, \det E') \neq 0$ by the Borel-Weil theorem. Hence $\det E' = \mathcal{O}_X$ i.e., $\xi_2 = 0$. Now let us observe: 1^0 $\dim_C V^{-\xi_1} \geqslant 2$, 2^0 an irreducible representation of a semi-simple Lie algebra is the tensor product of irreducible representations of simple Lie algebras, 3^0 the Dynkin diagram of a simple Lie algebra is connected. It follows, from the above observations and from $\xi_2 = 0$, that there exists a simple root β such that $(\xi_1, \beta) < 0$. This shows that ξ_1 does not belong to the Weyl chamber D. Hence by the Borel-Weil theorem $H^0(X, E') = 0$. If $\xi_2 \notin D$, then there exists a simple root $\beta \in \Pi$ such that $(\xi_2, \beta) < 0$. The lowest weight $-\xi_1$ is written in the form $-\xi_1 = -\frac{1}{r}\xi_2 + \sum_{\alpha \in \Pi_1} n_\alpha \alpha$ where n_α is a non-positive integer and r is the rank of E'.*) Therefore $(\xi_1, \beta) = \frac{1}{r}(\nu, \beta) - \sum_{\alpha \in \Pi_1} n_\alpha(\alpha, \beta)$ < 0. Hence $\xi_1 \notin D$ and the lemma follows from the Borel-Weil theorem.

DEFINITION (2.3). Let Y^n be a non-singular projective variety and H an ample line bundle over Y. A vector bundle E over Y is said to be H-stable (in the sense of Mumford and Takemoto) if for any coherent subsheaf F with $1 \leq \operatorname{rank} F < \operatorname{rank} E$, we have following inequality;

$$\frac{(c_1(F)\cdot H^{n-1})}{\operatorname{rank} F} < \frac{(c_1(E)\cdot H^{n-1})}{\operatorname{rank} E} .$$

THEOREM (2.4). *Let E be an irreducible homogeneous vector bundle over X. Then E is H-stable for any ample line bundle H over X.*

Proof. Let F be a subsheaf of E with $1 \leq \operatorname{rank} F < \operatorname{rank} E$ such that we have

$$\frac{(c_1(F)\cdot H^{n-1})}{\operatorname{rank} F} \geq \frac{(c_1(E)\cdot H^{n-1})}{\operatorname{rank} E} .$$

We shall show the existence of such F leads to a contradiction. Let s be the rank F. If we apply Λ^s to the exact sequence $0 \rightarrow F \rightarrow E$, we get $\Lambda^s F \rightarrow \Lambda^s E$ which is injective at the generic point of X. Since F is torsion free, $\Lambda^s F$ is isomorphic to a line bundle L over X minus a sub-

*) $\Pi_1 \subset \Pi$ is a simple root system of $\mathfrak{g}_1$.

variety of codimension 2. On the other hand L uniquely extends to a line bundle over X. We denote the extention again by L which is the first Chern class $c_1(F)$ of F. By tensoring L^{-1}, we get a generic injection $\mathcal{O} \to \Lambda^s E \otimes L^{-1}$ over X minus a subvariety of codimension 2. Hence $H^0(X, \Lambda^s E \otimes L^{-1}) \neq 0$. Now we recall the fact that a line bundle is homogeneous. It follows $\Lambda^s E \otimes L^{-1}$ is homogeneous. Let E' be an irreducible component of $\Lambda^s E$. The first Chern class $c_1(E')$ is given by Lemma (2.1) and equal to $\frac{s}{r}$ rank $E' \cdot c_1(E)$. Hence $c_1(E' \otimes L^{-1}) = \frac{s}{r}$ rank $E' \cdot c_1(E)$ $-$ rank $E' \cdot c_1(F) =$ rank $E'\left(\frac{s}{r} c_1(E) - c_1(F)\right)$. It follows $(c_1(E' \otimes L^{-1}) \circ H^{n-1}) \leq 0$. By Lemma (2.2) $H^0(X, E' \otimes L^{-1}) = 0$ if rank $E' \geq 2$. Hence the generic injection $\mathcal{O} \to \Lambda^s E \otimes L^{-1}$ is trivial onto the irreducible component E' if rank $E' \geq 2$. Therefore there exist line bundles M_i $1 \leq i \leq \ell$ such that $\bigoplus_{i=1}^{\ell} M_i$ is a direct summand of $\Lambda^s E \otimes L^{-1}$ and the map above factors through $\mathcal{O} \to \bigoplus_{i=1}^{\ell} M_i \to \Lambda^s E \otimes L^{-1}$. We choose M_i so that ℓ is minimum. The calculation above shows that $(M_i \cdot H^{n-1}) \leq 0$. On the other hand we have a generic injection $\mathcal{O} \to M_i$ for any $1 \leq i \leq \ell$. Hence $M_i = \mathcal{O}$ and the morphism $\mathcal{O} \to \bigoplus_{i=1}^{\ell} M_i = \bigoplus_{i=1}^{\ell} \mathcal{O}$ is given by a constant matrix. Tensoring L, we get $L \xrightarrow{f} \bigoplus_{i=1}^{\ell} L \xhookrightarrow{j} \Lambda^s E$. Let $-\xi_2$ be the weight of the representation of degree 1 of G_1 defining L. Since f is given by the constant matrix, f is induced by the homomorphism of G_1-modules $V^{-\xi_2} \to \bigoplus_{i=1}^{\ell} V^{-\xi_2}$. The homomorphism j of vector bundles is induced by the decomposition of the G_1-module $\Lambda^s V^{-\xi_1}$ where $\xi \in D_1$ and E is defined by G_1-module $V^{-\xi_1}$. We have proved that the homomorphism $j \circ f$ of the vector bundle L to the vector bundle $\Lambda^s E$ is induced by the homomorphism of the G_1-module $V^{-\xi_2}$ to the G_1-module $\Lambda^s V^{-\xi_1}$.

Now we notice the following; let ρ and ρ' be representations of G_1 we are given a homomorphism φ of G_1-module V^{ρ} to G_1-module $V^{\rho'}$. It induces a homomorphism Φ of vector bundle E^{ρ} to vector bundle $E^{\rho'}$. If we know Φ, by looking at Φ on a fibre we can recover φ.

By the remark above, we can recover the homomorphism of $V^{-\xi_2}$ to $\Lambda^s V^{\xi_1}$ from the homomorphism $L \to \Lambda^s E$ hence from the homomorphism $\Lambda^s F \to \Lambda^s E$ by looking at the homomorphism on a general fibre since these two homomorphism coincide on an open set of X. This shows that the image of $V^{-\xi_2}$ in $\Lambda^s V^{-\xi_1}$ is reduced i.e., written in the form $x_1 \wedge x_2 \wedge \cdots \wedge x_s$. The subspace generated by $x_1, x_2, \cdots, x_s$ in $V^{-\xi_1}$ is G_1-invariant. This contradicts the irreducibility of $V^{-\xi_1}$.

EXAMPLES (2.4). The universal bundle and the tangent bundle of the Grassmannian are H-stable. In particular the tangent bundle of the projective space P^n is H-stable.

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