

## NON-DEGENERATE REAL HYPERSURFACES IN COMPLEX MANIFOLDS ADMITTING LARGE GROUPS OF PSEUDO- CONFORMAL TRANSFORMATIONS II

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### Introduction

This is the continuation of our previous paper [3], and will complete, without homogeneity assumption, the classification of non-degenerate real hypersurfaces  $S$  of complex manifolds  $M$  for which the groups  $A(S)$  of pseudo-conformal transformations of  $S$  have either the largest dimension  $n^2 + 2n$  or the second largest dimension.

Our result is stated as follows

**THEOREM 3.4.** *Let  $M$  be a complex manifold of dimension  $n$ , let  $S$  be a connected non-degenerate (index  $r$ ) hypersurface of  $M$  ( $0 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$ ). Assume that  $A(S)$  attains the second largest dimension, then we have the following classification table:*

$(n, r)$	dim $A(S)$	$S$	
		homogeneous	inhomogeneous
$n = 3$ & $r = 1$	$11 (= n^2 + 2)$	$Q_1^*(1)$	
$n = 5$ & $r = 2$	$26 (= n^2 + 1)$	$Q_2^*(2)$ or $Q_2^*$	$Q_2 \setminus \{\bar{0}\}$
$n \geq 2$ & $r = 0$	$n^2 + 1$	$Q_0^*$	
otherwise	$n^2 + 1$	$Q_r^*$	$Q_r \setminus \{\bar{0}\}$

$$Q_r = \left\{ (z_0, \dots, z_n) \in P^n(\mathbb{C}) \mid -\sqrt{-1}z_0\bar{z}_n - \sum_{i=1}^r z_i\bar{z}_i + \sum_{i=r+1}^{n-1} z_i\bar{z}_i + \sqrt{-1}z_n\bar{z}_0 = 0 \right\},$$

$$Q_r^* = \{(z_0, \dots, z_n) \in Q_r \mid z_0 \neq 0\},$$

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$$\begin{aligned}
Q_1^*(1) &= \{(z_0, \dots, z_3) \in Q_1 \mid |z_0| + |z_1 - z_2| \neq 0\}, \\
Q_2^*(2) &= \{(z_0, \dots, z_6) \in Q_2 \mid |z_0| + |z_1 - z_4| + |z_2 - z_3| \neq 0\}, \\
\tilde{\sigma} &= (0, \dots, 0, 1) \in Q_r,
\end{aligned}$$

where  $P^n(\mathbb{C})$  is the complex projective space of dimension  $n$  with its homogeneous coordinate  $(z_0, \dots, z_n)$ .

This combined with Theorem 7.4 [3] gives the desired classification.

Section I is devoted to the classification of proper graded subalgebras of  $\mathfrak{g}(r)$  of the minimum codimension (The result of this section is already announced in Proposition 4.7 [3]). In section II we study the null ideals (cf. Definition 2.1) of  $\mathfrak{g}^0(r)$  and  $\mathfrak{g}^{**}(r, r)$ . In particular we will see that they are characteristic ideals. With these preparations we will prove Theorem 3.4 in III.

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### Preliminary remarks

Throughout this paper we always assume the differentiability of class  $C^\infty$ . We use the same notations and terminology in our previous paper [3].

### I. Graded subalgebras of $\mathfrak{g}(r)$ (cf. IV [3])

In this section we will determine the graded subalgebras  $\mathfrak{k}$  of  $\mathfrak{g}(r)$  of the minimum codimension without the homogeneity assumption (i.e.  $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$  and  $\mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r)$ ).

First recall the following which is purely computational: For  $\underline{a} \in \mathfrak{g}_{-2}(r)$ ,  $\underline{\xi}_i \in \mathfrak{g}_{-1}(r)$  ( $i = 1, 2$ ),  $X_0 = \begin{pmatrix} -\bar{u} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \in \mathfrak{g}_0(r)$ ,  $\tilde{w}_i \in \mathfrak{g}_1(r)$  ( $i = 1, 2$ ) and  $\tilde{b} \in \mathfrak{g}_2(r)$ , we have

$$(1.1) \quad [\underline{\xi}_1, \tilde{b}] = \tilde{b}\underline{\xi}_1 \quad (\text{resp. } [\tilde{w}_1, \underline{a}] = \underline{a}\tilde{w}_1),$$

$$(1.2) \quad [\underline{\xi}_1, \underline{\xi}_2] = -\underline{\text{Im}} \langle \underline{\xi}_1, \underline{\xi}_2 \rangle \quad (\text{resp. } [\tilde{w}_1, \tilde{w}_2] = \overline{\text{Im}} \langle \tilde{w}_1, \tilde{w}_2 \rangle),$$

$$(1.3) \quad [X_0, \underline{\xi}_1] = v\underline{\xi}_1 + \bar{u}\underline{\xi}_1 \quad (\text{resp. } [X_0, \tilde{w}_1] = \overline{v(w_1) - uw_1}),$$

$$(1.4) \quad [\tilde{w}_1[\tilde{w}_1, \underline{\xi}_1]] = -\sqrt{-1} \langle \tilde{w}_1, \tilde{w}_1 \rangle \underline{\xi}_1 - 2\sqrt{-1} \langle \underline{\xi}_1, \tilde{w}_1 \rangle \tilde{w}_1.$$

Now we will consider a graded subalgebra  $\mathfrak{k} = \sum_{p=-2}^2 \mathfrak{k}_p$  of  $\mathfrak{g}(r)$ .

**LEMMA 1.1.** *If  $\mathfrak{k}_{-2} = \mathfrak{k}_2 = \{0\}$ , then we have  $\text{codim } \mathfrak{k} \geq 2n$  (i.e.  $\dim \mathfrak{k} \leq n^2$ ).*

*Proof.* From (1.2) we see that the bracket operation  $\mathfrak{g}_{-1}(r) \times \mathfrak{g}_{-1}(r) \ni (\xi_1, \xi_2) \mapsto [\xi_1, \xi_2] \in \mathfrak{g}_{-2}(r)$  is non-degenerate. Hence we have  $\dim \mathfrak{k}_{-1} \leq \frac{1}{2} \dim \mathfrak{g}_{-1}(r) = n - 1$ . Similarly we have  $\dim \mathfrak{k}_1 \leq \frac{1}{2} \dim \mathfrak{g}_1(r) = n - 1$ . These facts show the above. Q.E.D.

We now consider the following three cases separately (Note that  $\dim \mathfrak{g}_{-2}(r) = \dim \mathfrak{g}_2(r) = 1$ ).

*Case 1.*  $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$  and  $\mathfrak{k}_2 = \{0\}$ ,

*Case 2.*  $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$  and  $\mathfrak{k}_2 = \mathfrak{g}_2(r)$ ,

*Case 3.*  $\mathfrak{k}_{-2} = \{0\}$  and  $\mathfrak{k}_2 = \mathfrak{g}_2(r)$ .

*Case 1.* First we have

**LEMMA 1.2** (cf. Lemma 4.1 [3]). *For any  $\tilde{w}_1, \tilde{w}_2 \in \mathfrak{k}_1$ , we have  $\langle w_1, w_2 \rangle = 0$ .*

*Proof.* Since  $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$ , we get from (1.1) that

$$\underline{w} \in \mathfrak{k}_{-1} \quad \text{for any } \tilde{w} \in \mathfrak{k}_1.$$

Hence from (1.2) and (1.4) we have

$$[\tilde{w}, [\tilde{w}[\tilde{w}, \underline{w}]]] = -\widetilde{6\langle w, w \rangle^2} \in \mathfrak{k}_2.$$

Since  $\mathfrak{k}_2 = \{0\}$  we get

$$\langle w, w \rangle = 0 \quad \text{for any } \tilde{w} \in \mathfrak{k}_1.$$

Let  $\tilde{w}_1, \tilde{w}_2 \in \mathfrak{k}_1$ . Then from

$$\begin{cases} \tilde{w}_1 + \tilde{w}_2 = \widetilde{w_1 + w_2} & \in \mathfrak{k}_1, \\ [\tilde{w}_1, \tilde{w}_2] = \widetilde{\sqrt{-1}(\langle w_2, w_1 \rangle - \langle w_1, w_2 \rangle)} & \in \mathfrak{k}_2, \end{cases}$$

we have  $\langle w_1 + w_2, w_1 + w_2 \rangle = 0$  and  $\langle w_1, w_2 \rangle = \langle w_2, w_1 \rangle$ . Hence we get  $\langle w_1, w_2 \rangle = 0$ . Q.E.D.

Next we consider the complexification  $\mathfrak{k}_1^c$  of  $\mathfrak{k}_1$ . More precisely we consider the complex vector subspace  $k^c$  of  $\mathbb{C}^{n-1}$  spanned by the vectors in  $k = \delta_r^{-1}(\mathfrak{k}_1)$  (i.e.  $k^c = k + \sqrt{-1}k$ ), where  $\delta_r: \mathbb{C}^{n-1} \ni w \mapsto \tilde{w} \in \mathfrak{g}_1(r)$ . We set  $\mathfrak{k}_1^c = \delta_r^{-1}(k^c)$ . Then we have

**LEMMA 1.3** (cf. Lemma 4.3 [3]).

- (i)  $\mathfrak{k}_1^c$  is an abelian subalgebra of  $\mathfrak{g}(r)$ .
- (ii)  $\delta_r^{-1}(\mathfrak{k}_1^c)$  is a complex isotropic subspace of  $(\mathbf{C}^{n-1}, \langle, \rangle)$ .
- (iii)  $[\mathfrak{k}_0, \mathfrak{k}_1^c] \subset \mathfrak{k}_1^c$ .

*Proof.* (i) and (ii) are obvious from (1.2) and Lemma 1.2. (iii) follows from (1.3).

We set  $\tilde{\mathfrak{k}}_0 = \{X \in \mathfrak{g}_0(r) \mid \text{ad}(X)(\mathfrak{k}_1^c) \subset \mathfrak{k}_1^c\}$ . Then obviously we have

$$\mathfrak{k} \subset \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r) \oplus \tilde{\mathfrak{k}}_0 \oplus \mathfrak{k}_1^c.$$

Hence from Proposition 4.6 [3], we have

PROPOSITION 1.4. *Let  $\mathfrak{k}$  be a graded subalgebra of  $\mathfrak{g}(r)$  satisfying  $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$  and  $\mathfrak{k}_2 = \{0\}$ . Then there exists  $\tau \in G'(r)$  such that  $\text{Ad}(\tau)$  preserves the grading of  $\mathfrak{g}(r)$  and  $\text{Ad}(\tau)(\mathfrak{k}) \subset \mathfrak{g}^*(r, s)$ , where  $2s = \dim \mathfrak{k}_1^c$ .*

Therefore in this case Proposition 4.7 [3] gives the list of the graded subalgebras of the minimum codimension.

*Case 2.* Let  $\tilde{\delta}_r$  be a linear isomorphism of  $\mathbf{C}^{n-1}$  onto  $\mathfrak{g}_{-1}(r)$  defined by  $\tilde{\delta}_r(\xi) = \tilde{\xi}$ ,  $\xi \in \mathbf{C}^{n-1}$ .

LEMMA 1.5. *We have  $[\mathfrak{k}_{-2}, \mathfrak{k}_1] = \mathfrak{k}_{-1}$  and  $[\mathfrak{k}_2, \mathfrak{k}_{-1}] = \mathfrak{k}_1$ . In particular  $\delta_r^{-1}(\mathfrak{k}_1) = \tilde{\delta}_r^{-1}(\mathfrak{k}_{-1}) \subset \mathbf{C}^{n-1}$ .*

*Proof.* Obviously we have  $[\mathfrak{k}_{-2}, \mathfrak{k}_1] \subset \mathfrak{k}_{-1}$  and  $[\mathfrak{k}_2, \mathfrak{k}_{-1}] \subset \mathfrak{k}_1$ . On the other hand we have  $[\mathfrak{k}_{-2}, \mathfrak{k}_2] = [\mathfrak{g}_{-2}(r), \mathfrak{g}_2(r)] = \mathbf{R}E_0$ , where  $E_0$  is the element of  $\mathfrak{g}_0(r)$  which defines the grading of  $\mathfrak{g}(r)$  (cf. I.3 [3]). Hence we get

$$[\mathfrak{k}_2, \mathfrak{k}_{-1}] \supset [\mathfrak{k}_2, [\mathfrak{k}_{-2}, \mathfrak{k}_1]] = [[\mathfrak{k}_2, \mathfrak{k}_{-2}], \mathfrak{k}_1] = \mathfrak{k}_1.$$

Therefore we have  $[\mathfrak{k}_2, \mathfrak{k}_{-1}] = \mathfrak{k}_1$ . Similarly we have  $[\mathfrak{k}_{-2}, \mathfrak{k}_1] = \mathfrak{k}_{-1}$ .

Q.E.D.

We set  $\mathbf{C}^{n-1} \supset k = \delta_r^{-1}(\mathfrak{k}_1) = \tilde{\delta}_r^{-1}(\mathfrak{k}_{-1})$ . Since we are classifying  $\mathfrak{k}$  under the group of automorphisms of the graded Lie algebra  $\mathfrak{g}(r)$ , we have only to classify  $k$  as a (real) subspace of  $(\mathbf{C}^{n-1}, \langle, \rangle)$  (cf. the proof of Lemma 4.4 [3]).

LEMMA 1.6.

- (i) *If there exists  $w_0 \in k$  such that  $\langle w_0, w_0 \rangle \neq 0$ , then  $k$  is a complex vector subspace of  $\mathbf{C}^{n-1}$ .*
- (ii) *Otherwise, we have*

$$\operatorname{Re} \langle w_1, w_2 \rangle = 0 \quad \text{for } w_1, w_2 \in k .$$

In particular we have  $\dim k \leq 2r$ .

*Proof.* (i) From (1.4), we have  $[\bar{w}_0[\bar{w}_0, \underline{w}]] = -3\sqrt{-1}\langle w_0, w_0 \rangle w_0 \in \mathfrak{k}_1$ . Hence we get  $\sqrt{-1}w_0 \in k$ . Moreover we have

$$[\bar{w}_0[\bar{w}_0, \underline{w}]] = -\sqrt{-1}\langle w_0, w_0 \rangle w - 2\sqrt{-1}\langle w, w_0 \rangle w_0 \in \mathfrak{k}_1$$

for  $w \in k$ . Therefore we get  $\sqrt{-1}w \in k$  for any  $w \in k$ .

(ii) We have  $\langle w, w \rangle = 0$  for any  $w \in k$ . Hence we have

$$\langle w_1 + w_2, w_1 + w_2 \rangle = \langle w_1, w_2 \rangle + \langle w_2, w_1 \rangle = 0 \quad \text{for } w_1, w_2 \in k .$$

This shows  $\operatorname{Re} \langle w_1, w_2 \rangle = 0$ . Note that  $\operatorname{Re} \langle w_1, w_2 \rangle$  defines an indefinite inner product of  $\mathbf{C}^{n-1}(= \mathbf{R}^{2(n-1)})$  of type  $(2r, 2(n-r-1))$ . Q.E.D.

In particular we note that  $k$  is necessarily a complex vector subspace in case  $r = 0$ .

Now we consider the complexification  $k^c$  of  $k$  (i.e.  $k^c = k + \sqrt{-1}k$ ). And we study the following two cases separately.

*Case 2.1.*  $k^c = \mathbf{C}^{n-1}$  and *Case 2.2*  $\dim_c k^c = s < n - 1$

*Case 2.1.* First we have

LEMMA 1.7. *If*  $k = k^c (= \mathbf{C}^{n-1})$ , *then*  $\mathfrak{k} = \mathfrak{g}(r)$ .

*Proof.*  $k = \mathbf{C}^{n-1}$  means  $\mathfrak{k}_1 = \mathfrak{g}_1(r)$  and  $\mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r)$ . Hence the assertion follows immediately from Lemma 4.1 [3]. Q.E.D.

Hence we further suppose  $k \subsetneq k^c$  in the following. Then  $k$  cannot be an arbitrary (real) subspace of  $\mathbf{C}^{n-1}$  as Lemma 1.6 shows.

Let  $\{e_i\}_{1 \leq i \leq n-1}$  be the natural base of  $\mathbf{C}^{n-1}$ . We set  $v_i = \frac{1}{\sqrt{2}}(e_i - e_{n-i})$  and  $w_i = \frac{\sqrt{-1}}{\sqrt{2}}(e_i + e_{n-i})$  ( $i = 1, 2, \dots, r$ ). Let  $k(r)$  be the  $2r$ -dimensional real vector subspace of  $\mathbf{C}^{n-1}$  spanned by the  $2r$  vectors  $v_1, \dots, v_r, w_1, \dots, w_r$ . Then we have

LEMMA 1.8. *If*  $k \subsetneq k^c = \mathbf{C}^{n-1}$ , *then we have*  $r = \frac{n-1}{2}$  ( $n$ : odd integer) *and*  $\dim k = 2r = n - 1$ . *Moreover there exists*  $\sigma \in U(I_r)$  *such that*  $\sigma(k) = k(r)$ .

*Proof.* From Lemma 1.6, we have  $\dim k \leq 2r \leq 2\left[\frac{n-1}{2}\right]$ . On the other hand we have  $\mathbf{C}^{n-1} = k^c = k + \sqrt{-1}k$ . Hence we have  $r = \frac{n-1}{2}$  and  $\dim k = 2r$ . In particular  $k$  is a real form of  $k^c = \mathbf{C}^{n-1}$ . Though the last assertion follows immediately using the Witt's theorem, we briefly sketch the proof of it. Let  $\zeta_1$  be an arbitrary element of  $k$ . Since  $\langle, \rangle$  is non-degenerate, there exists  $\eta_1 \in k$  such that  $\langle \zeta_1, \eta_1 \rangle = \sqrt{-1}$ . Let  $k_1$  be the real vector subspace of  $k$  spanned by  $\zeta_1$  and  $\eta_1$ . Then  $k_1^c$  is a non-degenerate subspace of  $\mathbf{C}^{n-1}$ . Hence we have

$$\mathbf{C}^{n-1} = k_1^c \oplus (k_1^c)^\perp \quad (\text{direct sum}).$$

We easily see that  $(k_1^c)^\perp = \overline{(k_1^c)^\perp}$ , where  $\overline{\phantom{x}}$  is the conjugation with respect to the real form  $k$  of  $\mathbf{C}^{n-1}$ . Hence there exists a subspace  $k'_1$  of  $k$  such that  $(k_1^c)^\perp = (k'_1)^c$ . Then we have

$$k = k_1 \oplus k'_1 \quad (\text{direct sum}).$$

For an arbitrary  $\zeta_2 \in k_1$  we repeat the above procedure for  $(k'_1)^c$ . Then we get the base  $\{\zeta_i, \eta_i\}_{1 \leq i \leq r}$  of  $k$  which satisfies  $\langle \zeta_i, \zeta_j \rangle = \langle v_i, v_j \rangle$ ,  $\langle \zeta_i, \eta_j \rangle = \langle v_i, w_j \rangle$ , and  $\langle \eta_i, \eta_j \rangle = \langle w_i, w_j \rangle$  ( $i, j = 1, 2, \dots, r$ ). Then we have only to define  $\sigma$  by  $\sigma(\zeta_i) = v_i$  and  $\sigma(\eta_i) = w_i$  ( $i = 1, 2, \dots, r$ ). Q.E.D.

We set  $\mathfrak{f}_{-1} = \delta_r^{-1}(k(r))$ ,  $\mathfrak{f}_1 = \delta_r^{-1}(k^c(r))$ ,  $\mathfrak{f}_0 = \{X \in \mathfrak{g}_0(r) \mid \text{ad}(X)(\mathfrak{f}_i) \subset \mathfrak{f}_i \ (i = -1, 1)\}$ , and  $\mathfrak{f}(r) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{f}_{-1} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{g}_2(r)$ .

From Lemma 1.8, we get

**PROPOSITION 1.9.** *Let  $\mathfrak{f}$  be a graded subalgebra of  $\mathfrak{g}(r)$  satisfying  $\mathfrak{f}_{-2} = \mathfrak{g}_{-2}(r)$ ,  $\mathfrak{f}_2 = \mathfrak{g}_2(r)$ ,  $k^c = \mathbf{C}^{n-1}$ , and  $k^c \supseteq k$ . Then we have  $r = \frac{n-1}{2}$  ( $n$ : odd number), and there exists  $\tau \in G'(r)$  such that  $\text{Ad}(\tau)$  preserves the grading of  $\mathfrak{g}(r)$  and  $\text{Ad}(\tau)\mathfrak{f} \subset \mathfrak{f}(r)$ .*

*In particular  $\dim \mathfrak{f} \leq \dim \mathfrak{f}(r) = (2r+1)(r+3) \leq n^2+1$ .*

The proof is quite similar to that of Lemmas 4.4 and 4.5 [3], hence is omitted. Note that  $\dim \mathfrak{f}(r) = n^2+1$  if and only if  $n=3$  (and  $r=1$ ).

*Case 2.2.* We set  $\mathfrak{f}_{-1}^c = \delta_r^{-1}(k^c)$ ,  $\mathfrak{f}_1^c = \delta_r^{-1}(k)$ ,  $\mathfrak{f}_0^c = \{X \in \mathfrak{g}_0(r) \mid \text{ad}(X)(\mathfrak{f}_i^c) \subset \mathfrak{f}_i^c \ (i = -1, 1)\}$ , and  $\mathfrak{f}^c = \mathfrak{f}_{-2}^c \oplus \mathfrak{f}_{-1}^c \oplus \mathfrak{f}_0^c \oplus \mathfrak{f}_1^c \oplus \mathfrak{f}_2^c$ . Then obviously we have (cf. (iii) of Lemma 1.3)

LEMMA 1.10.  $\mathfrak{k}$  is a proper graded subalgebra of  $\mathfrak{g}(r)$  such that  $\mathfrak{k} \subset \mathfrak{k}$ .

Hence we will classify such  $\mathfrak{k}$  under the group of automorphisms of  $\mathfrak{g}(r)$ . Note that  $\mathfrak{k}$  is completely determined by  $k^c \subset \mathbb{C}^{n-1}$ . Therefore in order to classify  $\mathfrak{k}$ , we have only to classify  $k^c$  under the group  $U(I_r)$  (cf. the proofs of Lemmas 4.4 and 4.5 [3]). However this is attained by the Witt's theorem as in the following.

Let  $s, a$  and  $b$  be natural numbers satisfying  $a + b \leq s < n - 1$ ,  $s - a \leq n - r$  and  $s - b \leq r$ . Let  $\{e_i\}_{1 \leq i \leq n-1}$  be the natural base of  $\mathbb{C}^{n-1}$ . We set  $\zeta_i = \frac{1}{\sqrt{2}}(e_i + e_{n-i})$  ( $i = 1, 2, \dots, s - (a + b)$ ). And let  $k^s(a, b)$  be the complex vector subspace of  $\mathbb{C}^{n-1}$  spanned by the  $s$  vectors  $\zeta_1, \dots, \zeta_{s-(a+b)}, e_{s+1-(a+b)}, \dots, e_{s-b}, e_{r+1}, \dots, e_{r+b}$ . Then  $k^s(a, b)$  is an  $s$ -dimensional subspace of  $\mathbb{C}^{n-1}$  and the restriction to  $k^s(a, b)$  of the hermitian inner product of  $\mathbb{C}^{n-1}$  defined by  $I_r$  is a (possibly degenerate) hermitian inner product of type  $(a, b)$  (cf. I.3 [3]). We say that the complex vector subspace of  $(\mathbb{C}^{n-1}, I_r)$  is of type  $(a, b)$  if the induced hermitian inner product from  $(\mathbb{C}^{n-1}, I_r)$  is of type  $(a, b)$ .

LEMMA B (WITT'S THEOREM). If  $\dim_{\mathbb{C}} k^c = s < n - 1$  and  $k^c$  is a subspace of type  $(a, b)$ , then there exists  $\sigma \in U(I_r)$  such that  $\sigma(k^c) = k^s(a, b)$ .

We set  $\mathfrak{d}_r^{-1}(s, a, b) = \tilde{\mathfrak{d}}_r(k^s(a, b))$ ,  $\mathfrak{d}_r^1(s, a, b) = \delta_r(k^s(a, b))$ ,  $\mathfrak{e}_r(s, a, b) = \{X \in \mathfrak{g}_0(r) \mid \text{ad}(X)(\mathfrak{d}^i) \subset \mathfrak{d}^i \text{ (} i = -1, 1)\}$  and  $\mathfrak{g}_r^0(s, a, b) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{d}_r^{-1} \oplus \mathfrak{e}_r \oplus \mathfrak{d}_r^1 \oplus \mathfrak{g}_2(r)$ . Then we have

PROPOSITION 1.11. Let  $\mathfrak{k}$  be a graded subalgebra of  $\mathfrak{g}(r)$  satisfying  $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$ ,  $\mathfrak{k}_2 = \mathfrak{g}_2(r)$  and  $\dim_{\mathbb{C}} k^c = s < n - 1$ . Then there exists  $\tau \in G^1(r)$  such that  $\text{Ad}(\tau)$  preserves the grading of  $\mathfrak{g}(r)$  and  $\text{Ad}(\tau)\mathfrak{k} \subset \mathfrak{g}_r^0(s, a, b)$ , where  $(a, b)$  is the type of  $k^c$  in  $(\mathbb{C}^{n-1}, I_r)$ . In particular  $\dim \mathfrak{k} \leq \dim \mathfrak{g}_r^0(s, a, b) = (a + b - s)^2 + 2s^2 - 2s(n - 3) + (n - 1)^2 + 3 \leq n^2 + 1$ .

The proof is quite similar to that of Lemmas 4.4 and 4.5 [3], hence is omitted. Note that  $\dim \mathfrak{g}_r^0(a, b) = n^2 + 1$  if and only if  $s = n - 2$ ,  $a = r - 1$  and  $b = n - r - 2$ . We will write  $\mathfrak{g}_r^0(n - 2, r - 1, n - r - 2)$  simply as  $\mathfrak{g}^0(r)$  (cf. Remark 4.8 [3]).

Case 3 can be treated quite similarly as Case 1.

And we obtain

**PROPOSITION 1.12.** *Let  $\mathfrak{k}$  be a graded subalgebra of  $\mathfrak{g}(r)$  satisfying  $\mathfrak{k}_{-2} = \{0\}$  and  $\mathfrak{k}_2 = \mathfrak{g}_2(r)$ . Then there exists  $\tau \in G'(r)$  such that  $\text{Ad}(\tau)$  preserves the grading of  $\mathfrak{g}(r)$  and  $\text{Ad}(\tau)(\mathfrak{k}) \subset \mathfrak{g}^{**}(r, s)$ , where  $2s = \dim \mathfrak{k}_{-1}$ .*

$$\mathfrak{g}^{**}(r, s) = \mathfrak{c}_s^*(r) \oplus \mathfrak{b}_s^*(r) \oplus \mathfrak{g}_1(r) \oplus \mathfrak{g}_2(r) .$$

Here

$$\mathfrak{c}_s^*(r) = \{\xi \in \mathfrak{g}_{-1}(r) \mid \xi \in \mathfrak{c}_s(r)\}$$

and

$$\mathfrak{b}_s^*(r) = \{X \in \mathfrak{g}_0(r) \mid \text{ad}(X)(\mathfrak{c}_s^*(r)) \subset \mathfrak{c}_s^*(r)\} \quad (= \mathfrak{b}_s(r)) .$$

Summarizing the above discussion we obtain the classification of the graded subalgebras of  $\mathfrak{g}(r)$  of the minimum codimension.

**PROPOSITION 1.13** (cf. Proposition 4.9 [3]). *Let  $\mathfrak{k}$  be a proper graded subalgebra of  $\mathfrak{g}(r)$ .*

- (1) *The case  $n = 3$  and  $r = 1$ .  $\dim \mathfrak{k} \leq n^2 + 2 = 11$ .*

*The equality holds if and only if there exists  $\tau \in G'(1)$  such that  $\text{Ad}(\tau)$  preserves the grading of  $\mathfrak{g}(1)$  and*

$$\text{Ad}(\tau)\mathfrak{k} = \mathfrak{g}^*(1, 1) \quad \text{or} \quad \mathfrak{g}^{**}(1, 1) .$$

- (2) *The case  $n = 5$  and  $r = 2$ .  $\dim \mathfrak{k} \leq n^2 + 1 = 26$ .*

*The equality holds if and only if there exists  $\tau \in G'(2)$  such that  $\text{Ad}(\tau)$  preserves the grading of  $\mathfrak{g}(2)$  and*

$$\text{Ad}(\tau)\mathfrak{k} = \mathfrak{g}^*(2, 2) , \quad \mathfrak{g}^{**}(2, 2) , \quad \mathfrak{g}^*(2) , \quad \mathfrak{g}'(2) \quad \text{or} \quad \mathfrak{g}^0(2) .$$

- (3) *The case  $n \geq 2$  and  $r = 0$ .  $\dim \mathfrak{k} \leq n^2 + 1$*

*The equality holds if and only if there exists  $\tau \in G'(0)$  such that  $\text{Ad}(\tau)$  preserves the grading of  $\mathfrak{g}(0)$  and*

$$\text{Ad}(\tau)\mathfrak{k} = \mathfrak{g}^*(0) \quad \text{or} \quad \mathfrak{g}'(0) .$$

- (4) *Otherwise.  $\dim \mathfrak{k} \leq n^2 + 1$ .*

*The equality holds if and only if there exists  $\tau \in G'(r)$  such that  $\text{Ad}(\tau)$  preserves the grading of  $\mathfrak{g}(r)$  and*

$$\text{Ad}(\tau)\mathfrak{k} = \mathfrak{g}^*(r) , \quad \mathfrak{g}'(r) \quad \text{or} \quad \mathfrak{g}^0(r) .$$



## II. Radicals and null ideals of $\mathfrak{g}^0(r)$ and $\mathfrak{g}^{**}(r, r)$

In this section we will seek the explicit form of the radicals and the null ideals (cf. Definition 2.1) of  $\mathfrak{g}^0(r)$  ( $r \geq 1$ ) and  $\mathfrak{g}^{**}(r, r)$  ( $r = \frac{n-1}{2}$ ).

**DEFINITION 2.1.** Let  $\mathfrak{g} = \sum_p \mathfrak{g}_p$  be a graded Lie algebra, and set  $\mathfrak{g}' = \sum_{p \geq 0} \mathfrak{g}_p$ . Then we call an ideal  $\mathfrak{n}$  the null ideal of  $\mathfrak{g}$  if it is the maximal ideal of  $\mathfrak{g}$  contained in  $\mathfrak{g}'$ .

Let  $\mathfrak{n}^0$  be the null ideal of  $\mathfrak{g}^0(r) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{d}^{-1}(r) \oplus \mathfrak{e}(r) \oplus \mathfrak{d}^1(r) + \mathfrak{g}_2(r)$ . Then we have

**LEMMA 2.2.**  $\mathfrak{n}^0 = \{X \in \mathfrak{e}(r) \mid \text{ad}(X)(\mathfrak{g}_{-2}(r)) = \text{ad}(X)(\mathfrak{d}^{-1}(r)) = 0\}$ . In particular  $\dim \mathfrak{n}^0 = 1$ .

*Proof.* Since  $\mathfrak{g}^0(r)$  contains  $E_0 \in \mathfrak{e}(r)$ , which defines the grading of  $\mathfrak{g}^0(r)$ , it is easily seen that any ideal of  $\mathfrak{g}^0(r)$  is a graded ideal of  $\mathfrak{g}^0(r)$ . Hence we have  $\mathfrak{n}^0 = \mathfrak{n}_0^0 \oplus \mathfrak{n}_1^0 \oplus \mathfrak{n}_2^0$ , where  $\mathfrak{n}_0^0 = \mathfrak{n}^0 \cap \mathfrak{e}(r)$ ,  $\mathfrak{n}_1^0 = \mathfrak{n}^0 \cap \mathfrak{d}^1(r)$ , and  $\mathfrak{n}_2^0 = \mathfrak{n}^0 \cap \mathfrak{g}_2(r)$ . From  $[\mathfrak{g}_{-2}(r), \mathfrak{n}^0] \subset \mathfrak{n}^0$ , we get

$$[\mathfrak{g}_{-2}(r), \mathfrak{n}_0^0] = 0, \quad [\mathfrak{g}_{-2}(r), \mathfrak{n}_1^0] = 0, \quad \text{and} \quad [\mathfrak{g}_{-2}(r), [\mathfrak{g}_{-2}(r), \mathfrak{n}_2^0]] = 0.$$

On the other hand we have  $[\mathfrak{g}_{-2}(r), \mathfrak{g}_2(r)] = \mathbf{RE}_0$  and the map  $\mathfrak{d}^1(r) \ni \tilde{w} \mapsto [\underline{g}, \tilde{w}] \in \mathfrak{d}^{-1}(r)$  is injective for  $\underline{g} \neq 0 \in \mathfrak{g}_{-2}(r)$ . Hence we have  $\mathfrak{n}_2^0 = \mathfrak{n}_1^0 = 0$  (i.e.  $\mathfrak{n}^0 \subset \mathfrak{e}(r)$ ). Moreover from  $[\mathfrak{d}^{-1}(r), \mathfrak{n}^0] \subset \mathfrak{n}^0$ , we get  $[\mathfrak{d}^{-1}(r), \mathfrak{n}^0] = 0$ . Hence  $\mathfrak{n}^0 \subset \mathfrak{M} = \{X \in \mathfrak{e}(r) \mid \text{ad}(X)(\mathfrak{g}_{-2}(r)) = \text{ad}(X)(\mathfrak{d}^{-1}(r)) = 0\}$ . It is easy to see that  $\mathfrak{M}$  is an ideal of  $\mathfrak{e}(r)$ ,  $[\mathfrak{M}, \mathfrak{d}^1(r)] = 0$  and  $[\mathfrak{M}, \mathfrak{g}_2(r)] = 0$ . Therefore  $\mathfrak{M}$  is an ideal of  $\mathfrak{g}^0(r)$ . The maximality of  $\mathfrak{n}^0$  implies  $\mathfrak{n}^0 = \mathfrak{M}$ . The last assertion follows from the explicit matrix representation of  $\mathfrak{n}^0$ .

Q.E.D.

Next we will study the radical  $\mathfrak{r}^0 = \sum_{p=-2}^2 \mathfrak{r}_p^0$  of  $\mathfrak{g}^0(r)$  (Note that  $\mathfrak{r}^0$  is a graded ideal).

**LEMMA 2.3.**  $\mathfrak{r}_{-2}^0 = \mathfrak{r}_2^0 = 0$

*Proof.* Since  $\mathfrak{g}^0(r)/\mathfrak{r}^0 = \mathfrak{g}_{-2}(r)/\mathfrak{r}_{-2}^0 \oplus \mathfrak{d}^{-1}(r)/\mathfrak{r}_{-1}^0 \oplus \mathfrak{e}(r)/\mathfrak{r}_0^0 \oplus \mathfrak{d}^1(r)/\mathfrak{r}_1^0 \oplus \mathfrak{g}_2(r)/\mathfrak{r}_2^0$  is a semi-simple graded Lie algebra, it is well known that  $\dim \mathfrak{g}_{-2}(r)/\mathfrak{r}_{-2}^0 = \dim \mathfrak{g}_2(r)/\mathfrak{r}_2^0$ . Hence if  $\mathfrak{r}_{-2}^0 \neq 0$  (i.e.  $\mathfrak{r}_{-2}^0 = \mathfrak{g}_{-2}(r)$ ), then we have  $\mathfrak{r}_2^0 = \mathfrak{g}_2(r)$ . On the other hand we have  $[\mathfrak{g}_{-2}(r), \mathfrak{g}_2(r)] = \mathbf{RE}_0$ . Setting  $\mathfrak{s} = \mathfrak{r}_{-2}^0 + [\mathfrak{r}_{-2}^0, \mathfrak{r}_2^0] + \mathfrak{r}_2^0$ , we get  $\mathfrak{s} \subset \mathfrak{r}^0$  and  $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ . This contradiction proves the Lemma.

Q.E.D.

Let  $\{e_i\}_{1 \leq i \leq n-1}$  be the natural base of  $\mathcal{C}^{n-1}$ . We set  $\zeta_1 = \frac{1}{\sqrt{2}}(e_1 + e_{n-1})$  and  $\eta_1 = \frac{1}{\sqrt{2}}(e_1 - e_{n-1})$ . Let  $D(r)$  be the  $(n-2)$ -dimensional subspace of  $\mathcal{C}^{n-1}$  spanned by the  $(n-2)$  vectors  $e_2, \dots, e_{n-2}$  and  $\zeta_1(D(r)) = k^{n-2}(r-1, n-r-1)$  in the notation of I). Hence by definition  $D(r) = \tilde{\delta}_r^{-1}(\mathfrak{b}^{-1}(r)) = \delta_r^{-1}(\mathfrak{b}^1(r))$ . We set  $D^\perp(r) = \{w \in D(r) \mid \langle w, \xi \rangle = 0 \text{ for any } \xi \in D(r)\}$ . Then obviously we have  $D^\perp(r) = \mathcal{C}\zeta_1$ .

LEMMA 2.4.  $\tilde{\delta}_r^{-1}(\mathfrak{r}_{-1}^0) = \delta_r^{-1}(\mathfrak{r}_1^0) \subset D^\perp(r)$

*Proof.* From  $[\mathfrak{g}_{-2}(r), \mathfrak{r}_1^0] \subset \mathfrak{r}_{-1}^0$  and  $[\mathfrak{g}_2(r), \mathfrak{r}_{-1}^0] \subset \mathfrak{r}_1^0$ , we get  $[\mathfrak{g}_{-2}(r), \mathfrak{r}_1^0] = \mathfrak{r}_{-1}^0$  and  $[\mathfrak{g}_2(r), \mathfrak{r}_{-1}^0] = \mathfrak{r}_1^0$  similarly as in Lemma 1.5. Hence we have  $\tilde{\delta}_r^{-1}(\mathfrak{r}_{-1}^0) = \delta_r^{-1}(\mathfrak{r}_1^0)$ . Let  $\xi \in D(r)$  and  $w \in \delta_r^{-1}(\mathfrak{r}_1^0)$ . From  $[\mathfrak{b}^1(r), \mathfrak{r}_1^0] \subset \mathfrak{r}_2^0 = \{0\}$ , we get  $[\tilde{\xi}, \tilde{w}] = 0$  and  $[\sqrt{-1}\tilde{\xi}, \tilde{w}] = 0$ . Then from (1.2) we obtain  $\langle \xi, w \rangle - \langle w, \xi \rangle = 0$  and  $\langle \sqrt{-1}\xi, w \rangle - \langle w, \sqrt{-1}\xi \rangle = 0$ . Hence we have  $\langle \xi, w \rangle = 0$ . Q.E.D.

We set  $\mathfrak{r}_{-1} = \tilde{\delta}_r(D^\perp(r))$ ,  $\mathfrak{r}_1 = \delta_r(D^\perp(r))$ ,  $\mathfrak{r}_0 = \{X \in \mathfrak{e}(r) \mid \text{ad}(X)(\mathfrak{g}_{-2}(r)) = 0 \text{ and } \text{ad}(X)(\mathfrak{b}^{-1}(r)) \subset \mathfrak{r}_{-1}\}$ , and  $\mathfrak{r} = \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1$ . Then we have

LEMMA 2.5.  $\mathfrak{r}^0 = \mathfrak{r}$

*Proof.* Obviously we have  $\mathfrak{r}^0 \subset \mathfrak{r}$ . It is also easy to see that  $\mathfrak{r}$  is an ideal of  $\mathfrak{g}^0(r)$ . Hence we have only to show that  $\mathfrak{r}$  is solvable. For this purpose we take a base  $\{e_i, \zeta_1, \eta_1\}_{2 \leq i \leq n-2}$  of  $\mathcal{C}^{n-1}$  explained before Lemma 2.4, and represent elements of  $\mathfrak{r}_0$  as matrices with respect to this base. Note that

$$\begin{aligned} \mathfrak{h}_0 &= \{X \in \mathfrak{g}_0(r) \mid \text{ad}(X)(\mathfrak{g}_{-2}(r)) = 0\} \\ &= \left\{ X \in \mathfrak{g}_0(r) \mid X = \begin{pmatrix} -\frac{1}{2} \text{tr } v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2} \text{tr } v \end{pmatrix}, v \in \mathfrak{u}(I_r) \right\} \end{aligned}$$

(which we several times denoted by  $\mathfrak{u}(I_r)$ ). Hence in this proof we identify  $\mathfrak{h}_0$  with  $\mathfrak{u}(I_r)$ . With respect to the base  $\{e_i, \zeta_1, \eta_1\}_{2 \leq i \leq n-2}$ ,  $I_r$  is represented as a matrix of the following form:

$$\begin{pmatrix} I_r^* & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{where} \quad I_r^* = \begin{pmatrix} -E_{r-1} & 0 \\ 0 & E_{n-r-2} \end{pmatrix}.$$

Then from (1.3),  $D(r) = \langle e_2, \dots, e_{n-2}, \zeta_1 \rangle_{\mathcal{C}}$  and  $\tilde{\delta}_r^{-1}(\mathfrak{r}_{-1}) = \mathcal{C}\zeta_1$ , we get

$$r_0 = \left\{ X \in \mathfrak{h}_0 \mid X = \begin{pmatrix} 0 & 0 & w \\ -\bar{w}I_r^* & \alpha & \sqrt{-1}a \\ 0 & 0 & -\bar{\alpha} \end{pmatrix} \alpha \in \mathbf{R}, \alpha \in \mathbf{C}, w \in \mathbf{C}^{n-3} \right\}.$$

On the other hand for  $\xi \in \mathfrak{g}_{-1}(r)$  we have  $[\zeta_1, \sqrt{-1}\zeta_1] = [\sqrt{-1}\zeta_1, \zeta_1] = 0$

$$[[\zeta_1, \zeta_1], \xi] = [[\sqrt{-1}\zeta_1, \sqrt{-1}\zeta_1], \xi] = -2\sqrt{-1}\langle \zeta_1, \xi \rangle \zeta_1 \quad (\text{cf. (4.2)[3]}).$$

Hence a direct calculation shows

$$\mathcal{D}r = [r, r] = r_{-1} + r'_0 + r_1 \quad \text{and} \quad \mathcal{D}^2r = [\mathcal{D}r, \mathcal{D}r] = r''_0$$

where  $r'_0 = \{X \in r_0 \mid \alpha = 0\}$  and  $r''_0 = \{X \in r_0 \mid \alpha = w = 0\}$ . Therefore we get  $\mathcal{D}^3r = \{0\}$ . Q.E.D.

In the above proof we note that  $r''_0 = n^0$ . Hence we obtain

**PROPOSITION 2.6.**  $n^0 = \mathcal{D}^2r^0$ . In particular  $n^0$  is a characteristic ideal of  $\mathfrak{g}^0(r)$ .

We add the following which is needed in III.

**LEMMA 2.7.** Let  $X \neq 0 \in n^0$ . For  $Y \in \mathfrak{m}(r) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r)$ ,  $[X \cdot Y] = 0$  if and only if  $Y \in \mathfrak{g}_{-2}(r) \oplus \mathfrak{b}^{-1}(r)$ .

*Remark 2.8.* As we will show in III,  $\mathfrak{g}^0(r)$  is isomorphic with  $\mathfrak{g}^*(r) = \sum_{p \leq 0} \mathfrak{g}_p(r)$  ( $r \neq 0$ ). On the other hand we have an obvious Levi decomposition of  $\mathfrak{g}^*(r)$  as follows; Let  $\mathfrak{z}$  be the center of  $\mathfrak{h}_0 = \mathfrak{u}(I_r)$  (as in the proof of Lemma 2.5 we identify  $\mathfrak{h}_0$  with  $\mathfrak{u}(I_r)$ ). We set  $r^* = \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r) \oplus \mathbf{R}E_0 \oplus \mathfrak{z}$ . Then  $r^*$  is obviously a solvable ideal. Hence a decomposition

$$\mathfrak{g}^*(r) = r^* + \mathfrak{su}(I_r)$$

gives a Levi decomposition of  $\mathfrak{g}^*(r)$ . In this connection  $n^0$  of  $\mathfrak{g}^0(r)$  corresponds to  $\mathfrak{g}_{-2}(r)$  of  $\mathfrak{g}^*(r)$ .

Now we will turn to the case of  $\mathfrak{g}^{**}(r, r)$ . First recall the following;  $\mathfrak{g}^{**}(r, r) = c_r^*(r) \oplus \mathfrak{b}_r^*(r) \oplus \mathfrak{g}_1(r) \oplus \mathfrak{g}_2(r)$ , where  $\tilde{\delta}_r^{-1}(c_r^*(r)) = C_r(r)$  ( $=k^r(0, 0)$  in the notation of I.) and

$$\mathfrak{b}_r^*(r) = \{X \in \mathfrak{g}_0(r) \mid \text{ad}(X)(c_r^*(r)) \subset c_r^*(r)\} \quad (\text{cf. Remark 4.8 [3]}).$$

We set  $n_0 = \{X \in \mathfrak{b}_r^*(r) \mid \text{ad}(X)(c_r^*(r)) = 0\}$  and  $n_1 = \{\bar{w} \in \mathfrak{g}_1(r) \mid \text{ad}(\bar{w})(c_r^*(r)) \subset n_0\}$ .

- LEMMA 2.9. (i)  $n_1 = \{\bar{w} \in \mathfrak{g}_1(r) \mid w \in (C_r(r))^\perp\}$ ,  
(ii)  $[n_0, \mathfrak{g}_1(r)] \subset n_1$ .

*Proof.* (i) By definition  $\bar{w} \in \mathfrak{g}_1(r)$  is in  $n_1$  if and only if  $[[\bar{w}, \xi_1] \xi_2] = 0$  for  $\xi_1, \xi_2 \in C_r(r)$ . On the other hand we have  $[[\bar{w}, \xi_1], \xi_2] = \sqrt{-1} \langle w, \xi_2 \rangle \xi_1 + \sqrt{-1} \langle w, \xi_1 \rangle \xi_2$ . (i) follows immediately from these facts.

- (ii) Let  $X = \begin{pmatrix} -\bar{u} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \in n_0$ . Then from (1.3)  $[X, \xi] = \underbrace{v(\xi) + \bar{u}\xi}_0 = 0$

for  $\xi \in C_r(r)$ . For  $\bar{w} \in \mathfrak{g}_1(r)$  we have  $[X, \bar{w}] = \overline{v(w) - uw}$ . On the other hand  $\langle v(w) - uw, \xi \rangle = -\langle w, v(\xi) + \bar{u}\xi \rangle = 0$  for  $\xi \in C_r(r)$ . Hence from (i) we get  $[X, \bar{w}] \in n_1$ . Q.E.D.

Let  $n^{**}$  be the null ideal of  $\mathfrak{g}^{**}(r, r)$ . Then  $n^{**}$  is a graded ideal since  $\mathfrak{g}^{**}(r, r)$  contains  $E_0 \in \mathfrak{h}_r^*(r)$ , which defines the grading of  $\mathfrak{g}^{**}(r, r)$ . We set  $n = n_0 \oplus n_1 \oplus \mathfrak{g}_2(r)$ . Then it is easily seen from Lemma 2.9 that  $n$  is an ideal of  $\mathfrak{g}^{**}(r, r)$  (Note  $C_r(r) \subset (C_r(r))^\perp$ ). Hence we obtain

LEMMA 2.10.  $n^{**} = n$ .

Next we will study the radical  $\mathfrak{r}^{**} = \sum_{p=-1}^2 \mathfrak{r}_p^{**}$  of  $\mathfrak{g}^{**}(r, r)$ . (Note that  $\mathfrak{r}^{**}$  is a graded ideal.)

LEMMA 2.11.  $\mathfrak{r}_{-1}^{**} = 0$

*Proof.* Assume the contrary, then we get  $\mathfrak{r}_{-1}^{**} = \mathfrak{c}_r^*(r)$  since  $\mathfrak{h}_r^*$  acts irreducibly on  $\mathfrak{c}_r^*(r)$ . Considering the semi-simple graded Lie algebra  $\mathfrak{g}^{**}(r, r)/\mathfrak{r}^{**}$ , we obtain  $\mathfrak{r}_1^{**} = \mathfrak{g}_1(r)$  and  $\mathfrak{r}_2^{**} = \mathfrak{g}_2(r)$  (cf. the proof of Lemma 2.3). On the other hand for  $\xi_0 \neq 0 \in \mathfrak{c}_r^*(r)$ , we can find  $\bar{w}_0 \in \mathfrak{g}_1(r)$  such that  $\langle \xi_0, w_0 \rangle = \sqrt{-1}$  and  $\langle w_0, w_0 \rangle = 0$  (Note  $\langle \xi_0, \xi_0 \rangle = 0$ ). Then it is easily seen that the subspace  $\mathfrak{s}$  of  $\mathfrak{r}^{**}$  spanned by the three elements  $\xi_0, \bar{w}_0, [\xi_0, \bar{w}_0] = E_0$  (cf. (4.2) [3]) forms a subalgebra satisfying  $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ . This contradicts the solvability of  $\mathfrak{r}^{**}$ . Q.E.D.

Hence we get  $\mathfrak{r}^{**} \subset n^{**}$ . More precisely we have

PROPOSITION 2.12. *If  $r = \frac{n-1}{2}$ , then  $n^{**} = \mathfrak{r}^{**}$ . In particular  $n^{**}$  is a characteristic ideal of  $\mathfrak{g}^{**}(r, r)$ .*

*Proof.* We have only to show  $n^{**}$  is solvable provided  $r = \frac{n-1}{2}$ .

However since  $n^{**} = n_0 \oplus n_1 \oplus \mathfrak{g}_2(r)$ , it is sufficient to show  $n_0$  is solvable. For this purpose we take a suitable base  $\{f_i\}_{1 \leq i \leq n-1}$  of  $\mathbf{C}^{n-1}$  such that  $\{f_i\}_{1 \leq i \leq r}$  forms a base of  $C_r(r)$  and that  $I_r$  is represented as a matrix of the following form

$$\begin{pmatrix} 0 & E_r \\ E_r & 0 \end{pmatrix}$$

Then each  $X \in n_0$  is represented as follows;

$$X = \begin{pmatrix} -a & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & a \end{pmatrix} \quad v = \begin{pmatrix} -aE_r & B \\ 0 & aE_r \end{pmatrix} \quad a \in \mathbf{R}, B \in u(r).$$

Hence  $n_0$  is obviously solvable.

Q.E.D.

*Remark 2.13.* In the case  $n = 5$  and  $r = 2$ , we get  $\dim \mathfrak{r}^0 = 11$  and  $\dim \mathfrak{r}^{**} = 10$  from Lemmas 2.5 and 2.10. Hence  $\mathfrak{g}^0(2)$  and  $\mathfrak{g}^{**}(2, 2)$  are not isomorphic.

Finally we add the following which is needed in III.

**LEMMA 2.14.** *Assume  $r = \frac{n-1}{2}$ . Let  $X \in n^{**}$  such that  $X \in [n^{**}, n^{**}]$ . Then for  $Y \in \mathfrak{m}(r) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r)$ ,  $\exp \operatorname{ad}(X)(Y) \equiv Y \pmod{\mathfrak{g}'(r)}$  if and only if  $Y \in \mathfrak{c}_r^*(r)$ .*

*Proof.* From the proof of Proposition 2.12 it is easily seen that  $[n_0, \mathfrak{g}_2(r)] = \mathfrak{g}_2(r)$ ,  $[n_0, n_1] = n_1$  and

$$[n_0, n_0] = \left\{ X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 0 \end{pmatrix} \in n_0 \mid v = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} B \in u(r) \right\}.$$

Hence  $X = X_0 + \tilde{w} + \tilde{b} \in n^{**}$  is not in  $[n^{**}, n^{**}]$  if and only if  $a \neq 0$ , where

$$X_0 = \begin{pmatrix} -a & 0 & 0 \\ 0 & v_0 & 0 \\ 0 & 0 & a \end{pmatrix} \quad v_0 = \begin{pmatrix} -aE_r & B \\ 0 & aE_r \end{pmatrix}.$$

Let  $Y = \underline{c} + \underline{\xi} \in \mathfrak{m}(r)$ . Then the  $\mathfrak{g}_{-2}(r)$ -component of  $\exp \operatorname{ad}(X)(Y)$  is equal to  $\exp \operatorname{ad}(X_0)(\underline{c}) = \underline{e}^{2a\underline{c}}$ . Hence if  $\exp \operatorname{ad}(X)(Y) \equiv Y \pmod{\mathfrak{g}'(r)}$ , we have  $c = 0$ . Moreover the  $\mathfrak{m}(r)$ -component of  $\exp \operatorname{ad}(X)(\underline{\xi})$  is equal to  $\exp \operatorname{ad}(X_0)(\underline{\xi})$ . Hence if  $\exp \operatorname{ad}(X)(\underline{\xi}) \equiv \underline{\xi} \pmod{\mathfrak{g}'(r)}$ , we have  $\exp \operatorname{ad}(X_0)(\underline{\xi})$

$= \xi$ , i.e.  $[X_0, \xi] = 0$ . On the other hand we have  $[X_0, \xi] = \underbrace{v_0(\xi)} + a\xi$ . If we represent  $\xi$  as  $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$  ( $\xi_i \in \mathcal{C}^r$   $i = 1, 2$ ) with respect to the base given in the proof of Proposition 2.12, we have  $v_0(\xi) + a\xi = \begin{pmatrix} B(\xi_2) \\ 2a\xi_2 \end{pmatrix}$ . Hence from  $[X_0, \xi] = 0$  we get  $\xi_2 = 0$  (i.e.  $\xi \in \mathfrak{c}_r^*(r)$ ). The converse is trivial since  $[\mathfrak{c}_r^*(r), \mathfrak{n}^{**}] \subset \mathfrak{n}^{**}$ . Q.E.D.

### III. The proof of the main theorem

In this section we use the same notation as in V [3].

Throughout this section we assume that  $S$  is a connected non-degenerate (index  $r$ ) hypersurface. Let  $(P, \omega, \bar{\ell})$  be the normal pseudo-conformal connection over  $S$  with the projection  $\pi$ . Let  $A(S)$  be the group of all the pseudo-conformal transformations of  $S$ . We set  $\mathfrak{a}(P) = \{X \in \mathcal{X}(P) \mid L_X \omega = 0, R_{\alpha_*} X = X \text{ for } \alpha \in G'(r) \text{ and } X \text{ is complete}\}$ . (cf. Proposition 5.6 [3]) Recall that  $\mathfrak{a}(P)$  is isomorphic with  $\mathfrak{a}(S)$ , the Lie algebra of  $A(S)$  (cf. II [3]).

From Proposition 1.13, Lemmas 2.3 and 5.5 [3], and Proposition 5.6 [3], we easily obtain

**PROPOSITION 3.1** (cf. Theorem 5.8 [3]). *Let  $M$  be a complex manifold of dimension  $n$ . Let  $S$  be a connected non-degenerate (index  $r$ ) hypersurface of  $M$ , and let  $p_0$  be an arbitrary point of  $S$ . Assume that  $\dim A(S) < n^2 + 2n$ ;*

(1) *The case  $n = 3$  and  $r = 1$ .  $\dim A(S) \leq n^2 + 2 = 11$ . The equality holds if and only if there exists  $z_0 \in \pi^{-1}(p_0)$  such that  $-\omega_{z_0}$  is a Lie algebra isomorphism of  $\mathfrak{a}(P)$  onto  $\mathfrak{g}^*(1, 1)$  or  $\mathfrak{g}^{**}(1, 1)$ .*

(2) *The case  $n = 5$  and  $r = 2$ .  $\dim A(S) \leq n^2 + 1 = 26$ . The equality holds if and only if there exists  $z_0 \in \pi^{-1}(p_0)$  such that  $-\omega_{z_0}$  is a Lie algebra isomorphism of  $\mathfrak{a}(P)$  onto  $\mathfrak{g}^*(2, 2)$ ,  $\mathfrak{g}^{**}(2, 2)$ ,  $\mathfrak{g}^*(2)$ ,  $\mathfrak{g}'(2)$  or  $\mathfrak{g}^0(2)$ .*

(3) *The case  $n \geq 2$  and  $r = 0$ .  $\dim A(S) \leq n^2 + 1$ . The equality holds if and only if there exists  $z_0 \in \pi^{-1}(p_0)$  such that  $-\omega_{z_0}$  is a Lie algebra isomorphism of  $\mathfrak{a}(P)$  onto  $\mathfrak{g}^*(0)$  or  $\mathfrak{g}'(0)$ .*

(4) *Otherwise.  $\dim A(S) \leq n^2 + 1$ . The equality holds if and only if there exists  $z_0 \in \pi^{-1}(p_0)$  such that  $-\omega_{z_0}$  is a Lie algebra isomorphism of  $\mathfrak{a}(P)$  onto  $\mathfrak{g}^*(r)$ ,  $\mathfrak{g}'(r)$  or  $\mathfrak{g}^0(r)$ .*

Now we will study again the model spaces given in VI [3]. Let  $G^*(r, r)$

and  $G^*(r)$  be the analytic subgroups of  $G(r)$  with the Lie algebra  $\mathfrak{g}^*(r, r)$  and  $\mathfrak{g}^*(r)$  respectively. Then  $Q_r$  has the orbital decomposition by these groups as follows (cf. Remark 6.3 [3]).

$$\begin{aligned} Q_r &= Q_r^*(r) \cup R_r^1(r) && \text{by } G^{**}(r, r) \\ Q_0 &= Q_0^* \cup \{\bar{o}\} && \text{by } G^*(0) \\ Q_r &= Q_r^* \cup R_r^2(0) \cup \{\bar{o}\} && \text{by } G^*(r) \ (r \geq 1) \end{aligned}$$

Recall that  $R_r^1(r)$  is a compact submanifold of dimension  $2r$  and  $R_r^2(0)$  is a (regular) submanifold of dimension  $2n - 3$  (cf. the proof (3) of Proposition 6.6 [3]). We naturally identify  $(G(r), \omega^r)$  with the normal pseudo-conformal connection over  $Q_r$ , where  $\omega^r$  is the Maurer-Cartan form on  $G(r)$  (cf. Proposition 6.4 [3]). Note that  $\alpha(G(r))$  coincides with the Lie algebra of all the right invariant vector fields on  $G(r)$ .

Then for the Lie algebras obtained in Proposition 3.1, we have

**PROPOSITION 3.2.** (1) *The case  $n = 3$  and  $r = 1$ .  $\mathfrak{g}^*(1, 1)$  and  $\mathfrak{g}^{**}(1, 1)$  are conjugate under an element of  $G(1)$ .*

(2) *The case  $n = 5$  and  $r = 2$ .  $\mathfrak{g}^*(2, 2)$  and  $\mathfrak{g}^{**}(2, 2)$  are conjugate under an element of  $G(2)$ .  $\mathfrak{g}^*(2)$ ,  $\mathfrak{g}'(2)$  and  $\mathfrak{g}^0(2)$  are mutually conjugate under elements of  $G(2)$ . Moreover  $\mathfrak{g}^*(2, 2)$  and  $\mathfrak{g}^*(2)$  are not isomorphic.*

(3) *The case  $n \geq 2$  and  $r = 0$ .  $\mathfrak{g}^*(0)$  and  $\mathfrak{g}'(0)$  are conjugate under elements of  $G(0)$ .*

(4) *Otherwise.  $\mathfrak{g}^*(r)$ ,  $\mathfrak{g}'(r)$  and  $\mathfrak{g}^0(r)$  are mutually conjugate under elements of  $G(r)$ .*

Here we say that two subalgebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  of  $\mathfrak{g}(r)$  are conjugate under  $\tau \in G(r)$  if  $\text{Ad}(\tau)\mathfrak{g}_1 = \mathfrak{g}_2$ .

*Proof of Proposition 3.2.* We consider the subalgebra  $\alpha(G)$  of  $\alpha(G(r))$  induced by  $G$  ( $= G^*(r, r)$  or  $G^*(r)$ ), i.e. each  $X \in \alpha(G)$  is a right invariant vector field on  $G(r)$  induced by the 1-parameter subgroup  $a(t) \in G$  such that  $a(0) = X_e$ , where  $e$  is the unit of  $G(r)$ . Note that  $\pi_{r*}X$  is an infinitesimal pseudo-conformal transformation of  $Q_r$  induced by the action of  $a(t)$  on  $Q_r$ , and that  $-\omega_e^r$  is a Lie algebra isomorphism of  $\alpha(G)$  onto  $\mathfrak{g}$ , the Lie algebra of  $G$ . Let  $\sigma \in G(r)$  and set  $\mathfrak{h}_\sigma = -\omega_\sigma^r(\alpha(G))$ . Then  $\mathfrak{h}_\sigma$  is a filtered subalgebra of  $\mathfrak{g}(r) = \mathcal{L}_{-2}(r)$  (cf. Proposition 2.4 [3]). Let  $\tilde{\alpha}(G)$  be the associated graded Lie algebra of  $\alpha(G)$ , and set  $\tilde{\mathfrak{h}}_\sigma = \nu_\sigma(\tilde{\alpha}(G))$  (cf.

Lemma 2.5 [3]).

We will only prove the case (2). The others can be proved similarly. First we consider the case  $G = G^*(2,2)$ . Then  $Q_2$  has the orbital decomposition by  $G^*(2,2)$ ;

$$Q_2 = Q_2^*(2) \cup R_2^1(2) .$$

Take an arbitrary point  $p_0 \in R_2^1(2)$ . Let  $\sigma \in \pi_2^{-1}(p_0)$  (i.e.  $\sigma(o) = p_0$ ). Since  $R_2^1(2)$  is a 4-dimensional orbit of  $G^*(2,2)$ , we have  $\dim \mathfrak{h}_\sigma / (\mathfrak{h}_\sigma)_0 = \dim ((\tilde{\mathfrak{h}}_\sigma)_{-2} \oplus (\tilde{\mathfrak{h}}_\sigma)_{-1}) = 4$ . On the other hand we have  $\dim \tilde{\mathfrak{h}}_\sigma = \dim \mathfrak{g}^*(2,2) = 26$ . Hence from Proposition 1.13,  $\tilde{\mathfrak{h}}_\sigma$  must be isomorphic with  $\mathfrak{g}^{**}(2,2)$ . (Note that  $\dim (\mathfrak{g}_{-2}(2) \oplus \mathfrak{g}_{-1}(2)) = 9$ ,  $\dim \mathfrak{c}_2^*(2) = 4$  and  $\dim (\mathfrak{g}_{-2}(2) \oplus \mathfrak{d}^{-1}(2)) = 7$ ). In other words there exists  $\sigma_0 \in \pi_2^{-1}(p_0)$  such that  $\mathfrak{h}_{\sigma_0} = \mathfrak{g}^{**}(2,2)$ . Then the composite  $(-\omega_{\sigma_0}) \circ (-\omega_e)^{-1}$  is a Lie algebra isomorphism of  $\mathfrak{g}^*(2,2)$  onto  $\mathfrak{g}^{**}(2,2)$ . Let  $A \in \mathfrak{a}(G^*(2,2))$  and set  $X = -\omega_e(A) \in \mathfrak{g}^*(2,2)$ ,  $Y = -\omega_{\sigma_0}(A) \in \mathfrak{g}^{**}(2,2)$ . Since  $A$  is a right invariant vector field we get  $Y = -\omega_{\sigma_0}(R_{\sigma_0^*} A_e) = -R_{\sigma_0^*}^* \omega_{\sigma_0}(A_e) = -\text{Ad}(\sigma_0^{-1}) \omega_e(A_e) = \text{Ad}(\sigma_0^{-1})(X)$ . Therefore  $\mathfrak{g}^{**}(2,2) = \text{Ad}(\sigma_0^{-1}) \mathfrak{g}^*(2,2)$ .

Next we consider the case  $G = G^*(2)$ . Then  $Q_2$  has the orbital decomposition by  $G^*(2)$ ;

$$Q_2 = Q_2^* \cup R_2^2(0) \cup \{\bar{o}\} .$$

Take an arbitrary point  $p_1 \in R_2^2(0)$ . Since  $R_2^2(0)$  is a 7-dimensional orbit of  $G^*(2)$ . We can conclude similarly as above that there exists  $\sigma_1 \in \pi_2^{-1}(p_1)$  such that  $\mathfrak{h}_{\sigma_1} = \mathfrak{g}^0(2)$ . Hence we have  $\mathfrak{g}^0(2) = \text{Ad}(\sigma_1^{-1}) \mathfrak{g}^*(2)$ . Take the point  $\bar{o} \in Q_2$ . Then since  $\bar{o}$  is a common fixed point of  $G^*(2)$ , we get similarly  $\mathfrak{g}'(2) = \text{Ad}(\sigma^{-1}) \mathfrak{g}^*(2)$  for any  $\sigma \in \pi_2^{-1}(\bar{o})$ . The last assertion follows immediately from Remark 2.13. Q.E.D.

Now we will mention about the ‘‘canonical metric’’ for the normal pseudo-conformal connection  $(P, \omega)$ , following I. Naruki [1], which is necessary for the proof of Theorem 3.4. Let us fix a positive definite inner product  $(,)$  on  $\mathfrak{g}(\mathfrak{r})$ . Since  $\omega$  defines an absolute parallelism on  $P$ , it defines a Riemannian metric  $g$  of  $P$  by

$$g_p(X, Y) = (\omega_p(X), \omega_p(Y)) \quad X, Y \in T_p(P)$$

$g$  is called the ‘‘canonical metric’’ for  $(P, \omega)$ . We denote by  $d_p$  the distance function of  $P$  with respect to the canonical metric for  $(P, \omega)$ . Note that each right translation  $R_\sigma(\sigma \in G'(\mathfrak{r}))$  is uniformly continuous with



respect to  $d_P$  (cf. Lemma 1.4 [1]). We consider the completion  $\hat{P}$  of  $P$  for the metric  $d_P$ . We call  $(\hat{P}, d_{\hat{P}})$  the completion of  $(P, \omega)$ . Then  $P$  is an open dense subset of  $\hat{P}$  and the right action of  $G'(r)$  on  $P$  extends uniquely to that of  $\hat{P}$ . In general  $\hat{P}$  is not a manifold, nor  $\hat{P}$  is a principal  $G'(r)$ -bundle.

Next we consider a closed submanifold  $R$  of  $S$  such that  $\dim R \leq \dim S - 2$ . Then  $S \setminus R$  is a connected open submanifold of  $S$ . Hence  $(\pi^{-1}(S \setminus R), \omega_R)$  is the normal pseudo-conformal connection over  $S \setminus R$ , where  $\omega_R$  is the restriction of  $\omega$  to  $\pi^{-1}(S \setminus R) = P \setminus \pi^{-1}(R)$ . Note that  $\pi^{-1}(R)$  is a closed submanifold of  $P$  such that  $\dim \pi^{-1}(R) \leq \dim P - 2$ . Then the completion of  $(\pi^{-1}(S \setminus R), \omega_R)$  coincides with that of  $(P, \omega)$  (cf. Lemma 1.1 [1]).

Now we study the normal pseudo-conformal connection over  $Q_r^*$  and  $Q_r^*(r)$ . Let  $\pi_r$  be the projection of  $G(r)$  onto  $Q_r$  (i.e.  $\pi_r(\sigma) = \sigma(o)$  for  $\sigma \in G(r)$ ). We set  $P^*(r) = \pi_r^{-1}(Q_r^*)$  and  $P_r^*(r) = \pi_r^{-1}(Q_r^*(r))$ . Then  $(P^*(r), \omega^r|_{P^*(r)})$  (resp.  $(P_r^*(r), \omega^r|_{P_r^*(r)})$ ) is the normal pseudo-conformal connection over  $Q_r^*$  (resp.  $Q_r^*(r)$ ). Note that  $\dim R_r^2(0) = 2n - 3 = \dim Q_r - 2$  and  $\dim R_r^1(r) = 2r < \dim Q_r - 2$ . Hence from the above argument we have  $\widehat{P^*(r)} = \widehat{G(r) \setminus \pi_r^{-1}(\bar{o})} = \widehat{G(r)}$  and  $\widehat{P_r^*(r)} = \widehat{G(r)}$ . On the other hand the canonical metric for  $(G(r), \omega^r)$  is nothing but the left invariant metric on  $G(r)$ . Hence  $G(r)$  is a homogeneous Riemannian manifold with respect to the canonical metric. In particular  $G(r)$  is complete, i.e.  $\widehat{G(r)} = G(r)$ . Therefore we obtain

$$\text{LEMMA 3.3. } \widehat{P^*(r)} = \widehat{P_r^*(r)} = G(r)$$

Now we will prove the main theorem of this paper

**THEOREM 3.4.** *Let  $M$  be a complex manifold of dimension  $n$ . Let  $S$  be a connected non-degenerate (index  $r$ ) hypersurface of  $M$   $\left(0 \leq r \leq \left\lfloor \frac{n-1}{2} \right\rfloor\right)$ .*

*Assume that  $\dim A(S) < n^2 + 2n$ ,*

(1) *The case  $n = 3$  and  $r = 1$ .  $\dim A(S) \leq n^2 + 2 = 11$ . The equality holds if and only if  $S$  is pseudo-conformally equivalent to  $Q_1^*(1)$ .*

(2) *The case  $n = 5$  and  $r = 2$ .  $\dim A(S) \leq n^2 + 1 = 26$ . The equality holds if and only if  $S$  is pseudo-conformally equivalent to  $Q_2^*(2)$ ,  $Q_2^*$  or  $Q_2 \setminus \{\bar{o}\}$ .*

(3) *The case  $n \geq 2$  and  $r = 0$ .  $\dim A(S) \leq n^2 + 1$ . The equality*

holds if and only if  $S$  is pseudo-conformally equivalent to the hyperconic  $Q_0^*$ .

(4) *Otherwise.*  $\dim A(S) \leq n^2 + 1$ . The equality holds if and only if  $S$  is pseudo-conformally equivalent to  $Q_r^*$  or  $Q_r \setminus \{\delta\}$ .

$Q_r^*(r)$  and  $Q_r^*$  are homogeneous hypersurfaces of  $P^n(\mathbf{C})$ , whereas  $Q_r \setminus \{\delta\}$  ( $r \geq 1$ ) is an inhomogeneous hypersurface of  $P^n(\mathbf{C})$  for which  $A(Q_r \setminus \{\delta\})$  coincides with  $A(Q_r^*)$  as a group of projective transformations (cf. Proposition 6.5 [3]).

*Proof of Theorem 3.4.* We will only prove the case (4). Others can be proved similarly. Let  $S$  be a connected non-degenerate (index  $r$ ) hypersurface such that  $\dim A(S) = n^2 + 1$ . Let  $p$  be an arbitrary point of  $S$ . Then from Proposition 3.1, there exists  $z \in \pi^{-1}(p)$  such that  $-\omega_z$  is a Lie algebra isomorphism of  $\mathfrak{a}(P)$  onto  $\mathfrak{g}^*(r)$ ,  $\mathfrak{g}'(r)$  or  $\mathfrak{g}^0(r)$ . (Note that from Proposition 3.2 (2), in the case (2), we have two cases (a) and (b) for a given  $S$ ;

$$(a) \quad -\omega_z(\mathfrak{a}(P)) = \mathfrak{g}^*(2, 2) \text{ or } \mathfrak{g}^{**}(2, 2)$$

$$(b) \quad -\omega_z(\mathfrak{a}(P)) = \mathfrak{g}^*(2), \mathfrak{g}'(2) \text{ or } \mathfrak{g}^0(2).$$

Now the proof is divided into several lemmas. Let  $\mathfrak{a}(S)$  be the Lie algebra of infinitesimal pseudo-conformal transformations which generate (global) 1-parameter subgroups of  $A(S)$ . Then  $\mathfrak{a}(S)$  is naturally isomorphic with the Lie algebra of  $A(S)$  and  $\pi_*$  is an isomorphism of  $\mathfrak{a}(P)$  onto  $\mathfrak{a}(S)$ . Hence we have

LEMMA 3.5. (i)  $-\omega_z(\mathfrak{a}(P)) = \mathfrak{g}^*(r)$  if and only if  $p$  belongs to an open orbit of  $A^0(S)$ .

(ii)  $-\omega_z(\mathfrak{a}(P)) = \mathfrak{g}'(r)$  if and only if  $p$  is a common fixed point of  $A^0(S)$ .

(iii)  $-\omega_z(\mathfrak{a}(P)) = \mathfrak{g}^0(r)$  if and only if  $p$  belongs to a  $(2n - 3)$ -dimensional orbit of  $A^0(S)$ .

Now we claim

LEMMA 3.6. *There exists an open orbit  $S_0$  of  $A^0(S)$ . Moreover  $S_0$  is pseudo-conformally equivalent to  $Q_r^*$ .*

*Proof.* Assume the contrary. Then the case (i) of Lemma 3.5 never occurs. Let  $N$  be the analytic subgroup of  $A^0(S)$  corresponding to  $\mathcal{D}^2\mathfrak{r}(S)$

of  $\alpha(S)$ , where  $\mathcal{D}^2\mathfrak{r}(S)$  is the second derived algebra of the radical  $\mathfrak{r}(S)$  of  $\alpha(S)$ . Then from Proposition 2.6 and Lemma 3.5,  $N$  acts trivially on  $S$ . Since  $A^0(S)$  acts effectively on  $S$ , this contradiction shows the existence of an open orbit  $S_0$ . Let  $\sigma \in A^0(S)$ . Then  $\sigma$  induces an automorphism  $\bar{\sigma}$  of  $(P, \omega)$ . From  $\sigma(S_0) = S_0$ , we have  $\bar{\sigma}(\pi^{-1}(S_0)) = \pi^{-1}(S_0)$ .  $(\pi^{-1}(S_0), \omega|_{\pi^{-1}(S_0)})$  is the normal pseudo-conformal connection over  $S_0$ . Hence if  $\sigma|_{S_0} = \text{id}_{S_0}$ ,  $\bar{\sigma}|_{\pi^{-1}(S_0)} = \text{id}_{\pi^{-1}(S_0)}$ . Since  $\bar{\sigma}$  is an automorphism of the absolute parallelism defined by  $\omega$ , we get  $\bar{\sigma} = \text{id}_P$ , i.e.  $\sigma = \text{id}_S$ . Hence  $A^0(S)$  acts effectively on  $S_0$ . Then  $S_0$  is a connected non-degenerate (index  $r$ ) homogeneous hypersurface such that  $\dim A(S_0) = n^2 + 1$ . Therefore from Theorem 7.2 [3],  $S_0$  is pseudo-conformally equivalent to  $Q_r^*$ . Q.E.D.

In the cases (1) and (a) of (2), we can prove the analogous assertion using Proposition 2.12 in place of Proposition 2.6.

Next we will show the regularity of singular orbits. First we have

LEMMA 3.7. *Let  $M$  be a manifold, and let  $N$  be a submanifold of  $M$ . Let  $f$  be a diffeomorphism of  $M$  onto  $M$  satisfying the following;*

- (1)  $f(x) = x$  for  $x \in N$
- (2) For  $X \in T_x(M)$ ,  $x \in N$ ,

$$f_*(X) = X \quad \text{if and only if} \quad X \in T_x(N).$$

*Then  $N$  is a regular submanifold of  $M$ . Moreover for each  $x \in N$ , there exists an open neighbourhood  $U$  of  $M$  at  $x$  such that  $f$  has no fixed point in  $U \setminus N$ .*

*Proof.* Take an arbitrary point  $x_0$  of  $N$ . From the implicit function theorem there exist a coordinate neighbourhood  $V$  of  $N$  at  $x_0$  with the coordinate  $(y^1, \dots, y^n)$  and a coordinate neighbourhood  $U$  of  $M$  at  $x_0$  with the coordinate  $(x^1, \dots, x^m)$  such that  $y^i = x^i \circ \iota$  ( $i = 1, 2, \dots, n$ ) and for  $x = (y^1, \dots, y^n) \in V$ , we have  $\iota(x) = (y^1, \dots, y^n, 0, \dots, 0)$ , and  $\iota(x_0) = (0, \dots, 0)$ , where  $n = \dim N$ ,  $m = \dim M$  and  $\iota$  is the inclusion of  $N$  into  $M$ . Moreover if we take  $V$  and  $U$  sufficiently small we may assume  $\iota(V) = \{x \in U \mid x^i(x) = 0 \text{ (} i = n + 1, \dots, m)\}$ . Now we will show that if we take  $U$  sufficiently small  $f$  has no fixed point in  $U \setminus \iota(V)$ . Then the assertion follows immediately. We set  $f^i = x^i \circ f$  ( $i = 1, 2, \dots, m$ ). From (1) we have

$$(1)' \quad \begin{cases} f^i(x^1, \dots, x^n, 0, \dots, 0) = x^i & (i = 1, 2, \dots, n) \\ f^i(x^1, \dots, x^n, 0, \dots, 0) = 0 & (i = n + 1, \dots, m). \end{cases}$$

If we represent  $f_* - \text{id}_{T_x(M)}$  with respect to the coordinate base, we have

$$\left( \frac{\partial f^i}{\partial x^j}(x) - \delta_j^i \right)_{1 \leq i, j \leq m} = \begin{pmatrix} 0 & \left( \frac{\partial f^i}{\partial x^j}(x) \right)_{\substack{1 \leq i \leq n \\ n+1 \leq j \leq m}} \\ 0 & \left( \frac{\partial f^i}{\partial x^j}(x) - \delta_j^i \right)_{n+1 \leq i, j \leq m} \end{pmatrix} \quad x \in \iota(V).$$

Since  $\left\{ \frac{\partial}{\partial x^i} \right\}_{1 \leq i \leq n}$  forms a base of  $T_x(N)$ , (2) is rewritten as follows

(2)'  $\{dx^i - df^i\}_{n+1 \leq i \leq m}$  are linearly independent at  $x \in \iota(V)$ .

Hence if we take  $U$  sufficiently small we may assume that  $\{dx^i - df^i\}_{n+1 \leq i \leq m}$  are independent on  $U$ . We set

$$V' = \{x \in U \mid x^i(x) = f^i(x) \ (i = n+1, \dots, m)\} \text{ and } F = \{x \in U \mid f(x) = x\}.$$

Then  $V'$  is an  $n$ -dimensional closed submanifold of  $U$ . Obviously we have  $\iota(V) \subset N \cap U \subset F \subset V'$ . On the other hand  $\iota(V)$  is an  $n$ -dimensional closed submanifold of  $U$ . Hence the connected component  $V_0$  of  $\iota(V)$  containing  $x_0$  must coincide with that of  $V'$ . Therefore if we take an open subset  $U_0$  of  $U$  such that  $U_0 \cap V' = V_0$ , we get  $N \cap U_0 = F \cap U_0 = V_0$ , i.e.  $f$  has no fixed point in  $U_0 \setminus \iota(V)$ . Q.E.D.

Let  $p \in S$  and  $z \in \pi^{-1}(p)$  be as in (ii) or (iii) of Lemma 3.5. Let  $A_p^0(S)$  be the isotropy subgroup of  $A^0(S)$  at  $p$ . We consider the linear isotropy representation of  $A_p^0(S)$  at  $p$ . Let  $\iota_z$  be the imbedding of  $A^0(S)$  into  $P$  defined by  $\iota_z(\sigma) = \tilde{\sigma}(z)$ , where  $\tilde{\sigma}$  is the automorphism of  $(P, \omega)$  induced by  $\sigma \in A^0(S)$ . Then  $\iota_z$  induces an injective homomorphism  $\rho_z$  of  $A_p^0(S)$  into  $G'(r)$ , i.e.  $\iota_z(\sigma) = z \cdot \rho_z(\sigma)$  for  $\sigma \in A_p^0(S)$ . (cf. Lemma 3.1 [3]). Note that  $\rho_{z*}(\mathfrak{a}_p(S)) = \mathfrak{g}'(r)$  (resp.  $\mathfrak{e}(r) \oplus \mathfrak{d}^1(r) \oplus \mathfrak{g}_2(r)$ ) in case (ii) (resp. in case (iii)) (cf. Lemma 3.1 [3] and the remark before Proposition 3.4 [3]). Let  $\alpha_z$  be a linear isomorphism of  $T_p(S)$  onto  $\mathfrak{m}(r) = \sum_{p < 0} \mathfrak{g}_p(r)$  defined by the following commutative diagram.

$$\begin{array}{ccc} T_z(P) & \xrightarrow{\omega_z} & \mathfrak{g}(r) \\ \pi_* \downarrow & & \downarrow p \\ T_p(S) & \xrightarrow{\alpha_z} & \mathfrak{m}(r) \end{array}$$

where  $p$  is the projection corresponding to the decomposition  $\mathfrak{g}(r) = \mathfrak{m}(r)$

$\oplus \mathfrak{g}'(r)$ . Let  $\ell; G'(r) \rightarrow GL(m(r))$  be the linear isotropy representation of  $G'(r)$  (cf. I.4 [3]). Then we have

LEMMA 3.8. For  $\sigma \in A_p^0(S)$ ,  $\alpha_z \circ \sigma_* = \ell(\rho_z(\sigma)) \circ \alpha_z$ .

*Proof.* This is a direct consequence of the following commutative diagram and  $\delta^*\omega = \omega$ .

$$\begin{array}{ccc} T_z(P) & \xrightarrow{R_{a_*^{-1} \circ \delta_*}} & T_z(P) \\ \omega_z \downarrow & & \downarrow \omega_z \\ \mathfrak{g}(r) & \xrightarrow{\text{Ad}(a)} & \mathfrak{g}(r) \end{array}$$

where  $a = \rho_z(\sigma)$ .

Q.E.D.

Now we have

LEMMA 3.9. If there are common fixed points of  $A^0(S)$ , they form a 0-dimensional regular submanifold  $F$  of  $S$ .

In other words each fixed point of  $A^0(S)$ , if there is, is isolated from others.

*Proof.* Let  $p$  be a fixed point of  $A^0(S)$ , and  $z \in \pi^{-1}(p)$  be as in (ii) of Lemma 3.5. From Lemma 3.8 and  $\rho_z(A^0(S)) = G'_0(r)$ , the identity component of  $G'(r)$ , it is easily seen that there exists  $\sigma_0 \in A^0(S)$  such that  $\sigma_{0*}(X) = X$  if and only if  $X = 0, X \in T_p(S)$  (e.g.  $\sigma_0 = \rho_z^{-1}(\exp E_0)$ ). Then from Lemma 3.7,  $p$  is an isolated fixed point of  $A^0(S)$ . Q.E.D.

LEMMA 3.10. If there are  $(2n - 3)$ -dimensional orbits of  $A^0(S)$ , they form a  $(2n - 3)$ -dimensional regular submanifold  $T$  of  $S$ .

This can be proved quite similarly as above using Lemma 2.7, hence the proof is omitted.

In the cases (1) and (a) of (2), we can prove the analogous assertion using Lemma 2.14 in place of Lemma 2.7.

Now we consider the completion  $\hat{P}$  of  $(P, \omega)$ . With the aid of  $\hat{P}$  we will imbed  $S$  pseudo-conformally into  $Q_r$ . Note that  $S \setminus (T \cup F)$  is connected (cf. VI Proposition D [3]). Hence  $S_0 = S \setminus (T \cup F)$ .

From Lemma 3.6 there exists a pseudo-conformal homeomorphism of  $S_0$  onto  $Q_r^*$ . We set  $P_0 = \pi^{-1}(S_0)$ . Then  $\varphi$  induces a bundle isomorphism  $\tilde{\varphi}$  of  $P_0$  onto  $P^*(r)$  satisfying  $\tilde{\varphi}^*\omega^r = \omega$ . Hence  $\tilde{\varphi}$  is an isometry

of  $P_0$  onto  $P^*(r)$ . Then  $\hat{\varphi}$  is extended uniquely to a distance preserving map  $\hat{\varphi}$  of  $\hat{P}_0$  onto  $\widehat{P^*(r)}$ . On the other hand from Lemmas 3.9 and 3.10 we have  $\hat{P}_0 = \widehat{P \setminus \pi^{-1}(F)} = \hat{P}$ . Moreover from Lemma 3.3 we have  $\widehat{P^*(r)} = G(r)$ . Hence  $\hat{\varphi}$  is a distance preserving map of  $\hat{P}$  onto  $G(r)$ . Since  $\varphi$  is a bundle map and  $P$  is an open dense subset of  $\hat{P}$ ,  $\hat{\varphi}$  commutes with the right action of  $G'(r)$ . Then  $P^* = \hat{\varphi}(P)$  is an open dense subset of  $G(r)$  which is invariant under the right action of  $G'(r)$ . We set  $Q = \pi_r(P^*)$ . Then  $Q$  is an open subset of  $Q_r$  and  $(P^*, \omega^r|_{P^*})$  is the normal pseudo-conformal connection over  $Q$ . Since  $\hat{\varphi}$  is a distance preserving map of  $P$  onto  $P^*$  and  $P^*$  is an open dense subset of  $G(r)$ ,  $\hat{\varphi}$  is an isometry of  $P$  onto  $P^*$  with respect to the canonical metrics for  $(P, \omega)$  and  $(P^*, \omega^r|_{P^*})$ . Moreover since  $P_0$  (resp.  $P^*(r)$ ) is an open dense subset of  $P$  (resp.  $P^*$ ) and  $\hat{\varphi}^* \omega^r = \omega$  on  $P_0$ , we get  $\hat{\varphi}^* \omega^r = \omega$  on  $P$ . Hence  $\hat{\varphi}$  induces a pseudo-conformal homeomorphism  $\psi$  of  $S$  onto  $Q$  such that  $\psi|_{S_0} = \varphi$ . On the other hand via  $\varphi: S_0 \rightarrow Q_r^*$ , each  $\sigma \in A^0(S)$  gives rise to a unique  $\tau \in G^*(r)$  (i.e.  $\tau = \varphi \circ \sigma \circ \varphi^{-1}$ ). Recall that  $\tau$  is a (global) projective transformation leaving  $Q_r$  invariant. Hence we must have  $\tau = \psi \circ \sigma \circ \psi^{-1}$  on  $Q$ . Therefore  $Q$  is invariant by the action of the subgroup  $G^*(r)$  of  $G(r)$ . Hence from the orbital decomposition of  $Q_r$  by  $G^*(r)$ , we must have  $Q = Q_r^*, Q_r \setminus \{\bar{o}\}$  or  $Q_r$ . However if  $Q = Q_r$ , we get  $\dim A(S) = n^2 + 2n$ . This contradiction shows that  $Q = Q_r^*$  or  $Q_r \setminus \{\bar{o}\}$ . Thus we have proved (4). Q.E.D.

*Remark 3.11.* In view of [2], our theorems are really the classifications of almost PC-manifolds admitting large groups of PC-automorphisms. In fact we don't use essentially the real analyticity in our proofs and the integrability condition  $\Omega_{-1} = 0$  for the normal connection is redundant (Note in the proof of Proposition 5.6 [3], the condition  $\Omega_{-1} = 0$  is not needed). In  $C^\infty$ -category, one must replace "pseudo-conformally equivalent" by "PC-equivalent" in the sense of [2]. Moreover in Theorem 7.4, one must assume that  $S$  is everywhere non-degenerate. However in Tanaka's "Fundamental theorem" (cf. Remark 1.2 [3]) and in Corollary 7.5 [3], the real analyticity assumption is indispensable.

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