A. Shimizu
Nagoya Math. J.
Vol. 66 (1977), 23-36

CONSTRUCTION OF A SOLUTION OF A CERTAIN EVOLUTION EQUATION

AKINOBU SHIMIZU

Let us consider the stochastic differential equation,

$$(1)\qquad \begin{aligned} du(t,x,\omega) &= \{\mathscr{L}_x u(t,x,\omega) + a(t)u(t,x,\omega) + b(t)\}dt \\ &\quad + \{c(t)u(t,x,\omega) + d(t)\}dB(t)\,, \qquad t \geqq 0\,, \qquad x \in R^d\,, \end{aligned}$$

with initial condition,

$$(2)\qquad u(0,x,\omega) = g(x)\,, \qquad x \in \boldsymbol{R}^d\,,$$

where $B_t, t \geqq 0$, is a one-dimensional Brownian motion, and $\mathscr{L}_x$ is a second order uniformly elliptic partial differential operator satisfying some additional conditions that will be described in §2. The existence and the uniqueness of solutions of the Cauchy problem have been established by B. L. Rozovskii [8].

The aim of this paper is to give an explicit expression of the solution in terms of Brownian motion. We are able to express the solution of the equation (1) as follows;

$$(3)\qquad \begin{aligned} u(t,x,\omega) = \Psi(t,\omega)\Big[p(t,x) &+ \int_0^t \Psi(s,\omega)^{-1}(b(s) - c(s)d(s))ds \\ &+ \int_0^t \Psi(s,\omega)^{-1}d(s)dB(s)\Big]\,, \end{aligned}$$

where

$$\Psi(t,\omega) = \exp\left\{\int_0^t c(s)dB(s) - \frac{1}{2}\int_0^t c(s)^2 ds + \int_0^t a(s)ds\right\}$$

and $p(t,x)$ is the solution of the Cauchy problem

$$(4)\qquad \frac{\partial}{\partial t}p(t,x) = \mathscr{L}_x p(t,x)\,, \qquad p(0,x) = g(x)\,, \qquad x \in R^d\,.$$

Received January 17, 1976.
Revised August 25, 1976.

We shall show in § 2 a method of construction of it as a functional of Brownian motion. To get this expression, we shall make use of the method by T. Hida [6][7], where the integral representation of the multiple Wiener integral is discussed systematically.

Once we got the formula (3), we are given much information on the asymptotic behaviour of the solution. For instance, suppose that $\mathscr{L}_x$ be the Laplace operator, and that $g(x)$ be a continuous function with compact support. Then, as is well-known, $p(t, x)$ converges to zero as t tends to infinity, so that we see that the process $u(t, x, \omega)$ is close enough to the process

$$\Psi(t, \omega)\left[\int_0^t \Psi(s, \omega)^{-1}(b(s) - c(s)d(s))ds + \int_0^t \Psi(s, \omega)^{-1}d(s)dB(s)\right]$$

for large t.

We shall further discuss the case where the space-parameter x varies in a bounded domain D in $\boldsymbol{R}^d$, whose boundary ∂D is sufficiently smooth. The solution $u(t, x, \omega)$ of the equation (1), with $x \in D$, satisfying the initial condition and the boundary condition,

$$\frac{\partial}{\partial n}u(t, x, \omega) = 0 , \qquad x \in \partial D , \tag{5}$$

is also expressed in the form (3), where (4) is replaced by

$$\begin{aligned} &\frac{\partial}{\partial t}p = \mathscr{L}_x p , && (t, x) \in (0, \infty) \times D , \\ &p(0, x) = g(x) , && x \in D , \\ &\frac{\partial}{\partial n}p = 0 , && (t, x) \in (0, \infty) \times \partial D . \end{aligned} \tag{6}$$

We are encouraged by B. L. Rozovskii [8], A. V. Balakrishnan [1], D. A. Dawson [2] and W. H. Fleming [4], to develop the method discussed in this paper.

§ 1. Existence and uniqueness

This section is devoted to a summary of known results for the stochastic differential equation (1). Let (Ω, F, P) be a probability space, endowed with a right-continuous increasing family F_t, and let $B_t(\omega)$, $t \geqq 0, \omega \in \Omega$, be a standard F_t-Brownian motion.

DEFINITION 1. A stochastic process $u(t, x, \omega)$ with space-time parameter $(t, x) \in [0, \infty) \times \boldsymbol{R}^d$ is called a solution of the equation (1) if the following conditions (i) $\sim$ (iv) are satisfied,

(i) $u(t, x, \omega)$ is F_t-measurable for any t and x,

(ii) $u(t, x, \omega)$ is continuous in (t, x) for almost all ω,

(iii) $u(t, x, \omega)$ is twice continuously (with respect to the pair (t, x)) differentiable in x,

(iv) $u(t, x, \omega)$ satisfies the equation (1).

DEFINITION 2. We say that the equation (1) has a unique solution, if there exists a set $\tilde{\Omega}$ satisfying $P(\tilde{\Omega}) = 1$, for any two solutions $u_i(t, x, \omega)$ $i = 1, 2$, such that the equality $u_1(t, x, \omega) = u_2(t, x, \omega)$ holds for any (t, x, ω) $\in [0, \infty) \times \boldsymbol{R}^d \times \tilde{\Omega}$.

The operator $\mathscr{L}_x$ is assumed to satisfy the following conditions (i) $\sim$ (iv),

(i) $\mathscr{L}_x u(t, x) = \sum_{i,j=1}^{d} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} u(t, x) + \sum_{i=1}^{d} b_i(t, x) \frac{\partial}{\partial x_i} u(t, x)$, $x \in \boldsymbol{R}^d$, $x \in$ $[0, \infty)$, is uniformly elliptic, that is, there exist positive numbers λ_0 and λ_1 such that the inequalities

$$\lambda_0 \|\xi\|^2 \leqq \sum_{i,j=1}^{d} a_{ij}(t, x) \xi_i \xi_j \leqq \lambda_1 \|\xi\|^2 , \qquad (t, x) \in [0, T] \times \boldsymbol{R}^d ,$$

hold for any vector $\xi \in \boldsymbol{R}^d$, and $(t, x) \in [0, T] \times \boldsymbol{R}^d$,

(ii) the coefficients $a_{ij}(t, x)$, $b_i(t, x)$ are bounded continuous in t,

(iii) $\{a_{ij}(t, x), x \in \boldsymbol{R}^d\}$ is equi-continuous in t,

(iv) the coefficients $a_{ij}(t, x)$, $b_i(t, x)$ have bounded continuous (in (t, x)) Hölder continuous (in x) derivatives in x.

Under these conditions, there exists the fundamental solution $p(t, x, s, y)$ of the parabolic equation,

$$\frac{\partial}{\partial t} u = \mathscr{L}_x u ,$$

and it has the following properties [3],

$$\left| \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} \right)^k \frac{\partial^m}{\partial x^m} p(t, x, s, y) \right| \leqq \frac{c}{(t - s)^{\alpha + m/2}} \exp \left\{ -\frac{\mu \|x - y\|^2}{t - s} \right\} , \tag{7}$$

$$\mu > 0, \ 0 \leqq m \leqq 2, \ 0 \leqq k \leqq 1 .$$

Let the coefficients $a(t), b(t), c(t)$ and $d(t)$ be continuous functions defined on $[0, \infty)$, and assume that the function $g(x)$ is bounded continuous, and that is has derivatives $g'(x)$ and $g''(x)$, which are bounded Hölder continuous. Under these assumptions, the Cauchy problem of the equation (1) is reduced to the following integral equation,

$$\begin{aligned} u(t, x, \omega) = p(t, x) &+ \int_0^t \int_{R^d} p(t, x, s, y)\{c(s)u(s, y, \omega) + d(s)\} dy dB(s) \\ &+ \int_0^t \int_{R^d} p(t, x, s, y)\{a(s)u(s, y, \omega) + b(s)\} dy ds \, . \end{aligned} \tag{8}$$

The existence of a solution of the Cauchy problem (1), (2) can be proved after we show the existence of a solution of (8). The uniqueness of the Cauchy problem can be proved without the expression (8). The following propositions, which give precise statements on these facts, have been established by B. L. Rozovskii [8]. Here, a solution of the equation (8) means a stochastic process $u(t, x, \omega)$ satisfying (i), (ii) in Definition 1 and the equation (8). The uniqueness of solutions of the equation (8) should be understood similarly to Definition 2.

PROPOSITION 1. *The equation* (8) *has a unique solution satisfying* $\sup_{(t,x)\in[0,T]\times R^d} E[u(t, x, \omega)^2] < \infty$ *for any* $T < \infty$, *and the solution satisfies* $\sup_{(t,x)\in[0,T]\times R^d} E[u(t, x, \omega)^{2k}] < \infty$ *for any* $T < \infty$, *and any positive integer* k.

The solution in Proposition 1 is obtained by the successive approximation, so that it is a 'strong solution'.

PROPOSITION 2. *The solution of the equation* (8) *satisfies the equation* (1).

PROPOSITION 3. *The equation* (1) *has a unique solution satisfying* $\sup_{(t,x)\in[0,T]\times R^d} E[u(t, x, \omega)^2] < \infty$ *for any* $T < \infty$.

§ 2. Construction of the solution

We are now in the position to show an algorithm to derive the formula (3). We shall make use of the notations and the results in T. Hida [6][7]. We denote by $\mathscr{S}$ the Schwartz space, and $\mathscr{S}^*$ means the space of tempered distributions. A probability measure $\mu(df)$ on $\mathscr{S}^*$ is defined in such a way that

$$\exp\left\{-\frac{1}{2}\|\xi\|^2\right\} = \int_{\mathscr{S}^*} \exp\,[i\langle f, \xi\rangle]\mu(df)\ , \qquad \xi \in \mathscr{S}\ ,$$

where $\|\cdot\|$ means L^2-norm. The measure $\mu(df)$ is defined on a σ-field $\mathfrak{B}$, which is generated by the sets,

$$\{f \in \mathscr{S}^*\,;\, (\langle f, \xi_1\rangle, \cdots, \langle f, \xi_n\rangle) \in B\}\ ,$$

where $\xi_i, i = 1, \cdots, n$, belong to $\mathscr{S}$, and where B is a Borel set in $\boldsymbol{R}^n$. Let $\chi_{[0,t]}$ be the indicator function of $[0, t]$, then we see that the process $\langle f, \chi_{[0,t]}\rangle$ is a Brownian motion on the space $(\mathscr{S}^*, \mathfrak{B}, \mu)$. Let $\mathfrak{B}_t$ be the σ-field generated by $\{\langle f, \chi_{[0,s]}\rangle, s \leq t\}$.

From now on, we shall take the system (Ω, F, P, F_t, B_t) to be $(\mathscr{S}^*, \mathfrak{B}, \mu, \mathfrak{B}_t, \langle f, \chi_{[0,t]}\rangle)$. The complex Hilbert space $(L^2) = L^2(\mathscr{S}^*, \mathfrak{B}, \mu)$ is decomposed as follows;

$$(L^2) = \sum_{n=0}^{\infty} \oplus\, \mathscr{H}_n\ ,$$

where $\mathscr{H}_n$ is the multiple Wiener integral of degree n.

Now, we consider the equation (8). In the first place, we assume $a(t) \equiv 0$ for simplicity. Let us project the both sides of the equation (8) on the space $\mathscr{H}_n$. We denote by $u_n(t, x, \omega)$ the projection of $u(t, x, \omega)$ on the space $\mathscr{H}_n$. Then, we get

$$\begin{aligned} u_n(t, x, \omega) &= \delta_{n,0}p(t, x) + \int_0^t \int_{\boldsymbol{R}^d} p(t, x, s, y)c(s)u_{n-1}(s, y, \omega)dydB(s) \\ &\quad + \delta_{n,1}\int_0^t d(s)dB(s) + \delta_{n,0}\int_0^t b(s)ds\ , \qquad n = 0, 1, 2, \cdots\ , \end{aligned} \tag{9}$$

that is,

$$\begin{aligned} u_0(t, x, \omega) &= p(t, x) + \int_0^t b(s)ds\ , \\ u_1(t, x, \omega) &= \int_0^t \int_{\boldsymbol{R}^d} p(t, x, s, y)c(s)u_0(s, y)dydB(s) + \int_0^t d(s)dB(s)\ , \\ u_n(t, x, \omega) &= \int_0^t \int_{\boldsymbol{R}^d} p(t, x, s, y)c(s)u_{n-1}(s, y)dydB(s) \qquad \text{for } n \geqq 2\ . \end{aligned}$$

Since $p(t, x, s, y)$ is the fundamental solution of (4), the equality $\int_{\boldsymbol{R}^d} p(t, x, s, y)p(s, y)dy = p(t, x)$ holds, so that we get

$$(10)\quad \begin{aligned} u_1 &= p(t,x)\int_0^t c(s)dB(s) + \int_0^t \left(c(s)\int_0^s b(\tau)d\tau\right)ds + \int_0^t d(s)\cdot dB(s)\ , \\ u_n &= p(t,x)\int_0^t c(\tau_n)\int_0^{\tau_n} c(\tau_{n-1})\cdots\int_0^{\tau_2} c(\tau_1)dB(\tau_1)dB(\tau_2)\cdots dB(\tau_n) \\ &\quad + \int_0^t c(\tau_n)\int_0^{\tau_n} c(\tau_{n-1})\cdots\int_0^{\tau_2}\left(c(\tau_1)\int_0^{\tau_1} b(\tau)d\tau\right)dB(\tau_1)dB(\tau_2)\cdots dB(\tau_n) \\ &\quad + \int_0^t c(\tau_n)\int_0^{\tau_n} c(\tau_{n-1})\cdots c(\tau_2)\int_0^{\tau_2} d(\tau_1)dB(\tau_1)dB(\tau_2)\cdots dB(\tau_n)\ , \\ &\qquad\qquad \text{for } n \geqq 2. \end{aligned}$$

We set

$$(11)\quad \begin{aligned} I_1 &= p(t,x) + \sum_{n=1}^{\infty} p(t,x)\int_0^t c(\tau_n)\int_0^{\tau_n} c(\tau_{n-1}) \\ &\qquad \cdots\int_0^{\tau_2} c(\tau_1)dB(\tau_1)dB(\tau_2)\cdots dB(\tau_n)\ , \\ I_2 &= \int_0^t b(s)ds + \sum_{n=1}^{\infty}\int_0^t c(\tau_n)\int_0^{\tau_n} c(\tau_{n-1}) \\ &\qquad \cdots\int_0^{\tau_2}\left(c(\tau_1)\int_0^{\tau_1} b(\tau)d\tau\right)dB(\tau_1)dB(\tau_2)\cdots dB(\tau_n)\ , \end{aligned}$$

and

$$I_3 = \sum_{n=1}^{\infty}\int_0^t c(\tau_n)\int_0^{\tau_n} c(\tau_{n-1})\cdots c(\tau_2)\int_0^{\tau_2} d(\tau_1)dB(\tau_1)dB(\tau_2)\cdots dB(\tau_n)\ .$$

Since $u(t,x,\omega) = \sum_{n=0}^{\infty} u_n(t,x,\omega) = I_1 + I_2 + I_3$, it is sufficient to calculate I_1, I_2 and I_3. It is known (See T. Hida [6][7]) that

$$(12)\quad \begin{aligned} I_1 &= p(t,x)\left\{1 + \sum_{n=1}^{\infty}\int_0^t c(\tau_n)\int_0^{\tau_n} c(\tau_{n-1})\cdots\int_0^{\tau_2} c(\tau_1)dB(\tau_1)dB(\tau_2)\cdots dB(\tau_n)\right\} \\ &= p(t,x)\exp\left\{\int_0^t c(u)dB(u) - \frac{1}{2}\|c\|_t^2\right\}\ , \end{aligned}$$

where $\|c\|_t^2 = \int_0^t c(u)^2 du$. In order to calculate I_2 and I_3, we need some lemmas on integral representations of the multiple Wiener integral.

The following operator $\mathscr{T}$ has been introduced by T. Hida-N. Ikeda [5],

$$(13)\qquad (\mathscr{T}\varphi)(\xi) = \int_{\mathscr{S}^*}\exp\,[i\langle f, \xi\rangle]\varphi(f)\mu(df)\ , \qquad \varphi\in(L^2)\ .$$

The collection $\mathscr{F} = \{\mathscr{T}\varphi\,;\,\varphi\in(L^2)\}$, which is made to be a reproducing kernel Hilbert space, is isomorphic to (L^2) under $\mathscr{T}$. If $\varphi(f) \in \mathscr{H}_n$, then

$$(\mathscr{T}\varphi)(\xi) = i^n \exp\left\{-\frac{1}{2}\|\xi\|^2\right\} \int_{R^n} \cdots \int F(u_1, \cdots, u_n)\xi(u_1) \cdots \xi(u_n) du_1 du_2 \cdots du_n \tag{14}$$

where $F \in \widehat{L^2(\boldsymbol{R}^n)} = \{\text{symmetric } L^2(R^n)\text{-functions}\}$, and the mapping

$$\varphi \to F \in \widehat{L^2(\boldsymbol{R}^n)} ,$$

is one to one. Besides,

$$\|\varphi\|_{(L^2)} = \sqrt{n!}\, \|F\|_{L^2(\boldsymbol{R}^n)} ,$$

holds. The function F is called the kernel of the integral representation of φ.

LEMMA 1. *Suppose that a function $F(u)$ is continuously differentiable on $[0, t]$. Then, the random variable $u_n(t)$ in $\mathscr{H}_n$, whose kernel of the integral representation is given by*

$$F_n(u_1, u_2, \cdots, u_n; t) = \frac{1}{n!} \prod_{i=1}^{n} f(u_i)\chi_{[0,t]}(u_i) F(u_1 \wedge u_2 \wedge \cdots \wedge u_n) \tag{15}$$

is equal to

$$F(0) H_n\left(\int_0^t f(u) dB(u); \|f\|_t^2\right) + \int_0^t H_n\left(\int_s^t f(u) dB(u); \|f\|_{s,t}^2\right) F'(s) ds , \tag{16}$$

where $H_n(z; \sigma^2)$ is the Hermite polynomial with parameter, and where $\|f\|_{s,t}^2 = \int_s^t f(u)^2 du < +\infty$ for $s \in [0, t]$.

Proof. By the equality $F(u) = F(0) + \int_0^u F'(u) du$, we have

the right hand side of (15)

$$= F(0)\frac{1}{n!} \prod_{i=1}^{n} f(u_i)\chi_{[0,t]}(u_i) + \frac{1}{n!} \prod_{i=1}^{n} f(u_i) \int_0^{u_1 \wedge u_2 \cdots \wedge u_n} F'(s) ds \prod_{j=1}^{n} \chi_{[0,t]}(u_j)$$

$$= F(0)\frac{1}{n!} \prod_{i=1}^{n} f(u_i)\chi_{[0,t]}(u_i) + \frac{1}{n!} \prod_{i=1}^{n} f(u_i) \int_0^t \prod_{j=1}^{n} \chi_{[s,t]}(u_j) F'(s) ds$$

$$= F(0)\frac{1}{n!} \prod_{i=1}^{n} f(u_i)\chi_{[0,t]}(u_i) + \int_0^t \frac{1}{n!} \prod_{i=1}^{n} f(u_i)\chi_{[s,t]}(u_i) F'(s) ds .$$

The random variable in $\mathscr{H}_n$, which correspond to the kernel $\frac{1}{n!}\prod_{i=1}^{n} f(u_i)$ $\chi_{[0,t]}(u_i)$ and $\frac{1}{n!}\prod_{i=1}^{n} f(u_i)\chi_{[s,t]}(u_i)$, are $H_n\left(\int_0^t f(u)dB(u)\,;\,\|f\|_t^2\right)$ and $H_n\left(\int_s^t f(u_i)dB(u)\,;\,\|f\|_{s,t}^2\right)$ respectively. Therefore we get

$$u_n(t) = F(0)H_n\left(\int_0^t f(u)dB(u)\,;\,\|f\|_t^2\right) + \int_0^t H_n\left(\int_s^t f(u)dB(u)\,;\,\|f\|_{s,t}^2\right)F'(s)ds\ .$$

Next, we consider the case where the function $F(u)$ in Lemma 1 is not smooth.

LEMMA 2. *Let $u_n(t), (n = 1, 2, \cdots)$, be random variables, whose kernels of the integral representation are of the form* (15), *where $F(u) = \frac{g(u)}{f(u)}$ with bounded functions $f(u)$ and $g(u)$, and where the set $\{u\,;\,f(u) = 0\}$ is of Lebesgue measure zero. Then, the equality,*

$$\begin{aligned}\sum_{n=1}^{\infty} u_n(t) = \exp&\left\{\int_0^t f(u)dB(u) - \frac{1}{2}\|f\|_t^2\right\}\\ &\left[\int_0^t f(u)F(u)\exp\left\{-\int_0^u f(\tau)dB(\tau) + \frac{1}{2}\|f\|_u^2\right\}dB(u)\right.\\ &\left.-\int_0^t f(u)^2F(u)\exp\left\{-\int_0^u f(\tau)dB(\tau) + \frac{1}{2}\|f\|_u^2\right\}du\right]\ ,\end{aligned}\tag{17}$$

holds.

Proof. As the first step, we consider the case where $F(u)$ is continuously differentiable. By Lemma 1, we have

$$u_n(t) = F(0)H_n\left(\int_0^t f(u)dB(u)\,;\,\|f\|_t^2\right) + \int_0^t H_n\left(\int_s^t f(u)dB(u)\,;\,\|f\|_{s,t}^2\right)F'(s)ds$$

so that we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(t) &= F(0)\exp\left\{\int_0^t f(u)dB(u) - \frac{1}{2}\|f\|_t^2\right\}\\ &\quad + \int_0^t \exp\left\{\int_s^t f(u)dB(u) - \frac{1}{2}\|f\|_{s,t}^2\right\}F'(s)ds\\ &= \exp\left\{\int_0^t f(u)dB(u) - \frac{1}{2}\|f\|_t^2\right\}\\ &\quad \times\left[F(0) + \int_0^t \exp\left\{-\int_0^s f(u)dB(u) + \frac{1}{2}\|f\|_s^2\right\}F'(s)ds\right]\end{aligned}$$

$$= \exp\left\{\int_0^t f(u)dB(u) - \frac{1}{2}\|f\|_t^2\right\}$$
$$\times \left[\exp\left\{-\int_0^t f(u)dB(u) + \frac{1}{2}\|f\|_t^2\right\}F(t)\right.$$
$$\left. - \int_0^t F(s)d_s \exp\left\{-\int_0^s f(u)dB(u) + \frac{1}{2}\|f\|_s^2\right\}\right]$$
$$= F(t) - \exp\left\{\int_0^t f(u)dB(u) - \frac{1}{2}\|f\|_t^2\right\}$$
$$\times \int_0^t F(s)d_s \exp\left\{-\int_0^s f(u)dB(u) + \frac{1}{2}\|f\|_s^2\right\},$$

where we set $u_0(t) = F(t)$. Hence we get

$$\sum_{n=1}^{\infty} u_n(t) = -\exp\left\{\int_0^t f(u)dB(u) - \frac{1}{2}\|f\|_t^2\right\}$$
$$\times \int_0^t F(s)d_s \exp\left\{-\int_0^s f(u)dB(u) + \frac{1}{2}\|f\|_s^2\right\}$$
$$= \exp\left\{\int_0^t f(u)dB(u) - \frac{1}{2}\|f\|_t^2\right\}$$
$$\times \left[\int_0^t f(u)F(u) \exp\left\{-\int_0^u f(u)dB(u) + \frac{1}{2}\|f\|_u^2\right\}dB(u)\right.$$
$$\left. - \int_0^t f(u)^2F(u) \exp\left\{-\int_0^u f(\tau)dB(\tau) + \frac{1}{2}\|f\|_u^2\right\}du\right].$$

We then come to the case $F(u) = \dfrac{g(u)}{f(u)}$, where $f(u)$ and $g(u)$ are bounded, and where the Lebesgue measure of the set $\{u; f(u) = 0\}$ is zero. We set

$$F^N(u) = \begin{cases} F(u)\,, & \text{if } |f(u)| > \dfrac{1}{N} \\ 0 \quad , & \text{if } |f(u)| \leqq \dfrac{1}{N} \end{cases}$$

Since $F^N(u)$ is bounded, we have a smooth function $F_N^N(u)$ such that $\|F^N(u) - F_N^N(u)\|_{L^2(0,t)} < \dfrac{1}{N}$. For this sequense $F_N^N(u), N = 1, 2, \cdots$, we have

$$\|f(u)F(u) - f(u)F_N^N(u)\|_{L^2(0,t)}$$
$$\leqq \|f(u)F(u) - f(u)F^N(u)\|_{L^2(0,t)} + \sup_{u\in[0,t]} |f(u)|\,\|F^N(u) - F_N^N(u)\|_{L^2(0,t)}$$

$$\leqq \sup_{u\in[0,t]} |g(u)| \times \text{Lebesgue measure of } \left\{u\,;\, |f(u)| \leqq \frac{1}{N}\right\} \cap [0,t]$$

$$+\frac{1}{N} \sup_{u\in[0,t]} |f(u)| = \varepsilon(N)$$

and $\varepsilon(N) \to 0$, as $N \to +\infty$. Furthermore, we see that

$$\|F_n^N(u_1, u_2, \cdots, u_n\,;\, t) - F_n(u_1, u_2, \cdots, u_n\,;\, t)\|^2_{L^2(R^n)} \leqq \frac{n}{(n\,!)^2} \|f\|_t^{2(n-)} \times \varepsilon(N)\ ,^{(*)}$$

so that we have

$$\sum_{n=1}^{\infty} n\,!\,\|F_n^N(u_1, \cdots, u_n\,; t) - F_n(u_1, \cdots, u_n\,; t)\|^2 \leqq \sum_{n=1}^{\infty} \frac{1}{(n-1)\,!} (\|f\|_t^2)^{n-1} \times \varepsilon(N)$$

$$\to 0, \text{ as } N \to +\infty\ .$$

Let $u_n^N(t)$ be the random variable with kernel $F_n^N(u_1, \cdots, u_n\,; t)$. From the argument above, we get $\|\sum_{n=1}^{\infty} u_n^N(t) - \sum_{n=1}^{\infty} u_n(t)\|_{(L^2)} \to 0$, as $N \to +\infty$. It is easy to see that the right hand side of (17) with smooth function F_N^N converges to the right hand side of (17) with function F, when N tends to infinity. Hence, we obtain the equality (17) for non-smooth function $F(u)$. The proof of Lemma 2 is complete.

Now, we can calculate I_2 and I_3. Let us begin with I_2. The kernel of the integral representation of

$$\int_0^t c(\tau_n) \int_0^{\tau_n} c(\tau_{n-1}) \cdots \int_0^{\tau_2} \left(c(\tau_1) \int_0^{\tau_1} b(\tau) d\tau\right) dB(\tau_1) dB(\tau_2) \cdots dB(\tau_n)$$

is equal to

$$\frac{1}{n\,!} \prod_{i=1}^{n} c(u_i) \chi_{[0,t]}(u_i) \int_0^{u_1\wedge u_2\wedge\cdots\wedge u_n} b(\tau) d\tau\ .$$

By Lemma 1, we get

$$\int_0^t c(\tau_n) \int_0^{\tau_n} c(\tau_{n-1}) \cdots \int_0^{\tau_2} \left(c(\tau_1) \int_0^{\tau_1} b(\tau) d\tau\right) dB(\tau_1) dB(\tau_2) \cdots dB(\tau_n)$$

$$= \int_0^t H_n\left(\int_s^t c(u) dB(u)\,;\, \|c\|_{s,t}^2\right) b(s) ds$$

Hence we obtain

$$I_2 = \int_0^t \exp\left\{\int_0^t c(u) dB(u) - \frac{1}{2}\|c\|_{s,t}^2\right\} b(s) ds\ . \tag{18}$$

(*) $F_n^N(u_1, u_2, \cdots, u_n;\, t) = \frac{1}{n\,!} \prod_{i=1}^{n} f(u_i) \chi_{[0,t]}(u_i) F_N^N(u_1 \wedge u_2 \wedge \cdots \wedge u_n)$

While, the kernel of the integral representation of

$$\int_0^t c(\tau_n)\int_0^{\tau_n} c(\tau_{n-1})\cdots c(\tau_2)\int_0^{\tau_2} d(\tau_1)dB(\tau_1)dB(\tau_2)\cdots dB(\tau_n)$$

is equal to

$$\frac{1}{n!}\prod_{i=1}^n c(u_i)\chi_{[0,t]}(u_i)\frac{d(u_1\wedge u_2\wedge\cdots\wedge u_n)}{c(u_1\wedge u_2\wedge\cdots\wedge u_n)}\,.$$

By Lemma 2, we obtain

$$\begin{aligned} I_3 &= \exp\left\{\int_0^t c(u)dB(u)-\frac{1}{2}\|c\|_t^2\right\} \\ &\quad\times\left[\int_0^t \exp\left\{-\int_0^s c(u)dB(u)+\frac{1}{2}\|c\|_s^2\right\}d(s)dB(s)\right. \\ &\quad\left.-\int_0^t \exp\left\{-\int_0^s c(u)dB(u)+\frac{1}{2}\|c\|_s^2\right\}c(s)d(s)ds\right]\,. \end{aligned} \tag{19}$$

By (12), (18) and (19), we are given an explicit expression of the solution,

$$\begin{aligned} u(t,x,\omega) &= \Psi(t,\omega)\left[p(t,x)+\int_0^t \Psi(s,\omega)^{-1}(b(s)-c(s)d(s))ds\right. \\ &\quad\left.+\int_0^t \Psi(s,\omega)^{-1}d(s)dB(s)\right]\,, \end{aligned} \tag{20}$$

where $\psi(t,\omega)=\exp\left[\int_0^t c(u)dB(u)-\frac{1}{2}\|c\|_t^2\right]$.

Finally, we consider the case where the coefficient $a(t)\not\equiv 0$. We set

$$v(t,x,\omega)=\exp\left\{-\int_0^t a(s)ds\right\}u(t,x,\omega)\,,$$

where the process $u(t,x,\omega)$ is the solution of the Cauchy problem of the equation (1). Then, the process $v(t,x,\omega)$ satisfies the following equation,

$$\begin{aligned} dv(t,x,\omega) &= \left\{\mathscr{L}_x v(t,x,\omega)+b(t)\exp\left(-\int_0^t a(u)du\right)\right\}dt \\ &\quad+\left\{c(t)v(t,x,\omega)+d(t)\exp\left(-\int_0^t a(u)du\right)\right\}dB(t)\,, \end{aligned} \tag{21}$$

and the initial condition,

$$v(t,x,\omega)=g(x)\,. \tag{22}$$

Therefore, this case is reduced to the case where $a(t)\equiv 0$.

Thus, we obtain

THEOREM 1. *Suppose that the following conditions are satisfied,*
(a) *the operator* $\mathscr{L}_x$ *satisfies the conditions* (i), (ii), (iii) *and* (iv) *in* § 1,
(b) *the coefficients* $a(t), b(t), c(t)$ *and* $d(t)$ *are bounded continuous on* $[0, \infty)$,
(c) *the Lebesgue measure of the set* $\{t; c(t) = 0\}$ *is equal to zero,*
(d) *the function* $g(x)$ *is bounded continuous, and it has derivatives* $g'(x)$ *and* $g''(x)$, *which are bounded Hölder continuous. Then, the solution of the Cauchy problem of the equation* (1) *is expressed in the form* (3).

Remark 1. Suppose that $c(t) \equiv 0$. By (11), it is easy to see that

$$I_1 = p(t, x) ,$$
$$I_2 = \int_0^t b(s)ds ,$$
$$I_3 = \int_0^t d(s)dB(s) .$$

Taking into account (21), we get

$$u(t, x, \omega) = \exp\left\{\int_0^t a(u)du\right\}\left[p(t, x) + \int_0^t \exp\left\{-\int_0^s a(u)du\right\}b(s)ds + \int_0^t \exp\left\{-\int_0^s a(u)du\right\}d(s)dB(s)\right] .$$

Remark 2. We can verify by Ito's formula that the process $u(t, x, \omega)$ given by (3) satisfies the equation (1). Carrying out this procedure, we can see that a sufficient condition for the process $u(t, x, \omega)$ to be a solution of the Cauchy problem is stated as follows;
(a′) the Cauchy problem $\frac{\partial}{\partial t}p = \mathscr{L}_x p$, $p(0, x) = g(x)$ has a smooth solution,
(b′) $a(t), b(t) \in L^1[0, T]$, and $c(t), d(t) \in L^2[0, T]$ for any $T < +\infty$.
The pair (a′) and (b′) is rather weaker than the quadruplet (a), (b), (c) and (d) in Theorem 1.

Remark 3. In case where the coefficients $a(t), b(t), c(t)$ and $d(t)$ are F_t-adapted process, the existence and the uniqueness of solutions of (1) and (2) have been proved by B. L. Rozovskii [8]. The solution is also expressed in the form (3). This fact can be verified by Ito's formula.

§ 3. The case where the space-parameter x varies in a bounded domain

Let us discuss the equation (1), when the space-parameter x runs through a bounded domain D in $\boldsymbol{R}^d$, whose boundary ∂D is sufficiently smooth. We consider the second initial-boundary value problem with initial condition

$$u(0, x, \omega) = g(x) , \qquad x \in D , \tag{2'}$$

and the boundary condition (5), Let us consider the integral equation (8), where $p(t, x, s, y)$ is replaced by the fundamental solution of the second initial-boundary value problem,

$$\begin{aligned} &\frac{\partial u}{\partial t} = \mathscr{L}_x u , && (t, x) \in (0, \infty) \times D , \\ &u(0, x) = g(x) , && x \in D , \\ &\frac{\partial u}{\partial n}(t, x) = 0 , && (t, x) \in (0, \infty) \times \partial D , \end{aligned} \tag{23}$$

and where $p(t, x)$ means the solution of (6). Then, we can prove the assertion corresponding to Proposition 1 also in this case. It seems to be plausible that the inequality (7) is valid. If the fundamental solution $p(t, x, s, y)$ has the property (7), then we can prove the assertions corresponding to Proposition 2 and 3. The algorithm to get an explicit solution, shown in § 2, remains true in this case. Hence, we get the formula (3) as an explicit expression of the solution of the equation (8) with boundary condition (5). Besides, using Ito's formula, we can verify that the process given by (3) satisfies the equation (1), the initial condition (2′) and the boundary condition (5).

Thus we obtain

THEOREM 2. *Suppose that the fundamental solution $p(t, x, s, y)$ of the second initial-boundary value problem* (23) *exists, and that the conditions* (b), (c) *and* (d) *in Theorem* 1 *are satisfied. Then, the solution of the equation* (8), *where $p(t, x, s, y)$ is replaced by the fundamental solution of the problem* (23), *is expressed in the form* (3), *where $p(t, x)$ is the solution of* (6). *Furthermore, the process given by* (3) *satisfies*

$$\begin{aligned} du(t, x, \omega) &= \{\mathscr{L}_x u(t, x, \omega) + a(t)u(t, x, \omega) + b(t)\}dt \\ &\quad + \{c(t)u(t, x, \omega) + d(t)\}dB(t) , \qquad (t, x) \in (0, \infty) \times D , \\ u(0, x, \omega) &= g(x) , \qquad x \in D , \end{aligned} \tag{24}$$

and

$$\frac{\partial u}{\partial n}(t, x, \omega) = 0 \ , \qquad (t, x) \in (0, \infty) \times \partial D \ .$$

REFERENCES

[1] A. V. Balakrishnan; Stochastic bilinear partial differential equations, the U.S.-Italy conference on variable structure systems, Oregon, May, 1974.

[2] D. A. Dawson; Stochastic evolution equations and related measure processes, J. Multivariate anal., Vol. **5**, No. **1** (1975).

[3] S. D. Eidel'man; Parabolic systems, Moscow, 1964.

[4] W. H. Fleming; Distributed parameter stochastic systems in population biology, m.s., Brown University, Providence, R. I.

[5] T. Hida and N. Ikeda; Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral, Proc. Fifth Berkeley Symp. on math. Statist. and Probability, Vol. **2**, Part **1** (1967), 117–143.

[6] T. Hida; Brownian motion (in Japanese), Iwanami Pub. Co., 1975.

[7] T. Hida; Analysis of Brownian functionals, Carleton University lecture notes, 1975.

[8] B. L. Rozovskii; On stochastic differential equations with partial derivatives (in Russian), Math. Sb., Vol. **96** (138), No. **2** (1975), 314–341.

Nagoya Institute of Technology