K. Yamaguchi
Nagoya Math. J.
Vol. 62 (1976), 55-96

NON-DEGENERATE REAL HYPERSURFACES IN COMPLEX MANIFOLDS ADMITTING LARGE GROUPS OF PSEUDO-CONFORMAL TRANSFORMATIONS. I

KEIZO YAMAGUCHI

Introduction

Let S (resp. S') be a (real) hypersurface (i.e. a real analytic submanifold of codimension 1) of an n-dimensional complex manifold M (resp. M'). A homeomorphism f of S onto S' is called a pseudo-conformal homeomorphism if it can be extended to a holomorphic homeomorphism of a neighborhood of S in M onto a neighborhood of S' in M. In case such an f exists, we say that S and S' are pseudo-conformally equivalent. A hypersurface S is called non-degenerate (index r) if its Levi-form is non-degenerate (and its index is equal to r) at each point of S.

In his paper [6], N. Tanaka has shown that if a hypersurface S is connected and non-degenerate at a point, then the group $A(S)$ of all pseudo-conformal transformations of S becomes a Lie transformation group of S with dim. $A(S) \leqq n^2 + 2n$.

The purpose of this paper is to determine, under pseudo-conformal equivalence, non-degenerate hypersurfaces S for which the groups $A(S)$ have either the largest dimension $n^2 + 2n$ or the second largest dimension.

Our main results are stated as follows;

THEOREM 7.2. *Let M be a complex manifold of dimension n. Let S be a connected non-degenerate* (*index r*) *homogeneous hypersurface* $\left(0 \leqq r \leqq \left[\frac{n-1}{2}\right]\right)$. *Then we have the following classification table*:

$$Q_r = \left\{(z_0, \cdots, z_n) \in P^n(C) \,|\, -\sqrt{-1} z_0 \bar{z}_n - \sum_{i=1}^{r} z_i \bar{z}_i + \sum_{i=r+1}^{n-1} z_i \bar{z}_i + \sqrt{-1} z_n \bar{z}_0 = 0 \right\},$$

Received August 4, 1975.

(n,r)	the case of the largest dimension		the case of the second largest dimension	
	dim. $A(S)$	S	dim. $A(S)$	S
$n=3$ & $r=1$	$15(=n^2+2n)$	Q_1	$11(=n^2+2)$	$Q_1{}^*(1)$
$n=5$ & $r=2$	$35(=n^2+2n)$	Q_2	$26(=n^2+1)$	$Q_2{}^*(2)$ or $Q_2{}^*$
otherwise	n^2+2n	Q_r	n^2+1	$Q_r{}^*$

$$Q_r^* = \{(z_0, \cdots, z_n) \in Q_r \mid z_0 \neq 0\} ,$$
$$Q_1^*(1) = \{(z_0, \cdots, z_3) \in Q_1 \mid |z_0| + |z_1 - z_2| \neq 0\} ,$$
$$Q_2^*(2) = \{(z_0, \cdots, z_5) \in Q_2 \mid |z_0| + |z_1 - z_4| + |z_2 - z_3| \neq 0\} ,$$

where $P^n(C)$ is the complex projective space of dimension n with its homogeneous coordinate $(z_0, \cdots, z_n)$.

This is a partial generalization of the results of E. Cartan [2] in the case $n = 2$.

THEOREM 7.4. *Let M be a complex manifold of dimension n. Let S be a connected hypersurface of M which is non-degenerate of index r at a point of S. If* dim. $A(S) = n^2 + 2n$, *then S is pseudo-conformally equivalent to Q_r.*

Now we will describe the method of proving our theorems. Let S be a non-degenerate (index r) hypersurface of a complex manifold, and let $A(S)$ be the group of all pseudo-conformal transformations of S and $\mathfrak{a}(S)$ be its Lie algebra. Then according to N. Tanaka [6], [7] we can associate with S a principal fibre bundle $P(S, G'(r))$ together with an infinitesimal structure ω on it, which is a Cartan connection of type $(G(r), G'(r))$, the so-called normal pseudo-conformal connection. Here $G(r)$ is the group of all projective transformations leaving Q_r invariant and $G'(r)$ is the isotropy subgroup of it at a point o of Q_r (cf. I). Let $\mathfrak{g}(r)$ be the Lie algebra of $G(r)$. If we fix a point p_0 of S, then the connection form ω induces an injective linear map of $\mathfrak{a}(S)$ (identified with the Lie algebra of right invariant vector fields of P leaving the Cartan connection invariant) into the graded Lie algebra $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$. So we can induce a filtration of $\mathfrak{a}(S)$ at p_0 via the map ω. With respect to this filtration $\mathfrak{a}(S) = \mathfrak{h}$ becomes a filtered Lie algebra. Moreover it is seen that the associated graded Lie algebra $\bar{\mathfrak{h}}$ of $\mathfrak{h}$ becomes a graded subalgebra of $\mathfrak{g}(r)$ (cf. II). So under the dimension hypothesis of $A(S)$

and the homogeneity assumption, we can determine explicitly the possibilities of $\tilde{\mathfrak{h}}$. In fact we determine the graded subalgebras of $\mathfrak{g}(r)$ of the minimum codimension satisfying a certain (homogeneity) condition (cf. IV). Moreover under the dimension hypothesis of $A(S)$ (more precisely if $\tilde{\mathfrak{h}}$ coincides with one of the graded subalgebras of $\mathfrak{g}(r)$ obtained in IV) we will see that S is flat, that is, the curvature form of the connection vanishes identically and that $\mathfrak{a}(S)$ is isomorphic with $\tilde{\mathfrak{h}}$ (cf V). Conversely let $\mathfrak{g}$ be one of the graded subalgebras of $\mathfrak{g}(r)$ obtained in IV. Then we can construct a model space Q corresponding to $\mathfrak{g}$ as follows; let G be the analytic subgroup of $G(r)$ corresponding to $\mathfrak{g}$. Q is defined as the orbit of G passing through $o \in Q_r$. Then Q is a connected non-degenerate (index r) homogeneous flat hypersurface of $P^n(C)$ for which G is the identity component of $A(Q)$ (cf. VI). On the other hand, the bundle $A(S)(S, A_{p_0}(S))$ can be regarded as a subbundle of $P(S, G'(r))$, if we assume that S is homogeneous. Moreover the structure equation of the connection determines the Maurer-Cartan equation of $A(S)$. From these facts we see that, in order to find a pseudo-conformal homeomorphism between two homogeneous hypersurfaces S and S', we have only to find a group isomorphism between $A(S)$ and $A(S')$ which satisfies certain additional conditions (cf. III). So under the dimension hypothesis we compare $A^0(S)$ with the corresponding G satisfying $\mathfrak{g} \cong \mathfrak{a}(S)$. In this way we see that S is pseudo-conformally equivalent to the corresponding Q (cf. VII).

The author is grateful to Prof. S. Kaneyuki who kindly read through the manuscript, and he is also grateful to Prof. N. Tanaka and Prof. H. Omoto for their constant encouragement and valuable advices during the preparation of this paper.

Preliminary remarks.

Throughout this paper we always assume the differentiability of class C^∞. We use the notations and terminology in S. Kobayashi-K. Nomizu [5] without special references (e.g. the differential of a mapping, fundamental vector fields, homomorphisms of fibre bundles).

Let I be a hermitian matrix of degree n. We denote by $U(I)$ the unitary group defined by I; $U(I) = \{\sigma \in GL(n, C) \mid {}^t\bar{\sigma} I \sigma = I\}$, where ${}^t\sigma$ is the transposed matrix of σ and $\bar{\sigma}$ is the complex conjugate matrix of σ. We denote by $\mathfrak{u}(I)$ the Lie algebra of $U(I)$. Moreover we denote by

$SU(I)$ the special unitary group defined by I; $SU(I) = \{\sigma \in U(I) \mid \det \sigma = 1\}$. We denote by $\mathfrak{su}(I)$ the Lie algebra of $SU(I)$.

I. Pseudo-conformal geometry.

In this section we will review the fundamental concepts of the pseudo-conformal geometry and state the results of Tanaka, following N. Tanaka [6], [7], which are necessary for later considerations.

1. The H-structure. Let M and M' be complex manifolds of dimension n ($n \geqq 2$). Let S (resp. S') be a (real) hypersurface, that is a $(2n-1)$-dimensional real analytic regular submanifold, of M (resp. M').

DEFINITION 1.1. A homeomorphism f of S onto S' is called a pseudo-conformal homeomorphism if it can be extended to a holomorphic homeomorphism of a neighborhood of S in M onto a neighborhood of S' in M'.

Let p be an arbitrary point of S. We denote by $T_p(S)$ the tangent space to S at p and by J the complex structure of M. We set

$$D_p = T_p(S) \cap J(T_p(S)) .$$

Then D_p is a maximal complex vector subspace of $T_p(M)$ contained in $T_p(S)$ and $\dim._C D_p = n-1$.

Take the natural base $\{e_i\}_{1 \leqq i \leqq n}$ of the n-dimensional complex number space C^n. We denote by $\mathfrak{m}$ the $(2n-1)$-dimensional real vector subspace of C^n spanned by the $2n-1$ vectors $e_1, \cdots, e_n, \sqrt{-1}e_1, \cdots, \sqrt{-1}e_{n-1}$ and by $\mathfrak{m}_*$ the $(n-1)$-dimensional complex vector subspace of C^n spanned by the $n-1$ vectors $e_1, \cdots, e_{n-1}$. We define a closed subgroup H of the general linear group $GL(n, C)$ by setting

$$H = \{\sigma \in GL(n, C) \mid \sigma(\mathfrak{m}) = \mathfrak{m}\} .$$

Each element of H is represented as a matrix of the following form

$$\begin{pmatrix} B & C \\ 0 & a \end{pmatrix}$$

where $a \in R \backslash \{0\}$, $B \in GL(n-1, C)$ and $C \in C^{n-1}$. Hence we get

$$H = \{\sigma \in GL(\mathfrak{m}) \mid \sigma(\mathfrak{m}_*) = \mathfrak{m}_* \text{ and } \sigma \mid \mathfrak{m}_* \text{ is complex linear}\}$$

We denote by $L(S)$ the bundle of linear frames of S. A linear frame

x at a point p of S is a linear isomorphism of $\mathfrak{m}$ onto $T_p(S)$, where we identify $\mathfrak{m}$ with $\boldsymbol{R}^{2n-1}$ through the natural isomorphism. We define a subbundle F of $L(S)$ by

$$F = \{x \in L(S) \mid x(\mathfrak{m}_*) = D_{\varpi(x)} \text{ and } x \mid \mathfrak{m}_* \text{ is complex linear}\} ,$$

where ϖ is the bundle projection of $L(S)$ onto S. Then F becomes a principal fibre bundle over S with the structure group H. $F(S, H)$ is called the pseudo-conformal H-bundle associated with the hypersurface S (cf. [6]).

Remark 1.2. The "Fundamental theorem" (i.e. Theorem 1 [6]) says that a C^ω-homeomorphism f of a hypersurface S onto another hypersurface S' is a pseudo-conformal homeomorphism if and only if f induces an isomorphism between the corresponding pseudo-conformal H-bundles, preserving the canonical 1-forms.

2. The Levi-form. Let θ^* be the canonical 1-form on F (cf. [5]), that is,

$$\theta_x^*(X) = x^{-1}(\varpi_*(X)) = \begin{pmatrix} \theta_1^*(X) \\ \vdots \\ \theta_n^*(X) \end{pmatrix} \in \mathfrak{m} \subset \boldsymbol{C}^n \qquad \text{for } x \in F, X \in T_x(F) ,$$

where θ_i^* $(i = 1, 2, \cdots, n)$ is the i-th component of θ^*. Note that θ_i^* $(i = 1, \cdots, n-1)$ is a $\boldsymbol{C}$-valued 1-form on F and θ_n^* is a $\boldsymbol{R}$-valued 1-form on F. We pay attention to θ_n^*, which characterizes the maximal complex tangent space D_p of $T_p(S)$. First we notice

LEMMA 1.3. *Let x be an arbitrary point of F, and let X and Y be tangent vectors at x. Then we have*

(i) $\theta_n^*(X) = 0$ *if and only if* $\varpi_*(X) \in D_{\varpi(x)}$

(ii) $d\theta_n^*(X, Y) = 0$ *if* $\varpi_*(X) \in D_{\varpi(x)}$ *and* $\varpi_*(Y) = 0$.

Lemma 1.3 is easily proved from the definition of F and the following

$$\begin{cases} R_\sigma^* \theta_n^* = a^{-1}\theta_n^* & \text{for } \sigma = \begin{pmatrix} B & C \\ 0 & a \end{pmatrix} \in H \\ \theta_n^*(A^*) = 0 & \text{for } A \in \text{the Lie algebra of } H \end{cases}$$

where R_σ is a right action on F induced by $\sigma \in H$ and A^* is the fundamental vector field corresponding to A (cf. [5]).

From Lemma 1.3 we can define a skew-symmetric bilinear mapping K_x of $D_p \times D_p$ into $\boldsymbol{R}$ by

$$K_x(X, Y) = -2\, d\theta^*_{nx}(X^*, Y^*) \qquad p = \varpi(x),\, X, Y \in D_p\,,$$

where X^* (resp. Y^*) is any vector at x such that $\varpi_*(X^*) = X$ (resp. $\varpi_*(Y^*) = Y$). One should note that we can also write

$$K_x(X, Y) = \theta^*_{nx}([X^*, Y^*])\,,$$

where X^* (resp. Y^*) is any vector field around x such that $\theta^*_n(X^*) = 0$ (resp. $\theta^*_n(Y^*) = 0$) and $\varpi_*(X^*_x) = X$ (resp. $\varpi_*(Y^*_x) = Y$). Hence from the integrability condition of the complex structure of the ambient space M we have

LEMMA 1.4. *Let x be an arbitrary point of F. Then*

$$K_x(X, Y) = K_x(JX, JY) \qquad \textit{for } X, Y \in D_{\varpi(x)}\,,$$

where J is the complex structure of M.

Now Lemma 1.3 and Lemma 1.4 imply

LEMMA 1.5 ([6]). *There exist a 1-form β and unique $\boldsymbol{C}$-valued functions L_{ij} $(i, j = 1, 2, \cdots, n-1)$ on F such that*

$$d\theta^*_n + \sum_{i,j=1}^{n-1} L_{ij}\theta^*_i \wedge \bar\theta^*_j + \beta \wedge \theta^*_n = 0 \;\; (L_{ij} + \bar L_{ji} = 0)\,,$$

where $\bar\theta^*_j$ is the complex conjugate 1-form of θ^*_j.

For $x \in F$, we set $L(x) = (L_{ij}(x))$. Then $\sqrt{-1}L(x)$ is a hermitian matrix of degree $n-1$. We call $\sqrt{-1}L(x)$ the Levi-form at $x \in F$. The Levi-form at x defines a hermitian inner product of $D_{\varpi(x)}$. In fact if we set;

$$L_x(X, Y) = K_x(JX, Y) + \sqrt{-1}K_x(X, Y) \qquad \text{for } X, Y \in D_{\varpi(x)}\,,$$

then we have easily

$$L_x(X, Y) = 2\sum_{i,j=1}^{n-1} \sqrt{-1}L_{ij}(x)\xi_i\bar\eta_j\,,$$

where

$$x^{-1}(X)=\begin{bmatrix}\xi_1\\ \vdots\\ \xi_{n-1}\\ 0\end{bmatrix},\qquad x^{-1}(Y)=\begin{bmatrix}\eta_1\\ \vdots\\ \eta_{n-1}\\ 0\end{bmatrix}\in\mathfrak{m}_* .$$

Now we will define the notion of a non-degenerate hypersurface and its index. Let p be a point of S. For $x\in\pi^{-1}(p)$, L_x is a hermitian inner product of D_p. Let $k(x)$ (resp. $l(x)$) be the dimension of a maximal subspace on which L_x is positive definite (resp. negative definite). We define an integer valued function $\lambda(p)$ on S by $\lambda(p)=$ minimum of $k(x)$ and $l(x)$. The integer $\lambda(p)$ is well-defined, that is, $\lambda(p)$ is independent of the choise of $x\in\pi^{-1}(p)$ ([6]), and satisfies $0\leqq\lambda(p)\leqq\left[\frac{n-1}{2}\right]$.

DEFINITION 1.6. Let p be a point of S.

(1) S is called non-degenerate at p if the Levi-form is non-degenerate at p.

(2) S is called of index r at p if $\lambda(p)=r$.

S is called a non-degenerate hypersurface if its Levi-form is non-degenerate at each point of S. Obviously the index of a non-degenerate hypersurface S is constant on each connected component of S.

3. Quadrics. Let us fix an integer r satisfying $0\leqq r\leqq\left[\frac{n-1}{2}\right]$. We will give the model space of non-degenerate (index r) hypersurface ([6]).

Let $P^n(C)$ be the n-dimensional complex projective space, and let $z_0, z_1,\cdots,z_n$ be the system of its homogeneous coordinates. We define the hermitian matrices I_r and $\tilde{I}_r$ of degree $n-1$ and $n+1$ by

$$I_r=\begin{pmatrix}-E_r & 0\\ 0 & E_{n-r-1}\end{pmatrix},\qquad \tilde{I}_r=\begin{pmatrix}0 & 0 & \sqrt{-1}\\ 0 & I_r & 0\\ -\sqrt{-1} & 0 & 0\end{pmatrix}$$

where E_s is the unit matrix of degree s.

Let Q_r be the quadric of $P^n(C)$ defined by $\tilde{I}_r$, that is,

$$Q_r=\Big\{(z_0,\cdots,z_n)\in P^n(C)\,|\,-\sqrt{-1}z_0\bar{z}_n - \sum_{i=1}^{r} z_i\bar{z}_i+\sum_{i=r+1}^{n-1} z_i\bar{z}_i+\sqrt{-1}z_n\bar{z}_0=0\Big\}.$$

It is known [6] that Q_r is a connected non-degenerate hypersurface of $P^n(C)$ and its index is r.

Let $P(n, C)$ be the group of all projective transformations. We consider the subgroup $G(r)$ of $P(n, C)$ which consists of all projective transformations leaving Q_r invariant. $G(r)$ acts effectively and transitively on Q_r as a group of pseudo-conformal transformations. Moreover if we identify $P(n, C)$ with $GL(n+1, C)/GL(1, C)$, the identity component of $G(r)$ is $U(\tilde{I}_r)/U(1) = SU(\tilde{I}_r)/\mathfrak{n}$, where $U(1)$ (resp. $\mathfrak{n}$) is the center of $U(\tilde{I}_r)$ (resp. $SU(\tilde{I}_r)$). $G(r)$ is connected in case $r \neq \frac{n-1}{2}$ and it has two connected components in case $r = \frac{n-1}{2}$ (n: odd integer). We denote by $G'(r)$ the isotropy subgroup of $G(r)$ at $o = (1, 0, \cdots, 0) \in Q_r$.

Now we will explain the graded structure of the Lie algebra $\mathfrak{g}(r)$ of $G(r)$. Since the identity component of $G(r)$ is $SU(\tilde{I}_r)/\mathfrak{n}$, $\mathfrak{g}(r)$ can be identified with $\mathfrak{su}(\tilde{I}_r)$, that is,

$$\mathfrak{g}(r) = \{X \in \mathfrak{gl}(n+1, C) \mid {}^t\bar{X}\tilde{I}_r + \tilde{I}_r X = 0, \text{ trace } X = 0\} .$$

$\mathfrak{g}(r)$ is isomorphic with $\mathfrak{su}(r+1, n-r)$, and so it is simple. Each element X of $\mathfrak{g}(r)$ can be written explicitly as a matrix of the form

$$\begin{pmatrix} -\bar{u} & -\sqrt{-1}\,{}^t\bar{w}I_r & w_n \\ \xi & v & w \\ \xi_n & \sqrt{-1}\,{}^t\bar{\xi}I_r & u \end{pmatrix}$$

where $\xi_n, w_n \in \boldsymbol{R}$, $\xi, w \in \boldsymbol{C}^{n-1}$, $v \in \mathfrak{u}(I_r)$, and $u - \bar{u} + \text{trace } v = 0$. For an element $E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ of $\mathfrak{g}(r)$, $\operatorname{ad}(E_0)$ (i.e. $\operatorname{ad}(E_0)(X) = [E_0, X]$) is a semi-simple endomorphism of $\mathfrak{g}(r)$. Its eigenvalues are $-2, -1, 0, 1$, and 2. We set $\mathfrak{g}_k(r) = \{X \in \mathfrak{g}(r) \mid \operatorname{ad}(E_0)(X) = kX\}$. Then $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$, and $\mathfrak{g}(r)$ becomes a graded Lie algebra with respect to this decomposition. More precisely $\{\mathfrak{g}_k(r)\}_{k \in Z}$ satisfies

$$[\mathfrak{g}_k(r), \mathfrak{g}_l(r)] \subset \mathfrak{g}_{k+l}(r) ,$$

where we set $\mathfrak{g}_k(r) = \{0\}$ for $|k| \geqq 3$. Moreover if we set

$$\begin{cases} \mathfrak{m}(r) = \sum_{k=-2}^{-1} \mathfrak{g}_k(r) , \\ \mathfrak{g}'(r) = \sum_{k=0}^{2} \mathfrak{g}_k(r) , \end{cases}$$

then we have $\mathfrak{g}(r) = \mathfrak{m}(r) \oplus \mathfrak{g}'(r)$. $\mathfrak{m}(r)$ and $\mathfrak{g}'(r)$ are subalgebras of $\mathfrak{g}(r)$. It is easily seen that $\mathfrak{g}'(r)$ coincides with the Lie algebra of $G'(r)$.

Remark 1.7. Let χ be the natural homomorphism of $GL(n+1, \boldsymbol{C})$ onto $P(n, \boldsymbol{C}) = GL(n+1, \boldsymbol{C})/GL(1, \boldsymbol{C})$. Setting $\hat{G}(r) = \chi^{-1}(G(r))$, we have

$$\hat{G}(r) = \{\sigma \in GL(n+1, \boldsymbol{C}) \mid {}^t\bar{\sigma}\tilde{I}_r\sigma = \pm\tilde{I}_r\} .$$

Hence we get

(1) if $r \neq \frac{n-1}{2}$ $\qquad \hat{G}(r) = U(\tilde{I}_r)$,

(2) if $r = \frac{n-1}{2}$ (n: odd integer) $\qquad \hat{G}(r) = U(\tilde{I}_r) \cup \sigma_0(U(\tilde{I}_r))$,

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_r^* & 0 \\ 0 & 0 & -1 \end{pmatrix} , \qquad I_r^* = \begin{pmatrix} 0 & E_r \\ E_r & 0 \end{pmatrix} .$$

In particular the Lie algebra of $\hat{G}(r)$ is $\mathfrak{u}(\tilde{I}_r)$. Note that the kernel of χ_* coincides with the center $\mathfrak{u}(1)$ of $\mathfrak{u}(\tilde{I}_r)$ and $\mathfrak{u}(\tilde{I}_r) = \mathfrak{u}(1) \oplus \mathfrak{su}(\tilde{I}_r)$ (direct sum). Moreover we have $\chi_* \circ \mathrm{Ad}_{\hat{G}(r)}(\sigma) = \mathrm{Ad}_{G(r)}(\chi(\sigma)) \circ \chi_*$ from $\chi \circ I_\sigma = I_{\chi(\sigma)} \circ \chi$ (I_σ is the inner automorphism induced by σ). Since we are identifying $\mathfrak{g}(r)$ with $\mathfrak{su}(\tilde{I}_r)$, $\mathrm{Ad}_{G(r)}(\chi(\sigma))$ is identified with the restriction of $\mathrm{Ad}_{\hat{G}(r)}(\sigma)$ to $\mathfrak{su}(\tilde{I}_r)$.

4. Pseudo-conformal $G'(r)$-bundles. First we consider the linear isotropy group of $G'(r)$. We identify the tangent space at o to $Q_r = G(r)/G'(r)$ with $\mathfrak{m}(r)$ ($\cong \mathfrak{g}(r)/\mathfrak{g}'(r)$). Moreover we identify $\mathfrak{m}(r)$ with $\mathfrak{m}$ via

$$\mathfrak{m} \ni \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ \xi_n & \sqrt{-1}\,{}^t\bar{\xi}I_r & 0 \end{pmatrix} \in \mathfrak{m}(r) \qquad \xi_n \in \boldsymbol{R},\ \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n-1} \end{pmatrix} \in \boldsymbol{C}^{n-1} .$$

We consider the linear isotropy representation $l; G'(r) \to GL(\mathfrak{m})$. Let $\tilde{G}(r) = l(G'(r))$ be the linear isotropy group of $G'(r)$. Then $\tilde{G}(r)$ is a closed subgroup of H. In fact let $\tau = \chi(\sigma)$ be an element of $G'(r)$, where σ is given by

$$\sigma = \begin{pmatrix} \bar{a}^{-1} & -\varepsilon\sqrt{-1}\bar{a}^{-1}\,{}^t\bar{C}I_rB & d \\ 0 & B & C \\ 0 & 0 & \varepsilon a \end{pmatrix}$$

$(\varepsilon = \pm 1, a, d \in \boldsymbol{C}\ ,\quad C \in \boldsymbol{C}^{n-1}\ ,\quad {}^t\bar{B}I_rB = \varepsilon I_r\ ,\quad \sqrt{-1}(\bar{a}d - a\bar{d}) = {}^t\bar{C}I_rC)$.

Then we have

$$l(\tau) = \begin{pmatrix} \bar{a}B & \bar{a}C \\ 0 & \varepsilon\,|a|^2 \end{pmatrix},$$

which is easily seen from the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g}(r) & \xrightarrow{\mathrm{Ad}\,(\tau)} & \mathfrak{g}(r) \\ p\big\downarrow & & \big\downarrow p \\ \mathfrak{m}(r) & \xrightarrow{l(\tau)} & \mathfrak{m}(r) \end{array} \qquad \tau \in G'(r)$$

(p is the projection of $\mathfrak{g}(r)$ onto $\mathfrak{m}(r)$ corresponding to $\mathfrak{g}(r) = \mathfrak{m}(r) \oplus \mathfrak{g}'(r)$). From this we get easily ([6])

$$\tilde{G}(r) = \left\{\sigma = \begin{pmatrix} B & C \\ 0 & a \end{pmatrix} \in H \,\middle|\, a^{-1}\,{}^t\bar{B}I_rB = I_r\right\}.$$

Let S be a hypersurface which is non-degenerate of index r at every point. Then at each point x of F the Levi-form $\sqrt{-1}L(x)$ is a hermitian matrix of signature $(r, n-r-1)$ or $(n-r-1, r)$, where we say that a hermitian matrix L is of signature (p, q) if L has p negative eigenvalues and q positive eigenvalues. We set

$$\tilde{F} = \{x \in F \mid \sqrt{-1}L(x) = I_r\}\ .$$

Then since $L(x\sigma) = a^{-1}\,{}^t\bar{B}L(x)B$ for $\sigma = \begin{pmatrix} B & C \\ 0 & a \end{pmatrix} \in H$ (cf. Lemma 4 [6]), $\tilde{F}$ becomes a principal fibre bundle over S with the structure group $\tilde{G}(r)$. Obviously $\tilde{F}(S, \tilde{G}(r))$ is a subbundle of $F(S, H)$ (therefore of $L(S)$). $\tilde{F}(S, \tilde{G}(r))$ is called the pseudo-conformal $\tilde{G}(r)$-bundle associated with S ([6], [7]).

Remark 1.8 (cf. [7]). Let $\tilde{\theta}_1, \cdots, \tilde{\theta}_n$ be the components of the canonical 1-form $\tilde{\theta}$ on $\tilde{F}$. Then from the definition of $\sqrt{-1}L(x)$ (cf. Lemma 1.5), we have

$$d\tilde{\theta}_n + \sqrt{-1}\sum_{i=1}^{n-1}\varepsilon_i\bar{\tilde{\theta}}_i \wedge \tilde{\theta}_i \equiv 0 \qquad \mathrm{mod}\ \tilde{\theta}_n\ ,$$

where

$$\varepsilon_i = \begin{cases} -1 & 1 \leqq i \leqq r , \\ 1 & \text{otherwise} . \end{cases}$$

Identifying $\mathfrak{m}$ with $\mathfrak{m}(r)$, we write the $\mathfrak{m}(r)$-valued 1-form $\tilde{\theta}$ in the form $\tilde{\theta} = \tilde{\theta}_{-2} + \tilde{\theta}_{-1}$, where $\tilde{\theta}_k$ is the $\mathfrak{g}_k(r)$-component of $\tilde{\theta}$ ($k = -2, -1$). Then we can write

$$d\tilde{\theta}_{-2} + \tfrac{1}{2}[\tilde{\theta}_{-1} \wedge \tilde{\theta}_{-1}] \equiv 0 \qquad \mod \tilde{\theta}_{-2} ,$$

where [,] is the bracket operation of $\mathfrak{m}(r)$.

5. Tanaka's theorem. Digressing from hypersurfaces we will now mention about the Cartan connection and its curvature (cf. [4]).

Let M be a manifold of dimension n. Let G be a Lie group, and G' be a closed subgroup of G with $\dim. G/G' = n$. We denote by $\mathfrak{g}, \mathfrak{g}'$ the Lie algebras of G and G' respectively.

DEFINITION 1.9. Let M, G and G' be as above. (P, ω) is called a Cartan connection of type (G, G') over M if P and ω satisfy the following

(1) P is a principal fibre bundle over M with the structure group G'.

(2) ω is a $\mathfrak{g}$-valued 1-form on P satisfying the following conditions.

(a) $R_a^*\omega = \operatorname{Ad}(a^{-1})\omega$ for $a \in G'$,

(b) $\omega(A^*) = A$ for $A \in \mathfrak{g}'$,

where A^* is the fundamental vector field corresponding to A.

(c) $\omega(X) = 0$ implies $X = 0$.

From (c) ω defines an absolute parallelism on P. Hence for $U \in \mathfrak{g}$, we can define a vector field U^* on P by $U_z^* = \omega_z^{-1}(U)$, $z \in P$. For $A \in \mathfrak{g}'$ it is obvious from (b) that A^* above coincides with the fundamental vector field corresponding to A.

The curvature form Ω of a Cartan connection (P, ω) is defined by

$$\Omega = d\omega + \tfrac{1}{2}[\omega \wedge \omega] .$$

DEFINITION 1.10. Let S be a non-degenerate (index r) hypersurface, and let $\tilde{F}(S, \tilde{G}(r))$ be the corresponding $\tilde{G}(r)$-bundle over S. A triplet $(P, \omega, \bar{l})$ is called a pseudo-conformal connection over S if P, ω and $\bar{l}$ satisfy the following

(1) (P, ω) is a Cartan connection of type $(G(r), G'(r))$ over S.

(2) $\bar{l}$ is a bundle homomorphism of $P(S, G'(r))$ onto $\tilde{F}(S, \tilde{G}(r))$ corresponding to $l; G'(r) \to \tilde{G}(r)$, which preserves the base space and satisfies

$\bar{l}^*\tilde{\theta} = \theta$, where $\tilde{\theta}$ is the canonical 1-form on $\tilde{F}$ and θ is the $\mathfrak{m}(r)$-component of ω.

Let Ω be the curvature form of a pseudo-conformal connection $(P, \omega, \bar{l})$. Let B be the Killing form of $\mathfrak{g}(r)$. We have $B(\mathfrak{g}_k(r), \mathfrak{g}_l(r)) = 0$ if $k + l \neq 0$. Moreover the bilinear mapping $\mathfrak{g}_k(r) \times \mathfrak{g}_{-k}(r) \ni (X, Y) \mapsto B(X, Y) \in \boldsymbol{R}$ gives a duality between $\mathfrak{g}_k(r)$ and $\mathfrak{g}_{-k}(r)$. Then the "Ricci" curvature Ω^*, which is a $\mathfrak{g}(r)$-valued 1-form on P, is defined by

$$\Omega^*_z(X) = \sum_{k=-2}^{-1} \sum_i [u_i^{-k}, \Omega_z((u_i^{-k})^*, X)] \qquad X \in T_z(P) ,$$

where $\{u_i^k\}_i$ is a base of $\mathfrak{g}_k(r)$ and $\{u_i^{-k}\}_i$ is the dual base of $\{u_i^k\}_i$.

Now we state the results of Tanaka.

THEOREM A [7]. *Let M and M' be complex manifolds of dimension n. Let S (resp. S') be a non-degenerate (index r) hypersurface of M (resp. M'). Then there exists a pseudo-conformal connection $(P, \omega, \bar{l})$ (resp. $(P', \omega', \bar{l}')$) over S (resp. S'), which satisfies*

$$\Omega_{-2} = \Omega_{-1} = \Omega^* = 0 \qquad (resp.\ \Omega'_{-2} = \Omega'_{-1} = \Omega'^* = 0) ,$$

where Ω_k (resp. Ω'_k) is the $\mathfrak{g}_k(r)$-component of Ω (resp. Ω').

And suppose that f is a pseudo-conformal homeomorphism of S onto S'. Then there corresponds a unique bundle isomorphism $\tilde{f}$ of $P(S, G'(r))$ onto $P'(S', G'(r))$ which induces the given f on S and satisfies $\tilde{f}^*\omega' = \omega$. Conversely every bundle isomorphism $\tilde{f}$ of $P(S, G'(r))$ onto $P'(S', G'(r))$ satisfying $\tilde{f}^*\omega' = \omega$ induces a pseudo-conformal homeomorphism of S onto S'.

The above $P(S, G'(r))$, whose existence and uniqueness (up to a isomorphism commuting with $\bar{l}$) are guaranteed in the theorem, is called the pseudo-conformal $G'(r)$-bundle associated with S and (P, ω) is called the normal pseudo-conformal connection.

Let S be a non-degenerate (index r) hypersurface, and let $P(S, G'(r))$ be the corresponding $G'(r)$-bundle over S. We now consider the Lie algebra $\tilde{\mathfrak{a}}(S)$ of all infinitesimal pseudo-conformal transformations of S. We set $\tilde{\mathfrak{a}}(P) = \{X \in \mathfrak{X}(P) \mid L_X\omega = 0, R_{a*}X = X \text{ for } a \in G'(r)\}$, where $\mathfrak{X}(P)$ is the Lie algebra of all vector fields on P and L_X is the Lie differentiation with respect to X. Then the infinitesimal version of Theorem A reads;

THEOREM A′. *Let S be a non-degenerate (index r) hypersurface, and* *let $P(S, G'(r))$ be the corresponding $G'(r)$-bundle over S. Let π be the bundle projection of P onto S. Then π_* is a Lie algebra isomorphism of $\tilde{\mathfrak{a}}(P)$ onto $\tilde{\mathfrak{a}}(S)$.*

II. Filtration of $\mathfrak{a}(S)$.

First we will examine the filtration of $\mathfrak{g}(r)$. For $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$, we set for each integer l

$$\begin{cases} \mathscr{L}_l(r) = \sum_{k=l}^{2} \mathfrak{g}_k(r) & (l = -2, -1, 0, 1, 2)\ , \\ \mathscr{L}_l(r) = \mathscr{L}_{-2}(r) & (l \leqq -3)\ , \qquad \mathscr{L}_l(r) = 0 \ \ (l \geqq 3)\ . \end{cases}$$

With respect to this filtration $\mathfrak{g}(r) = \mathscr{L}_{-2}(r)$ becomes a filtered Lie algebra, that is, $\{\mathscr{L}_k(r)\}_{k\in Z}$ satisfy $[\mathscr{L}_k(r), \mathscr{L}_l(r)] \subset \mathscr{L}_{k+l}(r)$.

LEMMA 2.1. *For $a \in G'(r)$, $\mathrm{Ad}\,(a)$ preserves this filtration.*

Proof. Recall that the Lie algebra of $G'(r)$ coincides with $\mathfrak{g}'(r) = \mathscr{L}_0(r)$.

(1) in case $G'(r)$ is connected $\left(\text{i.e. } r \neq \frac{n-1}{2}\right)$. For $X \in \mathfrak{g}'(r) = \mathscr{L}_0(r)$, $\mathrm{ad}\,(X)$ preserves the filtration. Hence $\mathrm{Ad}\,(\exp X) = \exp \mathrm{ad}\,(X)$ preserves the filtration.

(2) in case $G'(r)$ is not connected $\left(\text{i.e. } r = \frac{n-1}{2}\right)$. $G'(r)$ has two connected components. But in this case we can find an element $\tau_0 = \chi(\sigma_0)$ of $G'(r)$, which does not belong to the identity component, such that $\mathrm{Ad}\,(\tau_0)$ preserves the filtration, e.g.

$$\sigma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_r^* & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad I_r^* = \begin{pmatrix} 0 & E_r \\ E_r & 0 \end{pmatrix}.$$

(In fact $\mathrm{Ad}\,(\tau_0)$ preserves also the grading of $\mathfrak{g}(r)$.) Q.E.D.

From now on in this section let S be a non-degenerate (index r) hypersurface. And let $(P, \omega, \bar{l})$ be the normal pseudo-conformal connection over S.

Let us fix an arbitrary point z of P. Since each element of $\tilde{\mathfrak{a}}(P)$ is an infinitesimal automorphism of the absolute parallelism defined by (P, ω), it is known (cf. [5; p232 Lemma]) that the linear map $\omega_z : \tilde{\mathfrak{a}}(P) \ni X \mapsto \omega_z(X_z) \in \mathfrak{g}(r)$, is injective.

LEMMA 2.2. *For $X, Y \in T_z(P)$, we have*

(1) $\omega_z(X) \in \mathscr{L}_{-1}(r)$ *if and only if* $\pi_*(X) \in D_{\pi(z)}$,

(2) $\omega_z(x) \in \mathscr{L}_0(r) = \mathfrak{g}'(r)$ *if and only if* $\pi_*(X) = 0$,

(3) $\Omega_z(X, Y) = 0$ *if* $\pi_*(X) = 0$ *or* $\pi_*(Y) = 0$,

where Ω is the curvature form of the connection.

Proof. (1) and (2) follow immediately from $\bar{l}(z)\,(\mathfrak{g}_{-1}(r)) = D_{\pi(z)}$ and the following commutative diagram which is a direct consequence of the equality $\bar{l}^*\tilde{\theta} = \theta$ $(= p\omega)$;

$$\begin{array}{ccc} T_z(P) & \xrightarrow{\omega_z} & \mathfrak{g}(r) \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle p} \\ T_{\pi(z)}(S) & \xleftarrow{l(z)} & \mathfrak{m}(r) \end{array} .$$

In fact for $X \in T_z(P)$ we have

$$p\omega_z(X) = \theta_z(X) = \tilde{\theta}_{\bar{l}(z)}(\bar{l}_*X) = (\bar{l}(z))^{-1}(\varpi_*\bar{l}_*X) = (\bar{l}(z))^{-1}(\pi_*X) .$$

In order to prove (3), we have only to show $\Omega(U^*, A^*) = 0$ for $U \in \mathfrak{g}(r)$ and $A \in \mathfrak{g}'(r)$. First we note that $[U^*, A^*] = [U, A]^*$. In fact from $R_{a*}U^* = (\mathrm{Ad}\,(a^{-1})U)^*$, $a \in G'(r)$, we have

$$[U^*, A^*] = -L_{A^*}U^* = (-L_A U)^* = [U, A]^* .$$

Therefore, from the structure equation, we get $\Omega(U^*, A^*) = 0$. Q.E.D.

We set $\tilde{\mathfrak{a}}_z(P) = \{X \in \tilde{\mathfrak{a}}(P) \mid \pi_{*_z}(X) = 0\}$. Then

LEMMA 2.3. *For $X, Y \in \tilde{\mathfrak{a}}(P)$, we have*

$$-\omega_z([X, Y]) = [-\omega_z(X), -\omega_z(Y)] - 2\Omega_z(X, Y) .$$

In particular if either X or Y belongs to $\tilde{\mathfrak{a}}_z(P)$, then we have

$$-\omega_z([X, Y]) = [-\omega_z(X), -\omega_z(Y)] .$$

Proof. From $L_X\omega = 0$, we have $X\omega(Z) = \omega([X, Z])$ for $Z \in \mathfrak{X}(P)$. Hence the assertion is clear from the structure equation and Lemma 2.2 (3). Q.E.D.

Let $A(S)$ be the group of all pseudo-conformal transformations of S. We consider the subset $\mathfrak{a}(S)$ of $\tilde{\mathfrak{a}}(S)$ consisting of complete vector fields in $\tilde{\mathfrak{a}}(S)$. Then $\mathfrak{a}(S)$ is a subalgebra of $\tilde{\mathfrak{a}}(S)$ which is naturally isomorphic with the Lie algebra of $A(S)$. Moreover $\mathfrak{a}(S)$ can be regarded as a subalgebra $\mathfrak{h}$ of $\tilde{\mathfrak{a}}(P)$ via $\pi_*: \tilde{\mathfrak{a}}(P) \to \tilde{\mathfrak{a}}(S)$. In fact $\mathfrak{h}$ coincides with the subalgebra $\mathfrak{a}(P)$ of $\tilde{\mathfrak{a}}(P)$ which consists of complete vector fields in $\tilde{\mathfrak{a}}(P)$.

Now let us fix a point p_0 of S and choose a point z_0 of the fibre $\pi^{-1}(p_0)$ over p_0. We set for each integer k

$$\mathfrak{h}_k = \mathfrak{h} \cap \omega_{z_0}^{-1}(\mathscr{L}_k(r)) .$$

Then $\mathfrak{h}_k = \mathfrak{h}$ $(k \leqq -2)$ and $\mathfrak{h}_k = \{0\}$ $(k \geqq 3)$. Note that the above definition is independent of the choice of z_0 in $\pi^{-1}(p_0)$, which is easily seen from Lemma 2.1 and the equalities $R_a^*\omega = \mathrm{Ad}\,(a^{-1})\omega$ and $R_{a*}X = X$, $a \in G'(r)$, $X \in \tilde{\mathfrak{a}}(P)$. Hence the above defines a filtration of $\mathfrak{a}(S)$ at p_0. From Lemma 2.2 and Lemma 2.3 we have

Proposition 2.4. *With respect to the above filtration,* $\mathfrak{a}(S)$ *becomes a filtered Lie algebra. In particular* $(\mathfrak{a}(S))_{-1}$ *and* $(\mathfrak{a}(S))_0$ *are given by*

$$(\mathfrak{a}(S))_{-1} = \{X \in \mathfrak{a}(S) \mid X_{p_0} \in D_{p_0}\} ,$$
$$(\mathfrak{a}(S))_0 = \{X \in \mathfrak{a}(S) \mid X_{p_0} = 0\} .$$

Next we will consider the associated graded Lie algebra $\tilde{\mathfrak{h}}$ of the filtered Lie algebra $\mathfrak{h}$. Setting $\tilde{\mathfrak{h}}_k = \mathfrak{h}_k/\mathfrak{h}_{k+1}$ for each integer k (note $\tilde{\mathfrak{h}}_k = \{0\}$ for $|k| \geqq 3$), we define $\tilde{\mathfrak{h}}$ by

$$\tilde{\mathfrak{h}} = \sum_{k=-2}^{2} \tilde{\mathfrak{h}}_k \text{ (vector space direct sum) .}$$

The bracket operation of $\tilde{\mathfrak{h}}$ is defined in a natural manner. Obviously we have dim. $\mathfrak{h}$ = dim. $\tilde{\mathfrak{h}}$.

First observe that there exists an injective linear map $\nu_{z_0}^k$ of $\tilde{\mathfrak{h}}_k$ into $\mathfrak{g}_k(r)$ which satisfies the following commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_k & \xrightarrow{-p_k\omega_{z_0}} & \mathfrak{g}_k(r) \\ {\scriptstyle \mu_k}\downarrow & \nearrow {\scriptstyle \nu_{z_0}^k} & \\ \tilde{\mathfrak{h}}_k & & \end{array}$$

where μ_k is the natural projection of $\mathfrak{h}_k$ onto $\tilde{\mathfrak{h}}_k = \mathfrak{h}_k/\mathfrak{h}_{k+1}$ and p_k is the projection of $\mathfrak{g}(r)$ onto $\mathfrak{g}_k(r)$ corresponding to $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$. We define an injective linear map ν_{z_0} of $\tilde{\mathfrak{h}}$ into $\mathfrak{g}(r)$ by setting

$$\nu_{z_0} = \nu_{z_0}^{-2} \times \nu_{z_0}^{-1} \times \cdots \times \nu_{z_0}^{2} .$$

Lemma 2.5. *Notations being as above, the linear map* ν_{z_0} *is an injective homomorphism of* $\tilde{\mathfrak{h}}$ *into* $\mathfrak{g}(r)$.

Hence setting $\tilde{\mathfrak{h}}_{z_0} = \nu_{z_0}(\tilde{\mathfrak{h}})$, we see that $\tilde{\mathfrak{h}}_{z_0}$ is a graded subalgebra of $\mathfrak{g}(r)$ which is isomorphic with $\tilde{\mathfrak{h}}$ and satisfies dim. $\tilde{\mathfrak{h}}_{z_0} =$ dim. $\mathfrak{a}(S)$.

Proof of Lemma 2.5. It suffices to show $\nu_{z_0}([\tilde{X}_k, \tilde{Y}_l]) = [\nu_z(\tilde{X}_k), \nu_{z_0}(\tilde{Y}_l)]$ for $\tilde{X}_k \in \tilde{\mathfrak{h}}_k$ and $\tilde{Y}_l \in \tilde{\mathfrak{h}}_l$. Choose $X_k \in \mathfrak{h}_k$ (resp. $Y_l \in \mathfrak{h}_l$) such that $\tilde{X}_k = \mu_k(X_k)$ (resp. $\tilde{Y}_l = \mu_l(Y_l)$). Then

$$\nu_{z_0}([\tilde{X}_k, \tilde{Y}_l]) = -p_{k+l}\omega_{z_0}([X_k, Y_l]) .$$

Set $-\omega_{z_0}(X_k) = \sum_{i=k}^{2} \bar{X}_i$, $\bar{X}_i \in \mathfrak{g}_i(r)$ (resp. $-\omega_{z_0}(Y_l) = \sum_{i=l}^{2} \bar{Y}_i$, $\bar{Y}_i \in \mathfrak{g}_i(r)$). Then from the definition of ν_{z_0} and the graded structure of $\mathfrak{g}(r)$, we have

$$\nu_{z_0}(\tilde{X}_k) = \bar{X}_k , \qquad \nu_{z_0}(\tilde{Y}_l) = \bar{Y}_l$$

and

$$p_{k+l}([-\omega_{z_0}(X_k), -\omega_{z_0}(Y_l)]) = [\bar{X}_k, \bar{Y}_l]$$

(1) in case $k \geqq 0$ or $l \geqq 0$. From Lemma 2.3 we have $-\omega_{z_0}([X_k, Y_l]) = [-\omega_{z_0}(X_k), -\omega_{z_0}(Y_l)]$. Hence we get

$$\nu_{z_0}([\tilde{X}_k, \tilde{Y}_l]) = [\bar{X}_k, \bar{Y}_l] = [\nu_{z_0}(\tilde{X}_k), \nu_{z_0}(\tilde{Y}_l)] .$$

(2) otherwise. Non-trivial case is when $k = l = -1$. Form the above we have

$$\nu_{z_0}([\tilde{X}_{-1}, \tilde{Y}_{-1}]) = p_{-2}(-\omega_{z_0}([X_{-1}, Y_{-1}]) ,$$
$$[\nu_{z_0}(\tilde{X}_{-1}), \nu_{z_0}(\tilde{Y}_{-1})] = p_{-2}([-\omega_{z_0}(X_{-1}), -\omega_{z_0}(Y_{-1})]) .$$

In this case we have from Lemma 2.3

$$-\omega_{z_0}([X_{-1}, Y_{-1}]) = [-\omega_{z_0}(X_{-1}), -\omega_{z_0}(Y_{-1})] - 2\Omega_{z_0}(X_{-1}, Y_{-1}) .$$

But, due to Theorem A, the $\mathfrak{g}_{-2}(r)$-component Ω_{-2} of Ω vanishes identically. Hence we get $\nu_{z_0}([\tilde{X}_{-1}, \tilde{Y}_{-1}]) = [\nu_{z_0}(\tilde{X}_{-1}), \nu_{z_0}(\tilde{Y}_{-1})]$. Q.E.D.

Remark 2.6. Clearly the representation ν_{z_0} of $\tilde{\mathfrak{h}}$ into $\mathfrak{g}(r)$ is dependent on the choice of z_0 in $\pi^{-1}(p_0)$. Choose another point $z_1 = z_0 a$, if Ad(a) preserves the grading of $\mathfrak{g}(r)$, we get from $R_a^*\omega = \mathrm{Ad}\,(a^{-1})\omega$

$$\tilde{\mathfrak{h}}_{z_0 a} = \mathrm{Ad}\,(a^{-1})\tilde{\mathfrak{h}}_{z_0} .$$

Moreover if we define a vector subspace $\mathfrak{h}_{z_0}$ of $\mathfrak{g}(r)$ by $\mathfrak{h}_{z_0} = \omega_{z_0}(\mathfrak{h})$, we get similarly

$$\mathfrak{h}_{z_0 a} = \mathrm{Ad}\,(a^{-1})\mathfrak{h}_{z_0} , \qquad a \in G'(r) .$$

Remark 2.7. The discussion in this section can be well applied to a connected hypersurface S which is non-degenerate of index r *at a point*; Let S^* be the set of all points of S at which S is non-degenerate of index r. Obviously S^* is an open subset of S. Hence S^* is a non-degenerate (index r) hypersurface. Let $P^*(S^*, G'(r))$ be the corresponding $G'(r)$-bundle over S^*. We consider the restriction map *res* of $\mathfrak{a}(S)$ into $\tilde{\mathfrak{a}}(S^*)$. Since we are considering, exclusively, real analytic hypersurfaces, each infinitesimal pseudo-conformal transformation of S is a real analytic vector field on S. Hence the connectedness of S implies that res; $\mathfrak{a}(S) \to \tilde{\mathfrak{a}}(S^*)$ is an injective homomorphism. On the other hand $(\pi^*)_*$ is an isomorphism of $\tilde{\mathfrak{a}}(P^*)$ onto $\tilde{\mathfrak{a}}(S^*)$. Hence we can define a subalgebra $\mathfrak{h}$ of $\tilde{\mathfrak{a}}(P^*)$ by $\mathfrak{h} = (\pi^*)_*^{-1} \circ res\ (\mathfrak{a}(S))$. Then $\mathfrak{h}$ is isomorphic with $\mathfrak{a}(S)$. Therefore if we fix a point p_0 of S^*, we can define a filltration of $\mathfrak{h}$ (and consequently of $\mathfrak{a}(S)$) at p_0 similarly as in this section.

III. Relations between $A(S)$ $(S, A_{p_0}(S))$ and $P(S, G'(r))$.

Throughout this section we assume that S is a connected non-degenerate (index r) homogeneous (i.e. $A(S)$ acts transitively on S) hypersurface. Let $(P, \omega, \bar{l})$ be the normal pseudo-conformal connection over S. We denote by $\tilde{\sigma}$ the connection-preserving bundle isomorphism of $P(S, G'(r))$ induced by $\sigma \in A(S)$. Then from I. Theorem A, A(S) acts effectively on P as an automorphism group of the Cartan connection (P, ω).

Let us fix a point $p_0 \in S$ and take a point $z_0 \in \pi^{-1}(p_0)$. And we define ι_{z_0}; $A(S) \to P$ by $\iota_{z_0}(\sigma) = \tilde{\sigma}(z_0)$, $\sigma \in A(S)$. Then it is known ([4]) that ι_{z_0} is an imbedding of $A(S)$ as a closed submanifold of P.

Let $A_{p_0}(S)$ be the isotropy subgroup of $A(S)$ at $p_0 \in S$. Obviously we have

$$\iota_{z_0}(A_{p_0}(S)) \subset \pi^{-1}(p_0) .$$

On the other hand the fibre $\pi^{-1}(p_0)$ of $P(S, G'(r))$ is diffeomorphic with $G'(r)$ via a diffeomorphism γ_{z_0} of $G'(r)$ onto $\pi^{-1}(p_0)$, where $\gamma_{z_0}(a) = z_0 a$, $a \in G'(r)$. Therefore the composite map $\rho_{z_0} = \gamma_{z_0}^{-1} \circ \iota_{z_0}$ is an imbedding of $A_{p_0}(S)$ into $G'(r)$ and $\rho_{z_0}(A_{p_0}(S))$ is closed in $G'(r)$. Moreover we have

LEMMA 3.1. *The map* ρ_{z_0}; $A_{p_0}(S) \to G'(r)$ *is an injective homomorphism. And* $\rho_{z_0}(A_{p_0}(S))$ *is a closed subgroup of* $G'(r)$. *Moreover* $(\rho_{z_0 *})_e = \omega_{z_0} \cdot (\iota_{z_0 *})_e$, *where* e *is the unit of* $A_{p_0}(S)$.

Proof. Suppose $\rho_{z_0}(\sigma_i) = a_i$ $(i = 1, 2)$, that is, $\tilde{\sigma}_i(z_0) = z_0 \cdot a_i$, then

$$\iota_{z_0}(\sigma_1 \cdot \sigma_2) = \tilde{\sigma}_1 \cdot \tilde{\sigma}_2(z_0) = \tilde{\sigma}_1(z_0 \cdot a_2) = (z_0 \cdot a_1)a_2 = z_0(a_1 \cdot a_2) .$$

Hence we get $\rho_{z_0}(\sigma_1 \cdot \sigma_2) = a_1 \cdot a_2 = \rho_{z_0}(\sigma_1) \cdot \rho_{z_0}(\sigma_2)$. $\rho_{z_0}(A_{p_0}(S))$ is closed in $G'(r)$ since $A_{p_0}(S)$ is a closed subgroup of $A(S)$, $\iota_{z_0}(A(S))$ is a closed submanifold of P and $\pi^{-1}(p_0)$ is closed in P. In order to prove $(\rho_{z_0*})_e = \omega_{z_0} \cdot (\iota_{z_0*})_e$, it suffices to show $\omega_{z_0} = (\gamma_{z_0*})_{e'}^{-1}$, where e' is the unit element of $G'(r)$, which is clear from the definition of the fundamental vector field A^* corresponding to A and $\omega(A^*) = A$. Q.E.D.

Since $A(S)$ acts transitively on S, $A(S)$ is a principal $A_{p_0}(S)$-bundle over $S = A(S)/A_{p_0}(S)$. Then we have

PROPOSITION 3.2. *The imbedding* ι_{z_0}; $A(S) \to P$ *is an injective bundle homomorphism of* $A(S)(S, A_{p_0}(S))$ *into* $P(S, G'(r))$ *corresponding to* ρ_{z_0}; $A_{p_0}(S) \to G'(r)$, *which preserves the base space* S.

Hence $A(S)(S, A_{p_0}(S))$ can be regarded as a subbundle of $P(S, G'(r))$ via ι_{z_0}.

Proof of Proposition 3.2. Let τ be an element of $A_{p_0}(S)$. Let $\sigma \in A(S)$. Then we get easily the following commutative diagram

$$\begin{array}{ccc} A(S) & \xrightarrow{\ \iota_{z_0}\ } & P \\ {\scriptstyle R_\tau}\downarrow & & \downarrow{\scriptstyle R_{\rho_{z_0}(\tau)}}, \\ A(S) & \xrightarrow[\ \iota_{z_0}\ ]{} & P \end{array} \qquad \tau \in A_{p_0}(S) .$$

Therefore ι_{z_0} is a bundle homomorphism corresponding to ρ_{z_0}. Moreover ι_{z_0} induces the identity transformation of S, which follows from $\pi \cdot \iota_{z_0}(\sigma) = \pi \cdot \tilde{\sigma}(z_0) = \sigma \cdot \pi(z_0) = \sigma(p_0)$ for $\sigma \in A(S)$. Q.E.D.

Now we will consider the relation between the Maurer-Cartan form on $A(S)$ and the normal pseudo-conformal connection form ω on P. First observe

LEMMA 3.3. *Let* ω *be the connection form on* P *and let* Ω *be its curvature form. Then* $\iota_{z_0}^*\omega$ *and* $\iota_{z_0}^*\Omega$ *are* $\mathfrak{g}(r)$*-valued left invariant forms on* $A(S)$.

Proof. Let $\sigma \in A(S)$. We denote by L_σ the left translation of $A(S)$ by σ. Then we get easily the following commutative diagram.

$$\begin{array}{ccc} A(S) & \xrightarrow{\iota_{z_0}} & P \\ {\scriptstyle L_\sigma}\downarrow & & \downarrow{\scriptstyle \tilde\sigma} \\ A(S) & \xrightarrow{\iota_{z_0}} & P \end{array} \qquad \text{for } \sigma \in A(S) .$$

Therefore $\iota_{z_0}^*\omega$ is left invariant since $\tilde\sigma^*\omega = \omega$, $\sigma \in A(S)$. From the structure equation $d\omega + \frac{1}{2}[\omega \wedge \omega] = \Omega$, it is obvious that $\iota_{z_0}^*\Omega$ is also left invariant. Q.E.D.

In this section we denote by $\mathfrak{a}(S)$ the Lie algebra of $A(S)$. Then we have easily

$$\iota_{z_0}^*\omega(\mathfrak{a}(S)) = \omega_{z_0}(\mathfrak{h}) = \mathfrak{h}_{z_0} ,$$

where $\mathfrak{h} = \mathfrak{a}(P)$ (cf. II).

In case $\Omega = 0$ we have

PROPOSITION 3.4. *Suppose that the curvature form Ω of the normal pseudo-conformal connection vanishes identically. Then the linear map $\iota_{z_0}^*\omega\,;\, \mathfrak{a}(S) \to \mathfrak{g}(r)$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ into $\mathfrak{g}(r)$. Hence $\mathfrak{h}_{z_0}(= \iota_{z_0}^*\omega(\mathfrak{a}(S)))$ is a subalgebra of $\mathfrak{g}(r)$ which is isomorphic with $\mathfrak{a}(S)$. Moreover if we identify $\mathfrak{a}(S)$ with $\mathfrak{h}_{z_0}$, $\iota_{z_0}^*\omega$ is the Maurer-Cartan form of $A(S)$.*

Proof. From $\Omega = 0$ we get $d\iota_{z_0}^*\omega + \frac{1}{2}[\iota_{z_0}^*\omega \wedge \iota_{z_0}^*\omega] = 0$. Let $A, B \in \mathfrak{a}(S)$. Then we have

$$2\, d\iota_{z_0}^*\omega(A, B) = -\iota_{z_0}^*\omega([A, B]) ,$$

since $\iota_{z_0}^*\omega$ is left invariant. Hence we get $\iota_{z_0}^*\omega([A, B]) = [\iota_{z_0}^*\omega(A), \iota_{z_0}^*\omega(B)]$. Q.E.D.

Now we will consider an equivalence of two non-degenerate (index r) homogeneous hypersurfaces. Let M and M' be complex manifolds of dimension n. Let S (resp. S') be a connected non-degenerate (index r) homogeneous hypersurface of M (resp. M'). And let $(P, \omega, \bar{l})$ (resp. $(P', \omega', \bar{l}')$) be the normal pseudo-conformal connection over S (resp. S'). We denote by $A^0(S)$ the identity component of $A(S)$, and set $A_{p_0}^0(S) = A^0(S) \cap A_{p_0}(S)$. Note that the identity component $A^0(S)$ acts transitively on S.

PROPOSITION 3.5. *Notations being as above, let $p_0 \in S$ and $p_0' \in S'$. Suppose that for points, $z_0 \in \pi^{-1}(p_0)$, $z_0' \in \pi'^{-1}(p_0')$ suitably chosen, there exists a group isomorphism φ of $A^0(S)$ onto $A^0(S')$ satisfying* i), ii);

i) $\varphi(A^0_{p_0}(S)) = A^0_{p'_0}(S')$,

ii) $\varphi^* \iota^*_{z'_0} \omega' = \iota^*_{z_0} \omega$.

Then the bundle isomorphism φ *of* $A^0(S)$ $(S, A^0_{p_0}(S))$ *onto* $A^0(S')$ $(S', A^0_{p'_0}(S'))$ *induces a pseudo-conformal homeomorphism of* S *onto* S'.

Proof. From i) it is obvious that φ induces a bundle isomorphism of $A^0(S)(S, A^0_{p_0}(S))$ onto $A^0(S')(S', A^0_{p'_0}(S'))$. Since $A^0(S)(S, A^0_{p_0}(S))$ (resp. $A^0(S')(S', A^0_{p'_0}(S'))$) is a subbundle of P $(S, G'(r))$ (resp. $P'(S', G'(r))$), φ induces a bundle isomorphism $\tilde{\varphi}$ of $P(S, G'(r))$ onto $P'(S', G'(r))$ which satisfies the following commutative diagram

$$\begin{array}{ccc} A^0(S) & \xrightarrow{\varphi} & A^0(S') \\ \downarrow{\scriptstyle \iota_{z_0}} & & \downarrow{\scriptstyle \iota_{z'_0}} \\ P & \xrightarrow{\tilde{\varphi}} & P' \end{array} .$$

From ii) we get $\iota^*_{z_0} \tilde{\varphi}^* \omega' = \iota^*_{z_0} \omega$. Moreover, since $\tilde{\varphi}$ is a bundle isomorphism, we have $\tilde{\varphi}^* \omega' = \omega$. Therefore, from I. Theorem A, $\tilde{\varphi}$ induces a pseudo-conformal homeomorphism of S onto S'. Q.E.D.

IV. Graded subalgebras of $\mathfrak{g}(r)$.

First we will go into details about the structure of the graded Lie algebra $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$.

Identifying $\mathfrak{g}(r)$ with $\mathfrak{su}(\tilde{I}_r)$ we represent each element X of $\mathfrak{g}(r)$ as a matrix of the following form

$$X = \begin{pmatrix} -\bar{u} & -\sqrt{-1}\,{}^t\bar{w} I_r & w_n \\ \xi & v & w \\ \xi_n & \sqrt{-1}\,{}^t\bar{\xi} I_r & u \end{pmatrix},$$

where $\xi_n, w_n \in \boldsymbol{R}$, $u \in \boldsymbol{C}$ (and $\bar{u}$ is the complex conjugate of u), $\xi, w \in \boldsymbol{C}^{n-1}$, $v \in \mathfrak{u}(I_r)$ and $u - \bar{u} + \text{trace } v = 0$. For $\xi \in \boldsymbol{C}^{n-1}$, we define an element $\underset{\sim}{\xi} \in \mathfrak{g}_{-1}(r)$ and an element $\tilde{\xi} \in \mathfrak{g}_1(r)$ by

$$\underset{\sim}{\xi} = \begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & \sqrt{-1}\,{}^t\bar{\xi} I_r & 0 \end{pmatrix}, \qquad \tilde{\xi} = \begin{pmatrix} 0 & -\sqrt{-1}\,{}^t\bar{\xi} I_r & 0 \\ 0 & 0 & \xi \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover for $a \in \boldsymbol{R}$, we define an element $\underset{\approx}{a} \in \mathfrak{g}_{-2}(r)$ and an element $\tilde{\tilde{a}} \in \mathfrak{g}_2(r)$ by

$$\underset{\approx}{a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \qquad \tilde{\tilde{a}} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $\xi, w \in C^{n-1}$, we set $\langle \xi, w \rangle = {}^t\bar{\xi} I_r w$. $\langle\ ,\ \rangle$ is an indefinite hermitian inner product of C^{n-1} of type $(r, n-r-1)$. Then for $\tilde{\tilde{a}} \in \mathfrak{g}_2(r)$, $\tilde{w} \in \mathfrak{g}_1(r)$, $\underset{\sim}{\xi} \in \mathfrak{g}_{-1}(r)$ and $X_0 = \begin{pmatrix} -\bar{u} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \in \mathfrak{g}_0(r)$, we have

$$[\underset{\sim}{\xi}, \tilde{\tilde{a}}] = \widetilde{a\xi} \in \mathfrak{g}_1(r) \tag{4.1}$$

$$[\underset{\sim}{\xi}, \tilde{w}] = \begin{pmatrix} \sqrt{-1}\langle w, \xi \rangle & 0 & 0 \\ 0 & -\sqrt{-1}(\xi\, {}^t\bar{w} + w\, {}^t\bar{\xi}) I_r & 0 \\ 0 & 0 & \sqrt{-1}\langle \xi, w \rangle \end{pmatrix} \in \mathfrak{g}_0(r) \tag{4.2}$$

$$[X_0, \tilde{w}] = \widetilde{vw - uw} \in \mathfrak{g}_1(r) \tag{4.3}$$

$$[\tilde{w}_1, \tilde{w}_2] = \widetilde{\widetilde{\sqrt{-1}(\langle w_2, w_1 \rangle - \langle w_1, w_2 \rangle)}} \in \mathfrak{g}_2(r) . \tag{4.4}$$

From the above we easily obtain

LEMMA 4.1.

$$[\mathfrak{g}_{-1}(r), \mathfrak{g}_2(r)] = \mathfrak{g}_1(r) , \quad [\mathfrak{g}_1(r), \mathfrak{g}_1(r)] = \mathfrak{g}_2(r) , \quad [\mathfrak{g}_{-1}(r), \mathfrak{g}_1(r)] = \mathfrak{g}_0(r) .$$

Now we will consider a graded subalgebra $\mathfrak{k} = \sum_{k=-2}^{2} \mathfrak{k}_k$ of $\mathfrak{g}(r)$ which satisfies

$$\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r) \quad \text{and} \quad \mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r) .$$

First we have

LEMMA 4.2. *If* $\mathfrak{k}_2 \neq \{0\}$, *then* $\mathfrak{k} = \mathfrak{g}(r)$.

Proof. Since dim. $\mathfrak{g}_2(r) = 1$, we have $\mathfrak{k}_2 = \mathfrak{g}_2(r)$. Hence from $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$, $\mathfrak{k}_{-1}(r) = \mathfrak{g}_{-1}(r)$, and Lemma 4.1 we get $\mathfrak{k} = \mathfrak{g}(r)$. Q.E.D.

Therefore from now on we further assume $\mathfrak{k}_2 = \{0\}$. Let δ_r be a linear isomorphism of C^{n-1} onto $\mathfrak{g}_1(r)$ defined by $\delta_r(\xi) = \tilde{\xi}$, $\xi \in C^{n-1}$. Then we have

LEMMA 4.3. $\mathfrak{k}_1$ *is an abelian subalgebra of* $\mathfrak{g}(r)$; $\delta_r^{-1}(\mathfrak{k}_1)$ *is a complex isotropic vector subspace of the (indefinite) hermitian space* $(C^{n-1}, \langle\ \rangle)$. *In particular* dim. $\mathfrak{k}_1 = 2s \leqq 2r$.

Proof. Let $\tilde{w} \in \mathfrak{k}_1$ and $\underset{\sim}{\xi} \in \mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r)$. Then we have from (4.2) and (4.3)

$$\mathrm{ad}\,(\tilde{w})^2(\underset{\sim}{\xi}) = [\tilde{w}, [\tilde{w}, \underset{\sim}{\xi}]] = \widetilde{-\sqrt{-1}\langle w, w\rangle\xi - 2\sqrt{-1}\langle \xi, w\rangle w} \in \mathfrak{k}_1 .$$

Moreover from (4.4) we have

$$\mathrm{ad}\,(\tilde{w})^3(\underset{\sim}{\xi}) = \widetilde{\widetilde{-3(\langle \xi, w\rangle + \langle w, \xi\rangle)\langle w, w\rangle}} \in \mathfrak{k}_2 .$$

Since $\langle\ ,\ \rangle$ is a non-degenerate hermitian form, we can find $\xi_1 \in \boldsymbol{C}^{n-1}$ such that $\langle \xi_1, w\rangle = -\frac{1}{2}$. Hence from $\mathfrak{k}_2 = \{0\}$, we have

$$\mathrm{ad}\,(\tilde{w})^3(\underset{\sim}{\xi_1}) = \widetilde{\widetilde{3\langle w, w\rangle}} = 0 \quad (\text{i.e. } \langle w, w\rangle = 0) \quad \text{for any } \tilde{w} \in \mathfrak{k}_1 .$$

Moreover we have $\mathrm{ad}\,(\tilde{w})^2(\underset{\sim}{\xi_1}) = \widetilde{\sqrt{-1}w} \in \mathfrak{k}_1$. Therefore $\delta_r^{-1}(\mathfrak{k}_1)$ is a complex vector subspace of $\boldsymbol{C}^{n-1}$. On the other hand let $w_1, w_2 \in \delta_r^{-1}(\mathfrak{k}_1)$. Then from

$$\begin{cases} \tilde{w}_1 + \tilde{w}_2 = \widetilde{w_1 + w_2} \in \mathfrak{k}_1 , \\ [\tilde{w}_1, \tilde{w}_2] = \widetilde{\widetilde{\sqrt{-1}(\langle w_2, w_1\rangle - \langle w_1, w_2\rangle)}} \in \mathfrak{k}_2 , \end{cases}$$

we get $[\tilde{w}_1, \tilde{w}_2] = 0$ (i.e. $\langle w_1, w_2\rangle = \langle w_2, w_1\rangle$) and $\langle w_1 + w_2, w_1 + w_2\rangle = 0$. Hence we get $\langle w_1, w_2\rangle = 0$. Q.E.D.

Let $\{e_i\}_{1 \leq i \leq n-1}$ be the natural base of $\boldsymbol{C}^{n-1}$. Setting $w_i = e_i + e_{n-i}$ ($i = 1, 2, \cdots, s$), we consider a complex vector subspace of $\boldsymbol{C}^{n-1}$ spanned by the s vectors $w_1, \cdots, w_s$. This subspace is an s-dimensional complex isotropic subspace of the (indefinite) hermitian space $(\boldsymbol{C}^{n-1}, \langle\ ,\ \rangle)$. We denote by $\mathfrak{c}_s(r)$ its image under δ_r. Then $\mathfrak{c}_s(r)$ is an abelian subalgebra of $\mathfrak{g}(r)$ of dimension $2s$ contained in $\mathfrak{g}_1(r)$.

Now recall the following which is a direct consequence of Witt's theorem (cf. [1, p. 121]).

LEMMA B. *Let V_1 and V_2 be s-dimensional complex isotropic vector subspaces of the indefinite hermitian space $(\boldsymbol{C}^{n-1}, \langle\ ,\ \rangle)$. Then there exists an element σ of $U(I_r)$ which sends V_1 onto V_2.*

Then we have

LEMMA 4.4. *Let s be the complex dimension of $\delta_r^{-1}(\mathfrak{k}_1)$. Then there exists $\tau_1 \in G'(r)$ such that* Ad (τ_1) *preserves the grading of* $\mathfrak{g}(r)$ *and satisfies* Ad $(\tau_1)\mathfrak{k}_1 = \mathfrak{c}_s(r)$.

Proof. $\delta_r^{-1}(\mathfrak{k}_1)$ and $\delta_r^{-1}(\mathfrak{c}_s(r))$ are s-dimensional complex isotropic subspaces of $(C^{n-1}, \langle\ ,\ \rangle)$. Hence from Lemma B we can find $\sigma_1 \in U(I_r)$ such that $\sigma_1(\delta_r^{-1}(\mathfrak{k}_1)) = \delta_r^{-1}(\mathfrak{c}_s(r))$. Set $\sigma_1' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then σ_1' belongs to $U(\tilde{I}_r) \subset \hat{G}(r)$. Hence $\tau_1 = \chi(\sigma_1')$ is an element of $G'(r)$. In fact τ_1 belongs to the analytic subgroup of $G'(r)$ corresponding to the subalgebra $\mathfrak{g}_0(r)$ of $\mathfrak{g}'(r)$. In particular $\mathrm{Ad}(\tau_1)$ preserves the grading of $\mathfrak{g}(r)$. On the other hand

$$\mathrm{Ad}(\tau_1)\tilde{w} = \widetilde{\sigma_1 w} \qquad \text{for } \tilde{w} \in \mathfrak{g}_1(r)\ ,$$

so we can conclude $\mathrm{Ad}(\tau_1)\mathfrak{k}_1 = \mathfrak{c}_s(r)$. Q.E.D.

Next we will consider $\mathfrak{k}_0$. We define a subalgebra $\mathfrak{b}_s(r)$ of $\mathfrak{g}_0(r)$ by

$$\mathfrak{b}_s(r) = \{X \in \mathfrak{g}_0(r) \mid \mathrm{ad}(X)(\mathfrak{c}_s(r)) \subset \mathfrak{c}_s(r)\}\ .$$

Then we have

LEMMA 4.5. *Notations being the same as in Lemma* 4.4, *we have*
(i) $\mathrm{Ad}(\tau_1)\mathfrak{k}_0 \subset \mathfrak{b}_s(r)$ and $[\mathfrak{g}_{-1}(r), \mathfrak{c}_s(r)] \subset \mathfrak{b}_s(r)$
(ii) $\dim.\ \mathfrak{b}_s(r) = \dim.\ \mathfrak{g}_0(r) - s(2(n-1) - 3s)$.

Proof. (i) is clear from $\mathrm{Ad}(\tau_1)\mathfrak{k}_1 = \mathfrak{c}_s(r)$, $[\mathfrak{k}_0, \mathfrak{k}_1] \subset \mathfrak{k}_1$, (4.2) and (4.3). In order to prove (ii) we first note that $\mathfrak{g}_0(r)$ can be decomposed into the direct sum of $\langle\{E_0\}\rangle_R$ and $\mathfrak{u}(I_r)$, where $\langle\{E_0\}\rangle_R$ is the line spanned by

$$E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and $\mathfrak{u}(I_r)$ is identified with the subalgebra of $\mathfrak{g}_0(r)$ which consists of matrices of the form

$$\begin{pmatrix} -\frac{1}{2}\,\mathrm{trace}\, v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}\,\mathrm{trace}\, v \end{pmatrix} \quad \text{with} \quad {}^t\bar{v}I_r + I_r v = 0\ .$$

For $X = \begin{pmatrix} -\bar{u} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \in \mathfrak{g}_0(r)$, we have from (4.3)

$$\mathrm{ad}(X)(\tilde{w}) = \widetilde{vw - uw} \qquad \tilde{w} \in \mathfrak{c}_s(r)\ .$$

Since $\delta_r^{-1}(\mathfrak{c}_s(r))$ is a complex vector subspace of C^{n-1}, we have $\widetilde{uw} \in \mathfrak{c}_s(r)$. Hence X belongs to $\mathfrak{b}_s(r)$ if and only if $v(\delta_r^{-1}(\mathfrak{c}_s(r)) \subset \delta_r^{-1}(\mathfrak{c}_s(r))$. Obviously E_0 belongs to $\mathfrak{b}_s(r)$. Therefore in order to calculate the dimension of $\mathfrak{b}_s(r)$, we have only to calculate the dimension of a subalgebra of $\mathfrak{u}(I_r)$ which consists of all elements leaving the subspace $\delta_r^{-1}(\mathfrak{c}_s(r))$ invariant. A direct computation shows the above equality (ii). Q.E.D.

We set $\mathfrak{g}^*(r, s) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r) \oplus \mathfrak{b}_s(r) \oplus \mathfrak{c}_s(r)$. In the case $s = 0$, we write $\mathfrak{g}^*(r)$ instead of $\mathfrak{g}^*(r, 0)$, that is, $\mathfrak{g}^*(r) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r) \oplus \mathfrak{g}_0(r)$. Then from the above lemmas we have

PROPOSITION 4.6. *Let* $\mathfrak{k}$ *be a proper graded subalgebra of* $\mathfrak{g}(r)$ *satisfying* $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$ *and* $\mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r)$. *Then there exists* $\tau \in G'(r)$ *such that* $\mathrm{Ad}(\tau)$ *preserves the grading of* $\mathfrak{g}(r)$ *and* $\mathrm{Ad}(\tau)\mathfrak{k} \subset \mathfrak{g}^*(r, s)$, *where* $2s = \dim. \mathfrak{k}_1 (\leqq 2r)$.

From this we obtain $\dim. \mathfrak{k} \leqq \dim. \mathfrak{g}^*(r, s) = n^2 + 1 - s(2(n-2) - 3s)$. Since s is an integer satisfying $0 \leqq s \leqq r$, from the above considerations we obtain

PROPOSITION 4.7. *Let* $\mathfrak{k}$ *be a proper graded subalgebra of* $\mathfrak{g}(r)$ *satisfying* $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$ *and* $\mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r)$. *Then we have*

(1) *The case* $n = 3$ *and* $r = 1$

We have $\dim. \mathfrak{k} \leqq n^2 + 2 = 11$. *The equality holds if and only if there exists* $\tau \in G'(1)$ *such that* $\mathrm{Ad}(\tau)$ *preserves the grading of* $\mathfrak{g}(1)$ *and*

$$\mathrm{Ad}(\tau)\mathfrak{k} = \mathfrak{g}^*(1, 1) .$$

(2) *The case* $n = 5$ *and* $r = 2$

We have $\dim. \mathfrak{k} \leqq n^2 + 1 = 26$. *The equality holds if and only if there exists* $\tau \in G'(2)$ *such that* $\mathrm{Ad}(\tau)$ *preserves the grading of* $\mathfrak{g}(2)$ *and*

$$\mathrm{Ad}(\tau)\mathfrak{k} = \mathfrak{g}^*(2, 2) \quad \textit{or} \quad \mathfrak{g}^*(2) .$$

(3) *Otherwise*

We have $\dim. \mathfrak{k} \leqq n^2 + 1$. *The equality holds if and only if there exists* $\tau \in G'(r)$ *such that* $\mathrm{Ad}(\tau)$ *preserves the grading of* $\mathfrak{g}(r)$ *and*

$$\mathrm{Ad}(\tau)\mathfrak{k} = \mathfrak{g}^*(r) .$$

Remark 4.8. Let $D(r)$ be an $(n-2)$-dimensional complex vector subspace of C^{n-1} spanned by the $n-2$ vectors $w_1, e_2, \cdots$, and e_{n-2}, where $w_1 = e_1 + e_{n-1}$. We set $\mathfrak{d}^1(r) = \{\bar{\xi} \in \mathfrak{g}_1(r) \mid \xi \in D(r)\}$, $\mathfrak{d}^{-1}(r) = \{\underset{\sim}{\xi} \in \mathfrak{g}_{-1}(r) \mid \xi \in D(r)\}$,

$\mathfrak{e}(r) = \{X \in \mathfrak{g}_0(r) \mid \mathrm{ad}\,(X)(\mathfrak{d}^i(r)) \subset \mathfrak{d}^i(r)\ i = 1, 2\}$, $\mathfrak{c}_s^*(r) = \{\xi \in \mathfrak{g}_{-1}(r) \mid \xi \in \delta_r^{-1}(\mathfrak{c}_s(r))\}$ and $\mathfrak{b}_s^*(r) = \{X \in \mathfrak{g}_0(r) \mid \mathrm{ad}\,(X)(\mathfrak{c}_s^*(r)) \subset \mathfrak{c}_s^*(r)\}$ $(=\mathfrak{b}_s(r))$. Moreover we set

$$\begin{cases} \mathfrak{g}^0(r) = \mathfrak{g}_{-2}(r) + \mathfrak{d}^{-1}(r) + \mathfrak{e}(r) + \mathfrak{d}^1(r) + \mathfrak{g}_2(r)\,, \\ \mathfrak{g}^{**}(r, s) = \mathfrak{c}_s^*(r) + \mathfrak{b}_s^*(r) + \mathfrak{g}_1(r) + \mathfrak{g}_2(r)\,. \end{cases}$$

Then without the homogeneity assumption we have

PROPOSITION 4.9. *Let $\mathfrak{k}$ be a proper graded subalgebra of $\mathfrak{g}(r)$. Then we have*

(1) *The case $n = 3$ and $r = 1$;* $\dim.\, \mathfrak{k} \leqq n^2 + 2 = 11$. *The equality holds if and only if there exists $\tau \in G'(1)$ such that $\mathrm{Ad}\,(\tau)$ preserves the grading of $\mathfrak{g}(1)$ and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(1, 1) \quad or \quad \mathfrak{g}^{**}(1, 1)\,.$$

(2) *The case $n = 5$ and $r = 2$;* $\dim.\, \mathfrak{k} \leqq n^2 + 1 = 26$. *The equality holds if and only if there exists $\tau \in G'(2)$ such that $\mathrm{Ad}\,(\tau)$ preserves the grading of $\mathfrak{g}(2)$ and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(2, 2),\ \mathfrak{g}^{**}(2, 2),\ \mathfrak{g}^*(2),\ \mathfrak{g}'(2)\,, \quad or \quad \mathfrak{g}^0(2)\,.$$

(3) *The case $n \geqq 2$ and $r = 0$;* $\dim.\, \mathfrak{k} \leqq n^2 + 1$, *the equality holds if and only if there exists $\tau \in G'(0)$ such that $\mathrm{Ad}\,(\tau)$ preserves the grading of $\mathfrak{g}(0)$ and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(0) \quad or \quad \mathfrak{g}'(0)\,.$$

(4) *Otherwise;* $\dim.\, \mathfrak{k} \leqq n^2 + 1$. *The equality holds if and only if there exists $\tau \in G'(r)$ such that $\mathrm{Ad}\,(\tau)$ preserves the grading of $\mathfrak{g}(r)$ and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(r),\ \mathfrak{g}'(r) \quad or \quad \mathfrak{g}^0(r)\,.$$

V. Determination of $(\mathfrak{a}(S), \mathfrak{a}_{p_0}(S))$.

Throughout this section we assume that S is a connected non-degenerate (index r) homogeneous hypersurface. Let $(P, \omega, \bar{l})$ be the normal pseudo-conformal connection over S. Moreover we naturally identify the Lie algebra $\mathfrak{a}(S)$ of $A(S)$ with the Lie algebra of all infinitesimal pseudo-conformal transformations of S which generate (global) 1-parameter groups of pseudo-conformal transformations.

Now let us fix a point p_0 of S. As in the section II, we introduce the filtration of $\mathfrak{a}(S)$ at p_0 through the connection form ω. Notations being as in the section II, we first consider the associated graded Lie algebra $\tilde{\mathfrak{h}}$ of $\mathfrak{h}$.

LEMMA 5.1. *Let* $z_0 \in \pi^{-1}(p_0)$. *Suppose that* $A(S)$ *has the largest dimension* n^2+2n, *then* $\nu_{z_0}; \tilde{\mathfrak{h}} \to \mathfrak{g}(r)$ *is a Lie algebra isomorphism of* $\tilde{\mathfrak{h}}$ *onto* $\mathfrak{g}(r)$.

This lemma is clear from Lemma 2.5 and dim. $\mathfrak{g}(r) =$ dim. $\tilde{\mathfrak{h}}$ $(=n^2 + 2n)$.

Let z be an arbitrary point of $\pi^{-1}(p_0)$. Since $A(S)$ acts transitively on $S, \tilde{\mathfrak{h}}_z = \nu_z(\tilde{\mathfrak{h}})$ satisfies $(\tilde{\mathfrak{h}}_z)_{-2} = \mathfrak{g}_{-2}(r)$ and $(\tilde{\mathfrak{h}}_z)_{-1} = \mathfrak{g}_{-1}(r)$. Therefore from Proposition 4.7 and Remark 2.6 we get

LEMMA 5.2. *Suppose that* $A(S)$ *has the second largest dimension, then there exists* $z_1 \in \pi^{-1}(p_0)$ *such that*

(1) $\tilde{\mathfrak{h}}_{z_1} = \mathfrak{g}^*(1, 1)$ *if* $n = 3$ *and* $r = 1$,

(2) $\tilde{\mathfrak{h}}_{z_1} = \mathfrak{g}^*(2, 2)$ *or* $\mathfrak{g}^*(2)$ *if* $n = 5$ *and* $r = 2$,

(3) $\tilde{\mathfrak{h}}_{z_1} = \mathfrak{g}^*(r)$ *otherwise* .

As for $\mathfrak{h}_z = \omega_z(\mathfrak{h})$, we have

LEMMA 5.3. *Let* $z_0 \in \pi^{-1}(p_0)$. *Suppose that* $A(S)$ *has the largest dimension* $n^2 + 2n$, *then* $-\omega_{z_0}; \mathfrak{h} \to \mathfrak{g}(r)$ *is a linear isomorphism of* $\mathfrak{h}$ *onto* $\mathfrak{g}(r)$.

This lemma is also clear from dim. $\mathfrak{g}(r) =$ dim. $\mathfrak{h}$.

LEMMA 5.4. *Suppose that* $A(S)$ *has the second largest dimension, then there exists* $z_0 \in \pi^{-1}(p_0)$ *such that*

(1) $\mathfrak{h}_{z_0} = \mathfrak{g}^*(1, 1)$ *if* $n = 3$ *and* $r = 1$,

(2) $\mathfrak{h}_{z_0} = \mathfrak{g}^*(2, 2)$ *or* $\mathfrak{g}^*(2)$ *if* $n = 5$ *and* $r = 2$,

(3) $\mathfrak{h}_{z_0} = \mathfrak{g}^*(r)$ *otherwise,*

as vector subspaces of $\mathfrak{g}(r)$.

In order to prove Lemma 5.4, it suffices to show the following lemma. (Note that $\mathfrak{g}^*(r, s)(0 \leqq s \leqq r)$ contains E_0).

LEMMA 5.5. *If* $\tilde{\mathfrak{h}}_{z_1}$ *contains* $E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *for some point* z_1 *of* $\pi^{-1}(p_0)$, *then there exists a point* z_0 *of* $\pi^{-1}(p_0)$ *such that* $\mathfrak{h}_{z_0}$ *coincides with* $\tilde{\mathfrak{h}}_{z_1}$ *as a vector subspace of* $\mathfrak{g}(r)$.

Proof. Since the filtration of $\mathfrak{h}_z$ is given by $(\mathfrak{h}_z)_k = \mathfrak{h}_z \cap \mathscr{L}_k(r)$ ($\mathscr{L}_k(r) = \sum_{l=k}^{2} \mathfrak{g}_l(r)$), we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_k & \xrightarrow{-\omega_z} & (\mathfrak{h}_z)_k \subset \mathfrak{g}(r) \\ \mu_k \big\downarrow & & \big\downarrow p_k \\ \tilde{\mathfrak{h}}_k & \xrightarrow{\nu_z} & (\tilde{\mathfrak{h}}_z)_k \subset \mathfrak{g}_k(r) \end{array},$$

where p_k is the projection of $\mathfrak{g}(r)$ onto $\mathfrak{g}_k(r)$ corresponding to the decomposition $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$. From the assumption $(\tilde{\mathfrak{h}}_{z_1})_0$ contains E_0. Hence there exists $E \in (\mathfrak{h}_{z_1})_0$ such that $p_0(E) = E_0$. Since E belongs to $\mathscr{L}_0(r) = \sum_{k=0}^{2} \mathfrak{g}_k(r)$, there exist $\tilde{w}_0 \in \mathfrak{g}_1(r)$ and $\tilde{\tilde{c}}_0 \in \mathfrak{g}_2(r)$ such that $E = E_0 + \tilde{w}_0 + {}_0\tilde{\tilde{c}}$. Now we set $A_0 = \tilde{w}_0 + \frac{1}{2}\tilde{\tilde{c}}_0$. Then A_0 belongs to $\mathscr{L}_1(r)$ and satisfies $\mathrm{Ad}\,(\exp A_0)(E) = E_0$. Moreover $a_0 = \exp A_0$ is an element of $G'(r)$. Set $z_0 = z_1 a_0^{-1}$, then from Remark 2.6 we have $\mathfrak{h}_{z_0} = \mathrm{Ad}\,(a_0)\mathfrak{h}_{z_1}$. In particular $\mathfrak{h}_{z_0}$ contains E_0.

First we will see that $\tilde{\mathfrak{h}}_{z_0}$ coincides with $\tilde{\mathfrak{h}}_{z_1}$. From the above diagram we have $(\tilde{\mathfrak{h}}_{z_i})_k = p_k(\mathfrak{h}_{z_i} \cap \mathscr{L}_k(r))$ $(i = 0, 1)$. For $X \in \mathfrak{h}_{z_1} \cap \mathscr{L}_k(r)$, $\mathrm{Ad}\,(a_0)(X) = \exp \mathrm{ad}\,(A_0)(X)$ lies in $\mathfrak{h}_{z_0} \cap \mathscr{L}_k(r)$. This is obvious from $\mathfrak{h}_{z_0} = \mathrm{Ad}\,(a_0)\mathfrak{h}_{z_1}$ and Lemma 2.1. Moreover, since $A_0 \in \mathscr{L}_1(r)$, we have $\mathrm{ad}\,(A_0)(\mathscr{L}_k(r)) \subset \mathscr{L}_{k+1}(r)$. Hence we get $p_k(\mathrm{Ad}\,(a_0)(X)) = p_k(X)$. Therefore $(\tilde{\mathfrak{h}}_{z_0})_k = (\tilde{\mathfrak{h}}_{z_1})_k$.

Next we will see that $\mathfrak{h}_{z_0}$ coincides with $\tilde{\mathfrak{h}}_{z_0}$ as a vector subspace of $\mathfrak{g}(r)$. First one should note that Lemma 2.3 implies $[(\mathfrak{h}_{z_0})_0, \mathfrak{h}_{z_0}] \subset \mathfrak{h}_{z_0}$ and that $\mathfrak{h}_{z_0}$ contains E_0. Let X be an arbitrary element of $\mathfrak{h}_{z_0}$, and X_k $(k = -2, -1, \cdots, 2)$ be the $\mathfrak{g}_k(r)$-component of X. From $[(\mathfrak{h}_{z_0})_0, \mathfrak{h}_{z_0}] \subset \mathfrak{h}_{z_0}$ and $(\mathfrak{h}_{z_0})_0 \ni E_0$, we obtain

$$\begin{cases} -X_{-2} + X_2 = \frac{1}{6}(\mathrm{ad}\,(E_0)^3(X) - \mathrm{ad}\,(E_0)(X)) \in \mathfrak{h}_{z_0} \\ X_{-2} + X_2 = \frac{1}{12}(\mathrm{ad}\,(E_0)^4(X) - \mathrm{ad}\,(E_0)^2(X)) \in \mathfrak{h}_{z_0} \\ -X_{-1} + X_1 = \frac{1}{3}(4\mathrm{ad}\,(E_0)(X) - \mathrm{ad}\,(E_0)^3(X)) \in \mathfrak{h}_{z_0} \\ X_{-1} + X_1 = \frac{1}{3}(4\mathrm{ad}\,(E_0)^2(X) - \mathrm{ad}\,(E_0)^4(X)) \in \mathfrak{h}_{z_0} . \end{cases}$$

Hence we get $X_{-2}, X_{-1}, X_1, X_2 \in \mathfrak{h}_{z_0}$. Therefore X_k $(k = -2, -1, 0, 1, 2)$ lies in $\mathfrak{h}_{z_0}$, that is, $\mathfrak{h}_{z_0}$ decomposes as follows

$$\mathfrak{h}_{z_0} = \sum_{k=-2}^{2} \mathfrak{h}_{z_0} \cap \mathfrak{g}_k(r) .$$

In other words, $\mathfrak{h}_{z_0}$ is a graded subspace of $\mathfrak{g}(r)$. Then from the construction of the assoicated graded Lie algebra, we have $(\tilde{\mathfrak{h}}_{z_0})_k = \mathfrak{h}_{z_0} \cap \mathfrak{g}_k(r)$.

Therefore we obtain $\mathfrak{h}_{z_0} = \tilde{\mathfrak{h}}_{z_0}$. Q.E.D.

Next we will see that the curvature form Ω of the normal pseudo-conformal connection of S vanishes identically if $A(S)$ has either the largest dimension $n^2 + 2n$ or the second largest dimension. First we will show the following proposition.

PROPOSITION 5.6. *If* $\mathfrak{h}_{z_0}$ *contains* $E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *for some point* z_0 *of* $\pi^{-1}(p_0)$, *then* $\Omega_z = 0$ *for any* $z \in \pi^{-1}(p_0)$.

Proof. The proof is quite analogous to that of IV. Theorem 3.2 of [4]. Recall that $\mathfrak{h} = \mathfrak{a}(P) = \{X \in \mathfrak{X}(P) \mid L_X\omega = 0,\ R_{a*}X = X,\ a \in G'(r)$, and X is complete$\}$ (see II). Since $\mathfrak{h}_{z_0} = \omega_{z_0}(\mathfrak{a}(P))$, there exists $X_0 \in \mathfrak{a}(P)$ such that $(X_0)_{z_0} = \omega_{z_0}^{-1}(E_0) = (E_0)^*_{z_0}$. First we know

LEMMA C (cf. [5; p. 233]). *For the curvature form* $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$, *we have*

(1) $A^*(\Omega(\xi^*, \eta^*)) = -[A, \Omega(\xi^*, \eta^*)] + \Omega([A, \xi]^*, \eta^*) + \Omega(\xi^*, [A, \eta]^*)$ *for* $\xi, \eta \in \mathfrak{g}(r)$, $A \in \mathfrak{g}'(r)$,

(2) $L_X\Omega = 0$ *and* $X(\Omega(\xi^*, \eta^*)) = 0$ *for* $X \in \mathfrak{a}(P)$, $\xi, \eta \in \mathfrak{g}(r)$.

Applying the above lemma to $(X_0)_{z_0} = (E_0)^*_{z_0}$, we obtain

$$[E_0, \Omega_{z_0}(\xi^*, \eta^*)] = \Omega_{z_0}([E_0, \xi]^*, \eta^*) + \Omega_{z_0}(\xi^*, [E_0, \eta]^*) . \tag{5.1}$$

Since $\Omega(U^*, A^*) = 0$ for $U \in \mathfrak{g}(r)$ and $A \in \mathfrak{g}'(r)$ (cf. II. Lemma 2.2), we have only to show $\Omega(\xi^*, \eta^*) = 0$ for $\xi, \eta \in \mathfrak{m}(r) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r)$. For the sake of simplicity we show the above equality in the case $\xi, \eta \in \mathfrak{g}_{-1}(r)$. Let Ω_k $(k = -2, -1, \cdots, 2)$ be the $\mathfrak{g}_k(r)$-component of Ω. From I. Theorem A, we have $\Omega_{-1} = 0$ and $\Omega_{-2} = 0$. Hence from (5.1) we get

$$(\Omega_1)_{z_0}(\xi^*, \eta^*) + 2(\Omega_2)_{z_0}(\xi^*, \eta^*) = -2(\Omega_0 + \Omega_1 + \Omega_2)_{z_0}(\xi^*, \eta^*) , \qquad \xi, \eta \in \mathfrak{g}_{-1}(r) .$$

From this it follows $(\Omega_k)_{z_0}(\xi^*, \eta^*) = 0$ $(k = 0, 1, 2)$. Therefore we obtain $\Omega_{z_0} = 0$. For any $z \in \pi^{-1}(p_0)$, there exists $a \in G'(r)$ such that $z_0 = za$. Then from $R_a^*\omega = \mathrm{Ad}\,(a^{-1})\omega$, we have $\Omega_z = \mathrm{Ad}\,(a)R_a^*\Omega_{z_0} = 0$. Q.E.D.

From Lemma 5.3, Lemma 5.4 and Proposition 5.6 we get

PROPOSITION 5.7. *Let* S *be a non-degenerate homogeneous hypersurface. If* $A(S)$ *has either the largest dimension* $n^2 + 2n$ *or the second largest dimension, then* S *is flat, that is, the curvature form of the* *normal pseudo-conformal connection vanishes identically.*

Hence from Proposition 3.4, $\iota_z^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ into $\mathfrak{g}(r)$ for any $z \in P$.

Summarizing the results of this section we obtain.

THEOREM 5.8. *Let M be a complex manifold of dimension n. Let S be a connected non-degenerate (index r) homogeneous hypersurface of M. Let p_0 be an arbitrary point of S.*

(1) *If* $\dim. A(S) = n^2 + 2n$, *then $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto $\mathfrak{g}(r)$ for any $z_0 \in \pi^{-1}(p_0)$.*

(2) *If* $\dim. A(S) < n^2 + 2n$, *we have the following three cases.*

(i) *The case $n = 3$ and $r = 1$; We have* $\dim. A(S) \leqq n^2 + 2 = 11$. *The equality holds if and only if there exists $z_0 \in \pi^{-1}(p_0)$ such that $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto* $\mathfrak{g}^*(1, 1)$.

(ii) *The case $n = 5$ and $r = 2$; We have* $\dim. A(S) \leqq n^2 + 1 = 26$. *The equality holds if and only if there exists $z_0 \in \pi^{-1}(p_0)$, such that $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto* $\mathfrak{g}^*(2, 2)$ *or* $\mathfrak{g}^*(2)$.

(iii) *Otherwise; We have* $\dim. A(S) \leqq n^2 + 1$. *The equality holds if and only if there exists $z_0 \in \pi^{-1}(p_0)$ such that $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto* $\mathfrak{g}^*(r)$.

VI. Model spaces.

We consider the analytic subgroups (i.e. connected Lie subgroups) of $G(r)$ corresponding to $\mathfrak{g}(r)$ and $\mathfrak{g}^*(r, s)$ $(0 \leqq s \leqq r)$. The identity component $G^0(r)$ of $G(r)$ corresponds to $\mathfrak{g}(r)$. We denote by $G^*(r, s)$ the analytic subgroup of $G(r)$ corresponding to $\mathfrak{g}^*(r, s)$. In particular we set $G^*(r) = G^*(r, 0)$.

First we will characterize $G^*(r, s)$ geometrically. Let χ be the natural homomorphism of $U(\tilde{I}_r)$ onto $G^0(r)\,(= U(\tilde{I}_r)/U(1))$. We set $\hat{G}^*(r, s) = \chi^{-1}(G^*(r, s))$. Take the natural base $\{e_i\}_{0 \leqq i \leqq n}$ of C^{n+1} and set $w_i = e_i + e_{n-i}$ $(i = 1, 2, \cdots, s)$. We denote by $C_s(r)$ the $(s + 1)$-dimensional complex vector subspace of C^{n+1} spanned by the $(s + 1)$ vectors $w_1, w_2, \cdots, w_s$ and e_n. Then $C_s(r)$ is an $(s + 1)$-dimensional complex isotropic subspace of the indefinite hermitian space $(C^{n+1}, \tilde{I}_r)$.

LEMMA 6.1.

$$\hat{G}^*(r, s) = \{\sigma \in U(\tilde{I}_r) \,|\, \sigma(C_s(r)) = C_s(r)\} .$$

Proof. Since we are identifying $\mathfrak{g}(r)$ with $\mathfrak{su}(\tilde{I}_r)$, χ_* is identified with the projection of $\mathfrak{u}(\tilde{I}_r)$ onto $\mathfrak{su}(\tilde{I}_r)$ corresponding to the decomposition $\mathfrak{u}(\tilde{I}_r)=\mathfrak{u}(1)\oplus\mathfrak{su}(\tilde{I}_r)$, where $\mathfrak{u}(1)$ is the center of $\mathfrak{u}(\tilde{I}_r)$. For $X\in\mathfrak{u}(\tilde{I}_r)$;

$$X=\begin{pmatrix}-\bar{u} & -\sqrt{-1}\,{}^t\bar{w}I_r & w_n\\ \xi & v & w\\ \xi_n & \sqrt{-1}\,{}^t\bar{\xi}I_r & u\end{pmatrix}\qquad \xi_n, w_n\in\boldsymbol{R},\ \xi, w\in\boldsymbol{C}^{n-1},\ v\in\mathfrak{u}(I_r),\ \text{we note}$$

$$\mathfrak{g}^*(r,s)\ni\chi_*(X)\ \text{if and only if}\begin{cases}w_n=0\\ w\in\delta_r^{-1}(\mathfrak{c}_s(r))\ ,\\ v(\delta_r^{-1}(\mathfrak{c}_s(r))\subset\delta_r^{-1}(\mathfrak{c}_s(r))\ .\end{cases}$$

On the other hand for $(0,\eta,z_n)\in C_s(r)$ we have

$$X\begin{pmatrix}0\\ \eta\\ z_n\end{pmatrix}=\begin{pmatrix}-\sqrt{-1}\langle w,\eta\rangle+w_nz_n\\ v\eta+z_nw\\ \sqrt{-1}\langle\xi,\eta\rangle+uz_n\end{pmatrix}.$$

Hence we have

$$X(C_s(r))\subset C_s(r)\ \text{if and only if}\begin{cases}-\sqrt{-1}\langle w,\eta\rangle+w_nz_n=0\\ v\eta+z_nw\in\delta_r^{-1}(\mathfrak{c}_s(r))\end{cases}$$
$$\text{for } z_n\in\boldsymbol{C},\ \eta\in\delta_r^{-1}(\mathfrak{c}_s(r))\ .$$

From the above $\mathfrak{g}^*(r,s)\ni\chi^*(X)$ if and only if $X(C_s(r))\subset C_s(r)$. We set $K=\{\sigma\in U(\tilde{I}_r)\,|\,\sigma(C_s(r))=C_s(r)\}$. From $G^*(r,s)=\hat{G}^*(r,s)/U(1)$, we see that $\hat{G}^*(r,s)$ is connected. In fact, $\hat{G}^*(r,s)$ is the analytic subgroup of $U(\tilde{I}_r)$ corresponding to $\chi_*^{-1}(\mathfrak{g}^*(r,s))$. Therefore $\hat{G}^*(r,s)$ coincides with the identity component of K.

In order to prove $\hat{G}^*(r,s)=K$, we have only to show that K is connected. For this we take a base $\{f_i\}_{0\leqq i\leqq n}$ of $\boldsymbol{C}^{n+1}$ such that $\{f_i\}_{0\leqq i\leqq s}$ forms a base of $C_s(r)$ and with respect to this base the hermitian form $\tilde{I}_r$ is represented as a matrix of the following form

$$\tilde{I}_r=\begin{pmatrix}0 & E_{s+1} & 0\\ E_{s+1} & 0 & 0\\ 0 & 0 & I_s^*\end{pmatrix},\qquad I_s^*=\begin{pmatrix}-E_{r-s} & 0\\ 0 & E_{n-(r+s+1)}\end{pmatrix}.$$

(The existence of such a base is guaranteed by the Witt's theorem). Then each $\sigma\in K$ is represented as a matrix of the form

$$\begin{pmatrix} A & -\frac{1}{2}A(C + {}^t\bar{K}I_s^*K) & -A\,{}^t\bar{K}I_s^*B \\ 0 & {}^t\bar{A}^{-1} & 0 \\ 0 & K & B \end{pmatrix};$$

$$A \in GL(s+1, C),\ B \in U(I_s^*),\ {}^t\bar{C} + C = 0 .$$

From this we see that K is homeomorphic with $GL(s+1, C) \times U(I_s^*) \times \mathfrak{u}(s+1) \times M(n-2s-1, s+1; C)$, where $M(n-2s-1, s+1; C)$ is the set of all complex $(n-2s-1) \times (s+1)$ matrices. In particular K is connected. Q.E.D.

Now we consider the orbit of $G^0(r)$ or $G^*(r, s)$ passing through o of Q_r as the model space corresponding to $\mathfrak{g}(r)$ or $\mathfrak{g}^*(r, s)$.

Since $G^0(r)$ acts transitively on Q_r, the model space corresponding to $\mathfrak{g}(r)$ is Q_r itself. We denote by $Q_r^*(s)$ the model space corresponding to $\mathfrak{g}^*(r, s)$. In particular we set $Q_r^* = Q_r^*(0)$.

LEMMA 6.2.

$$Q_r^* = \{(z_0, z_1, \cdots, z_n) \in Q_r \,|\, z_0 \neq 0\} ,$$

and

$$Q_r^*(s) = \{(z_0, z_1, \cdots, z_n) \in Q_r \,|\, |z_0| + |z_1 - z_{n-1}| + \cdots + |z_s - z_{n-s}| \neq 0\} \quad (s \geqq 1) .$$

Proof. We consider the orbital decomposition of Q_r by $G^*(r, s)$. We denote by $(\ ,\ )$ the indefinite hermitian inner product of C^{n+1} defined by $\tilde{I}_r$. And set $(C_s(r))^{\perp} = \{\zeta \in C^{n+1} \,|\, (\zeta, \eta) = 0 \text{ for } \eta \in C_s(r)\}$. Then from Lemma 6.1 we see that each $\sigma \in \hat{G}^*(r, s)$ leaves $(C_s(r))^{\perp}$ invariant as well. On the other hand we have $Q_r = \{\zeta = (\zeta_0, \cdots, \zeta_n) \,|\, (\zeta, \zeta) = 0\}$ in homogeneous coordinate. Then using the arguments in the proof of the Witt's theorem ([1, p. 121]), we easily see that Q_r is decomposed by $G^*(r, s)$ into the following three orbits;

$$\begin{aligned} R_r^0(s) &= \{\kappa(\zeta) \in Q_r \,|\, \zeta \notin (C_s(r))^{\perp}\} , \\ R_r^1(s) &= \{\kappa(\zeta) \in Q_r \,|\, \zeta \in C_s(r)\} , \\ R_r^2(s) &= \{\kappa(\zeta) \in Q_r \,|\, \zeta \in (C_s(r))^{\perp} \backslash C_s(r)\} , \end{aligned}$$

where κ is the projection of $C^{n+1} \backslash \{0\}$ onto $P^n(C)$. From $o = \kappa(e_0)$, $e_n \in C_s(r)$ and $(e_0, e_n) = \sqrt{-1} \neq 0$, we see $o \in R_r^0(s)$. Hence we have $Q_r^*(s) = R_r^0(s)$. Q.E.D.

Remark 6.3. From the above we have the orbital decomposition of Q_r by $G^*(r, s)$;

$$Q_r = Q_r^*(s) \cup R_r^1(s) \cup R_r^2(s) .$$

Note that

(1) $R_r^1(s) = \{\tilde{o}\}$ if and only if $s = 0$, where $\tilde{o} = \kappa(e_n)$,

(2) $R_r^2(s) = \emptyset$ if and only if $s = r$.

Hence we have

$$Q_r = Q_r^* \cup \{\tilde{o}\} \cup R_r^2(0) \qquad \left(1 \leqq r \leqq \left[\frac{n-1}{2}\right]\right),$$
$$Q_0 = Q_0^* \cup \{\tilde{o}\} ,$$
$$Q_r = Q_r^*(r) \cup R_r^1(r) .$$

From Lemma 6.2 we see that $Q_r^*(s)$ is a connected open subset of Q_r, hence it is a connected non-degenerate (index r) homogeneous flat hypersurface of $P^n(C)$.

Next we will determine the groups $A(Q_r), A(Q_r^*(s))$ of all pseudo-conformal transformations of $Q_r, Q_r^*(s)$.

PROPOSITION 6.4 ([6]). $A(Q_r) = G(r)$.

Proof. Let us fix a frame $x_0 \in F(Q_r, \tilde{G}(r))$ at o. For $\tau \in G(r)$ we set $\bar{l}_0(\tau) = \tau_*(x_0)$. Then $\bar{l}_0$ is a bundle homomorphism of $G(r)$ $(Q_r, G'(r))$ onto $\tilde{F}(Q_r, \tilde{G}(r))$ corresponding to l, $G'(r) \to \tilde{G}(r)$, which preserves the base space Q_r. It is known ([6; Theorem 6]) that $G(r)$ $(Q_r, G'(r))$ together with $\bar{l}_0$ is the pseudo-conformal $G'(r)$-bundle over Q_r and that the Maurer-Cartan form on $G(r)$ coincides with the normal pseudo-conformal connection form. Hence we have $A(Q_r) = G(r)$ as a Lie transformation group. Q.E.D.

PROPOSITION 6.5

(1) *In the case* $r \neq \dfrac{n-1}{2}$, $A(Q_r^*(s)) = G^*(r, s)$,

(2) *In the case* $r = \dfrac{n-1}{2}$ $(n\,;\,\text{odd})$, $A(Q_r^*(s)) = G^*(r, s) \cup \tau_s(G^*(r, s))$,

where $\tau_s = \chi(\sigma_s)$;

$$\sigma_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_s^* & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_s^* = \begin{pmatrix} 0 & 0 & E_s \\ 0 & I_s^{**} & 0 \\ E_s & 0 & 0 \end{pmatrix}, \quad I_s^{**} = \begin{pmatrix} 0 & E_{r-s} \\ E_{r-s} & 0 \end{pmatrix}.$$

Proof. Let π_r be the projection of $G(r)$ onto Q_r (i.e. $\pi_r(\tau) = \tau(o)$ for $\tau \in G(r)$). Since $Q_r^*(s)$ is an open subset of Q_r, the restriction $\pi_r^{-1}(Q_r^*(s))$ $(Q_r^*(s), G'(r))$ of $G(r)$ $(Q_r, G'(r))$ is the pseudo-conformal $G'(r)$-bundle over $Q_r^*(s)$ and the restriction ω_s of the Maurer-Cartan form of $G(r)$ coincides with the normal pseudo-conformal connection form. Hence we get $A(Q_r^*(s)) = \{\tau \in G(r) \mid \tau(Q_r^*(s)) = Q_r^*(s)\}$. On the other hand we have $Q_r^*(s) = \{\zeta = (\zeta_0, \cdots, \zeta_n) \in Q_r \mid \zeta \notin (C_s(r))^{\perp}\}$ and $G^*(r, s) = \{\chi(\sigma) \in G^0(r) \mid \sigma(C_s(r)) = C_s(r)\}$. From these we see easily $A(Q_r^*(s)) \cap G^0(r) = G^*(r, s)$. In case $G(r)$ is not connected $\left(\text{i.e. in case } r = \dfrac{n-1}{2}\right)$, we can find an element $\tau_s \in A(Q_r^*(s))$ which does not belong to $G^0(r)$. Q.E.D.

From the above we have $P(Q_r^*(s), G'(r)) = \pi_r^{-1}(Q_r^*(s))$ $(Q_r^*(s), G'(r))$ and $A^0(Q_r^*(s)) = G^*(r, s)$. Let $e \in \pi_r^{-1}(o)$ be the unit element of $G(r)$. Then the natural inclusion ι_e of $G^*(r, s)$ into $G(r)$ induces the imbedding ι_e of $A^0(Q_r^*(s))$ into $P(Q_r^*(s), G'(r))$ in the sense of Proposition 3.2. In fact, letting z_0 and ρ_{z_0} be the same as in Proposition 3.2 we may take e as z_0, then ρ_{z_0} coincides with the natural inclusion of the isotropy subgroup of $G^*(r, s)$ at o into $G'(r)$. Moreover $\iota_e^*\omega_s$ is just the Maurer-Cartan form on $G^*(r, s)$. In particular we have $\mathfrak{h}_e = \mathfrak{g}^*(r, s)$, where the notation $\mathfrak{h}_e$ is the same as in Proposition 3.4.

Now we will investigate in detail the model spaces $Q_r, Q_r^*(s)$ and their groups $G^0(r), G^*(r, s)$ of pseudo-conformal transformations.

First we have

PROPOSITION 6.6. *Let us fix an integer r with $0 \leqq r \leqq \left[\dfrac{n-1}{2}\right]$ $(n \geqq 2)$. Then $P^n(C) \supset Q_r$, $Q_r^*(s)$ $(0 \leqq s \leqq r)$ are all simply connected.*

Proof. (1) Simply connectedness of Q_r; We consider

$$Q'_r = \left\{(z_0, \cdots, z_n) \in P^n(C) \,\middle|\, -\sum_{i=0}^{r} z_i\bar{z}_i + \sum_{i=r+1}^{n} z_i\bar{z}_i = 0\right\}.$$

Then Q'_r and Q_r are projectively equivalent (hence they are pseudo-conformally equivalent). One should note that Q'_0 is the $(2n-1)$-dimensional unit sphere in $C^n = \{(z_0, \cdots, z_n) \in P^n(C) \mid z_0 \neq 0\}$. We will show the simply connectedness of Q'_r $(r \geqq 1)$. From Proposition 6.4 we know $A^0(Q'_r) = U(r+1, n-r)/U(1)$. Moreover it is easily seen that the maximal compact subgroup $K = U(r+1) \times U(n-r)$ of $U(r+1, n-r)$ acts transitively on Q'_r, where

$$K = \left\{\sigma \in U(r+1, n-r) \middle| \sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \sigma_1 \in U(r+1),\ \ \sigma_2 \in U(n-r)\right\}.$$

Let o' be a point of Q'_r with homogeneous coordinate $(1, 0, \cdots, 0, 1)$. Then the isotropy subgroup L of K at o' is given by

$$L = \left\{\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \in K \middle| \sigma_1 = \begin{pmatrix} \exp(\sqrt{-1}\theta) & 0 \\ 0 & \tilde{\sigma}_1 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} \tilde{\sigma}_2 & 0 \\ 0 & \exp(\sqrt{-1}\theta) \end{pmatrix} \right.$$
$$\left. \tilde{\sigma}_1 \in U(r),\ \tilde{\sigma}_2 \in U(n-r-1)\right\}.$$

Hence L is isomorphic with $U(1) \times U(r) \times U(n-r-1)$. From the above Q'_r is homeomorphic with K/L. Then the following homotopy exact sequence of the principal fibre bundle $K(Q'_r, L)$ shows the simply connectedness of Q'_r;

$$\longrightarrow \pi_1(L, e) \xrightarrow{i_*} \pi_1(K, e) \xrightarrow{p_*} \pi_1(Q'_r, o') \xrightarrow{\Delta} \pi_0(L, e).$$

In fact, the arcwise connectedness of L implies $\pi_0(L, e) = \{0\}$. Hence we have only to check that i_* is onto. Since we suppose $r \geqq 1$, we have

$$\begin{cases} \pi_1(K, e) = \pi_1(U(r+1), e) \times \pi_1(U(n-r), e)\ \ (\cong \mathbf{Z} \oplus \mathbf{Z}), \\ \pi_1(L, e) = \pi_1(U(1), e) \times \pi_1(U(r), e) \times \pi_1(U(n-r-1), e)\ \ (\cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}). \end{cases}$$

Moreover the generator of $\pi_1(U(r), e) \subset \pi_1(L, e)$ is also the generator of $\pi_1(U(r+1), e) \subset \pi_1(K, e)$ and similarly the generator of $\pi_1(U(n-r-1, e)) \subset \pi_1(L, e)$ is also the generator of $\pi_1(U(n-r), e) \subset \pi_1(K, e)$. Hence i_* is onto.

(2) Simply connectedness of Q^*_r; We identify C^n with the set of points of $P^n(C)$ for which $z_0 \neq 0$. Then from $Q^*_r = Q_r \cap C^n$, we have

$$Q^*_r = \left\{(z'_1, \cdots, z'_n) \in C^n \middle| \operatorname{Im} z'_n = \frac{1}{2}\left(-\sum_{i=1}^{r} |z'_i|^2 + \sum_{i=r+1}^{n-1} |z'_i|^2\right)\right\},$$

where $\operatorname{Im} z'_n$ is the imaginary part of z'_n. Hence it is clear that Q^*_r is diffeomorphic with R^{2n-1}. In particular Q^*_r is simply connected.

(3) Simply connectedness of $Q^*_r(s)$ $(1 \leqq s \leqq r)$; From Lemma 6.2 we have the orbital decomposition of Q_r by $G^*(r, s)$; $Q_r = Q^*_r(s) \cup R^1_r(s) \cup R^2_r(s)$. From $\dim._C C_s(r) = s + 1$ we have $\dim. R^1_r(s) = 2s \leqq 2r$. Moreover from $\dim._C (C_s(r))^\perp = n - s$, we have $\dim. R^2_r(s) = 2(n-s) - 3$ provided that $s < r$ (if $s = r$, $R^2_r(r) = \emptyset$). Hence if $s \geqq 1$, both $R^1_r(s)$ and $R^2_r(s)$ are regular submanifolds of Q_r of codimension greater than or

equal to 3. Obviously $R_r^1(s)$ is closed in Q_r and $R_r^2(s)$ is closed in $Q_r\backslash R_r^1(s)$. Therefore the simply connectedness of $Q_r^*(s)$ follows from (1) and the next proposition.

PROPOSITION D (cf. [3; VII Proposition 9.6]). *Let M be a connected manifold, and let S be a closed submanifold of M with* $\dim. S \leqq \dim. M - 3$. *Then $M\backslash S$ is connected and $\pi_1(M)$ is isomorphic with $\pi_1(M\backslash S)$.*

Q.E.D.

Next we consider $G^0(r)$ and $G^*(r, s)$. We set $G_0'(r) = G^0(r) \cap G'(r)$. Since $Q_r = G^0(r)/G_0'(r)$ is simply connected, $G_0'(r)$ is connected.

PROPOSITION 6.7. *$G^0(r)$ satisfies the following;*

(1) *There exists an element τ_0 of $G^0(r)$ such that o is the only fixed point of τ_0 in Q_r.*

(2) *The center $Z(G^0(r))$ of $G^0(r)$ is reduced to the unit.*

(3) *The normalizer $N_{G^0(r)}(G_0'(r))$ of $G_0'(r)$ in $G^0(r)$ coincides with $G_0'(r)$.*

Proof. (1) Let κ be the projection of $C^{n+1}\backslash\{0\}$ onto $P^n(C)$. Let $\sigma \in U(\tilde{I}_r)$ and $p = \kappa(\zeta) \in Q_r$ (i.e. $(\zeta, \zeta) = 0$). Then for $\chi(\sigma) \in G^0(r)$ we have

$$\chi(\sigma)(p) = p \text{ if and only if } \sigma(\zeta) = \lambda\zeta \qquad \text{for some } \lambda \in C\backslash\{0\}\,.$$

Hence $\chi(\sigma)$ fixes a point $p = \kappa(\zeta)$ of Q_r if and only if ζ is an isotropic eigenvector of σ. Therefore finding an element of $G^0(r)$ having $o = \kappa(e_0)$ as the only fixed point in Q_r is equivalent to finding an element of $U(\tilde{I}_r)$ having $\langle e_0\rangle_C$ as the only isotropic eigenline. Here we mean by an eigenline of σ a 1-dimensional subspace invariant by σ. Using the Witt's theorem one can easily construct such an element $\sigma \in U(\tilde{I}_r)$.

(2) Let $\tau \in Z(G^0(r))$ and let τ_0 be as in (1). From $\tau_0\cdot\tau = \tau\cdot\tau_0$ we have $\tau_0(\tau(o)) = \tau(\tau_0(o)) = \tau(o)$. Hence $\tau(o)$ is a fixed point of τ_0. But τ_0 fixes o alone. Therefore $\tau(o) = o$. Since $G^0(r)$ acts transitively on Q_r, we see easily τ fixes every point of Q_r. Then since $G^0(r)$ acts effectively on Q_r, τ is the unit of $G^0(r)$.

(3) Let $\tau \in G^0(r)$. Since $G_0'(r)$ is the isotropy subgroup of $G^0(r)$ at $o \in Q_r$, $\tau(G_0'(r))\tau^{-1}$ is the isotropy subgroup of $G^0(r)$ at $\tau(o)$. Hence each element of $\tau(G_0'(r))\tau^{-1}$ fixes $\tau(o)$. Now let $\tau_1 \in N_{G^0(r)}(G_0'(r))$, and let τ_0 be as in (1). Since $\tau_1(G_0'(r))\tau_1^{-1} = G_0'(r)$, each element of $G_0'(r)$ fixes $\tau_1(o)$. In particular $G_0'(r) \ni \tau_0$ fixes $\tau_1(o)$. Hence we have $\tau_1(o) = o$, that is, $\tau_1 \in G_0'(r)$. Therefore we get $N_{G^0(r)}(G_0'(r)) \subset G_0'(r)$. The opposite inclusion is obvious.

Q.E.D.

Let $G_o^*(r, s)$ be the isotropy subgroup of $G^*(r, s)$ at $o \in Q_r^*(s)$. Since $Q_r^*(s) = G^*(r, s)/G_o^*(r, s)$ is simply connected, $G_o^*(r, s)$ is connected.

PROPOSITION 6.8. *$G^*(r, s)$ $(0 \leqq s \leqq r)$ satisfies the following*

(1) *There exists an element τ_0^s of $G^*(r, s)$ such that o is the only fixed point of τ_0^s in $Q_r^*(s)$.*

(2) *The center $Z(G^*(r, s))$ of $G^*(r, s)$ is reduced to the unit.*

(3) *The normalizer $N_{G^*(r,s)}(G_o^*(r, s))$ of $G_o^*(r, s)$ in $G^*(r, s)$ coincides with $G_o^*(r, s)$.*

Proof. (1) Since $Q_r^*(s) = \{\zeta = (\zeta_0, \cdots, \zeta_n) \in Q_r \mid \zeta \notin (C_s(r))^\perp\}$ and $\hat{G}^*(r, s) = \{\sigma \in U(\tilde{I}_r) \mid \sigma(C_s(r)) = C_s(r)\}$, we have only to find an element σ_0^s of $U(\tilde{I}_r)$ which satisfies

(i) $\sigma_0^s(C_s(r)) = C_s(r)$

(ii) $\langle e_0 \rangle_C$ is the only isotropic eigenline of σ_0^s that is not included in $(C_s(r))^\perp$.

(cf. the proof of (1) Proposition 6.7). Using the Witt's theorem one can easily construct such an element $\sigma_0^s \in U(\tilde{I}_r)$.

Since $G^*(r, s)$ acts effectively and transitively on $Q_r^*(s)$, in view of (1), (2) and (3) can be proved similarly as in Proposition 6.7. Q.E.D.

VII. Determination of $(A(S), A_{p_0}(S))$.

In this section let $\mathfrak{g}$ be $\mathfrak{g}(r)$ or $\mathfrak{g}^*(r, s)$ $(s = 0, 1, \cdots, r)$. Let G be the analytic subgroup of $G(r)$ with Lie algebra $\mathfrak{g}$, and let Q be the model space corresponding to $\mathfrak{g}$ which is defined in VI. Moreover let G' be the isotropy subgroup of G at $o \in Q$, and let $\mathfrak{g}'$ be its Lie algebra. Hence in the case $\mathfrak{g} = \mathfrak{g}(r)$ (resp. $\mathfrak{g}^*(r, s)$), we have $G = G^0(r)$ (resp. $G^*(r, s)$), $Q = Q_r$ (resp. $Q_r^*(s)$) and $G' = G_0'(r)$ (resp. $G_o^*(r, s)$). From Propositions 6.6, 6.7 and 6.8 we have

(1) *$Q = G/G'$ is connected and simply connected.*

(2) *The center $Z(G)$ of G is reduced to the unit.*

(3) *The normalizer $N_G(G')$ of G' in G coincides with G'.*

(4) *$\mathfrak{g}'$ contains $E_0 \in \mathfrak{g}(r)$ which defines the grading of $\mathfrak{g}(r)$.*

As we see in VI, Q is a connected non-degenerate (index r) homogeneous flat hypersurface of $P^n(C)$ for which G is the identity component of $A(Q)$.

Now we have

PROPOSITION 7.1. *Let* $\mathfrak{g}, \mathfrak{g}', Q, G$ *and* G' *be as above. Let* S *be a connected non-degenerate (index* r*) homogeneous hypersurface, and let* $(P, \omega, \bar{l})$ *be the normal pseudo-conformal connection over* S*. For* $p_0 \in S$ *we suppose that there exists a point* $z_1 \in \pi^{-1}(p_0)$ *such that* $\tilde{\mathfrak{h}}_{z_1} = \mathfrak{g}$*. Then* S *is pseudo-conformally equivalent to* Q*.*

Proof. Since $\mathfrak{g}'$ contains E_0, we see from Lemma 5.5, Proposition 5.6 and Proposition 3.4 that there exists a point $z_0 \in \pi^{-1}(p_0)$ such that $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto $\mathfrak{g}$. In particular we have $\iota_{z_0}^*\omega(\mathfrak{a}_{p_0}(S)) = \mathfrak{g}'$. On the other hand, from Lemma 3.1 we have $(\rho_{z_0*})_e = \omega_{z_0}(\iota_{z_0*})_e$, that is, $\rho_{z_0*} = \iota_{z_0}^*\omega$ as a Lie algebra homomorphism. Let $(A_{p_0}(S))^0$ be the identity component of $A_{p_0}(S)$. Then ρ_{z_0} is a group isomorphism of $(A_{p_0}(S))^0$ onto G'.

Next we compare $A^0(S)$ with G. Since G is connected and $Z(G) = \{e\}$, the adjoint representation Ad_G of G is an isomorphism of G onto the adjoint group $\mathrm{Int}(\mathfrak{g})$. Hence the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ of $\mathfrak{g}$ is also faithful. On the other hand the adjoint representation $\mathrm{Ad}_{A^0(S)}$ of $A^0(S)$ is a homomorphism of $A^0(S)$ onto $\mathrm{Int}(\mathfrak{a}(S))$. Set $h = \iota_{z_0}^*\omega$. Then since h is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto $\mathfrak{g}, h$ naturally induces a group isomorphism $\tilde{h}$ of $\mathrm{Int}(\mathfrak{a}(S))$ onto $\mathrm{Int}(\mathfrak{g})$. More precisely we set $(\tilde{h}(\tau))(X) = h \cdot \tau \cdot h^{-1}(X)$ for $\tau \in \mathrm{Int}(\mathfrak{a}(S))$, $X \in \mathfrak{g}$. Then we have $\tilde{h}_* \cdot \mathrm{ad}_{\mathfrak{a}(S)} = \mathrm{ad}_{\mathfrak{g}} \cdot h$.

Now we set $\varphi = (\mathrm{Ad}_G)^{-1} \cdot \tilde{h} \cdot \mathrm{Ad}_{A^0(S)}$. Then φ is a homomorphism of $A^0(S)$ onto G such that $\varphi_* = h$. Moreover we consider a mapping ψ of $A^0(S)/\varphi^{-1}(G')$ onto Q which satisfies the following commutative diagram

$$\begin{array}{ccc} A^0(S) & \xrightarrow{\varphi} & G \\ \downarrow & & \downarrow \\ A^0(S)/\varphi^{-1}(G') & \xrightarrow{\psi} & Q = G/G' . \end{array}$$

Then ψ is a C^ω-homeomorphism of $A^0(S)/\varphi^{-1}(G')$ onto Q. Since $\varphi_* = h$, we have $\varphi_*(\mathfrak{a}_{p_0}(S)) = \mathfrak{g}'$. Hence the Lie algebra of $\varphi^{-1}(G')$ coincides with $\mathfrak{a}_{p_0}(S)$. On the other hand $\varphi^{-1}(G')$ is connected since Q (therefore $A^0(S)/\varphi^{-1}(G')$) is simply connected. Hence we have $\varphi^{-1}(G') = (A_{p_0}(S))^0$. From $N_G(G') = G'$ and the connectedness of G', we see that G' is the only Lie subgroup of G with Lie algebra $\mathfrak{g}'$. On the other hand $\varphi(A_{p_0}^0(S))$ is a Lie subgroup of G with Lie algebra $\varphi_*(\mathfrak{a}_{p_0}(S)) = \mathfrak{g}'$. Hence we have $\varphi(A_{p_0}^0(S)) = G'$. In particular $A_{p_0}^0(S) \subset \varphi^{-1}(G') = (A_{p_0}(S))^0$. Therefore we conclude $A^0_{p_0}(S)=(A_{p_0}(S))^0$, that is, $A^0_{p_0}(S)$ is connected. Moreover comparing the restriction of φ to $A^0_{p_0}(S)$ with ρ_{z_0}, we have $\varphi_* = \rho_{z_0 *} = h$. Hence we get $\varphi|_{A^0_{p_0}(S)} = \rho_{z_0}$. In particular $\varphi|_{A^0_{p_0}(S)}$ is an isomorphism of $A^0_{p_0}(S)$ onto G'.

Now from $\varphi^{-1}(G')=A^0_{p_0}(S)$ and $S=A^0(S)/A^0_{p_0}(S)$, the above diagram can be rewritten as follows

$$\begin{array}{ccc} A^0(S) & \xrightarrow{\varphi} & G \\ \downarrow & & \downarrow \\ S & \xrightarrow{\psi} & Q\,. \end{array}$$

Since ψ is a C^ω-homeomorphism of S onto Q and the restriction of φ to $A^0_{p_0}(S)$ is an isomorphism of $A^0_{p_0}(S)$ onto G', φ becomes a bundle isomorphism of $A^0(S)$ $(S, A^0_{p_0}(S))$ onto $G(Q, G')$. Hence φ is a group isomorphism of $A^0(S)$ onto G.

Now we compare two (connected non-degenerate (index r) homogeneous) hypersurface S and Q. Let $(\pi_r^{\prime -1}(Q), \omega_Q, \bar{l}_0)$ be the normal pseudo-conformal connection over Q (for the notations see Proposition 6.5). If we choose points $z_0 \in \pi^{-1}(p_0)$ and $e \in \pi_r^{-1}(o)$, then φ satisfies the assumption of Proposition 3.5 since $\varphi(A^0_{p_0}(S)) = G'$, $\varphi_* = \iota^*_{z_0}\omega$ (as Lie algebra isomorphisms) and $\iota^*_e\omega_Q$ is the Maurer-Cartan form of G. Therefore ψ is a pseudo-conformal homeomorphism of S onto Q. Q.E.D.

From Theorem 5.8 and the above proposition, we have the main theorem of this paper.

THEOREM 7.2. *Let M be a complex manifold of dimension n. Let S be a connected non-degenerate* (*index r*) *homogeneous hypersurface of M.*

(1) *If* dim. $A(S)=n^2+2n$, *then S is pseudo-conformally equivalent to*

$$Q_r = \left\{(z_0, \cdots, z_n) \in P^n(C) \,\middle|\, -\sqrt{-1}z_0\bar{z}_n - \sum_{i=1}^{r} z_i\bar{z}_i + \sum_{i=r+1}^{n-1} z_i\bar{z}_i + \sqrt{-1}z_n\bar{z}_0 = 0\right\}.$$

(2) *If* dim. $A(S) < n^2+2n$, *we have the following three cases.*

(i) *the case $n=3$ and $r=1$; We have* dim. $A(S) \leqq n^2+2=11$. *The equality holds if and only if S is pseudo-conformally equivalent to*

$$Q_1^*(1) = \{(z_0, \cdots, z_3) \in Q_1 \,|\, |z_0| + |z_1 - z_2| \neq 0\} .$$

(ii) *the case* $n = 5$ *and* $r = 2$; *We have* dim. $A(S) \leqq n^2 + 1 = 26$. *The equality holds if and only if S is pseudo-conformally equivalent to*

$$Q_2^*(2) = \{(z_0, \cdots, z_5) \in Q_2 \,|\, |z_0| + |z_1 - z_4| + |z_2 - z_3| \neq 0\}$$

or

$$Q_2^* = \{(z_0, \cdots, z_5) \in Q_2 \,|\, z_0 \neq 0\} .$$

(iii) *otherwise*; *We have* dim. $A(S) \leqq n^2 + 1$. *The equality holds if and only if S is pseudo-conformally equivalent to*

$$Q_r^* = \{(z_0, \cdots, z_n) \in Q_r \,|\, z_0 \neq 0\} .$$

In Theorem 7.2, if we specify the ambient space M, then the question arises whether a hypersurface S with dim. $A(S) = n^2 + 2n$ (or $n^2 + 1$) exists in M, in other words, whether Q_r (or Q_r^*) can be pseudo-conformally imbedded in M or not. In general this is a very hard problem. Concerning with this we observe

COROLLARY 7.3. *Let* C^n *be the complex number space of dimension* n. *Let S be a connected non-degenerate (index r) homogeneous hypersurface of* C^n. *Then we have*

(1) *In the case* $r = 0$ (*i.e. in the case S is strongly pseudo-convex*) $A(S)$ *has the largest dimension* $n^2 + 2n$, *if and only if S is pseudo-conformally equivalent to the unit sphere* S^{2n-1}. *And* $A(S)$ *has the second largest dimension* $n^2 + 1$, *if and only if S is pseudo-conformally equivalent to the hyperconic*

$$Q_0^* = \left\{(z_1, \cdots, z_n) \in C^n \,\middle|\, \operatorname{Im} z_n = \frac{1}{2} \sum_{i=1}^{n-1} |z_i|^2 \right\} .$$

(2) *In the case* $1 \leqq r < \left[\dfrac{n-1}{2}\right]$

$A(S)$ *has the largest dimension* $n^2 + 1$, *if and only if S is pseudo-conformally equivalent to*

$$Q_r^* = \left\{(z_1, \cdots, z_n) \in C^n \,\middle|\, \operatorname{Im} z_n = \frac{1}{2}\left(-\sum_{i=1}^{r} |z_i|^2 + \sum_{i=r+1}^{n-1} |z_i|^2\right)\right\} .$$

(3) *In the case* $r = \left[\dfrac{n-1}{2}\right]$, *we have the following three cases.*

(i) $n = 3$ *We have* dim. $A(S) \leqq n^2 + 2 = 11$.

(ii) $n = 5$ *We have* dim. $A(S) \leqq n^2 + 1 = 26$.

(iii) *otherwise*; *$A(S)$ has the largest dimension $n^2 + 1$, if and only if S is pseudo-conformally equivalent to Q_r^*.*

Before the proof, recall the following

PROPOSITION E (cf. [5; VII Proposition 4.6], [6; Corollary to Theorem 5]). *Let S be a compact hypersurface of C^n. Then there exists a point p_0 of S such that S is strongly pseudo-convex at p_0.*

Proof of Corollary 7.3. If dim. $A(S) = n^2 + 2n$, S is pseudo-conformally equivalent to Q_r from Theorem 7.2. Hence S is compact. Then r must be zero as the above proposition shows. In other words, if $r \geqq 1$, Q_r cannot be realized as a hypersurface of C^n. On the other hand from the proof of Proposition 6.6, we know that Q_0 is projectively equivalent to S^{2n-1}. Other assertions of the corollary is obvious from Theorem 7.2. Q.E.D.

We don't know whether $Q_1^*(1)$ (resp. $Q_2^*(2)$) can be pseudo-conformally imbedded into C^3 (resp. C^5).

Finally we will see that in the case dim. $A(S) = n^2 + 2n$, the homogeneity assumption is dispensable. In fact we have

THEOREM 7.4. *Let M be a complex manifold of dimension n. Let S be a connected hypersurface of M which is non-degenerate of index r at a point $p_0 \in S$. If* dim. $A(S) = n^2 + 2n$, *then S is pseudo-conformally equivalent to Q_r.*

Proof. We denote by $\mathfrak{a}(S)$ the Lie algebra of all infinitesimal pseudo-conformal transformations of S which generate global 1-parameter groups of transformations. Then $\mathfrak{a}(S)$ is naturally isomorphic with the Lie algebra of $A(S)$. Let S^* be the set of points of S at which S is non-degenerate of index r. Obviously S^* is an open subset of S containing p_0. Hence S^* is a non-degenerate (index r) hypersurface. Let (P^*, ω^*, l^*) be the normal pseudo-conformal connection over S^*. We consider the Lie algebra $\tilde{\mathfrak{a}}(S^*)$ of all infinitesimal pseudo-conformal transformations of S^*. Since S^* is an open subset of S and each element of $\mathfrak{a}(S)$ is a real analytic vector field on S, the restriction map *res* of $\mathfrak{a}(S)$ into $\tilde{\mathfrak{a}}(S^*)$ is an injective homomorphism. Set $\tilde{\mathfrak{a}}(P^*) = \{X \in \mathfrak{X}(P^*) \mid L_X\omega^* = 0, R_{a*}X = X \; a \in G'(r)\}$. Since $(\pi^*)_*$ is an isomorphism of $\tilde{\mathfrak{a}}(P^*)$ onto $\tilde{\mathfrak{a}}(S^*)$, we have dim. $\tilde{\mathfrak{a}}(S^*) \leqq n^2 + 2n$. On the other hand from the assumption we have dim. $\mathfrak{a}(S) = n^2 + 2n$. Hence *res* is an isomorphism of $\mathfrak{a}(S)$ onto $\tilde{\mathfrak{a}}(S^*)$. In particular *res* maps the isotropy subalgebra $\mathfrak{a}_{p_0}(S)$ of $\mathfrak{a}(S)$ at p_0 onto the isotropy subalgebra $\tilde{\mathfrak{a}}_{p_0}(S^*)$ of $\tilde{\mathfrak{a}}(S^*)$ at p_0. Then from dim. $\tilde{\mathfrak{a}}_{p_0}(S^*) = n^2 + 1$, we have dim. $\mathfrak{a}_{p_0}(S) = n^2 + 1$.

Now we consider the orbit S^{**} of $A^0(S)$ passing through p_0. Then as is easily seen from dim. $\mathfrak{a}(S) = n^2 + 2n$ and dim. $\mathfrak{a}_{p_0}(S) = n^2 + 1$, $S^{**} = A^0(S)/A^0_{p_0}(S)$ is an open submanifold of S. Hence S^{**} is a connected non-degenerate (index r) homogeneous hypersurface. Moreover we have dim. $A(S^{**}) = n^2 + 2n$. In fact we have only to show that $A^0(S)$ acts effectively on S^{**}, which is clear since S^{**} is an open subset of S and pseudo-conformal transformations of S are C^ω-homeomorphisms of S. Therefore from Theorem 7.2 S^{**} is pseudo-conformally equivalent to Q_r. In particular S^{**} is compact. On the other hand S^{**} is an open subset of a connected hypersurface S. Hence we must have $S = S^{**}$. Therefore S is pseudo-conformally equivalent to Q_r. Q.E.D.

COROLLARY 7.5. *Let S be a compact connected hypersurface of $\boldsymbol{C}^n$. If* dim. $A(S) = n^2 + 2n$, *then S is pseudo-conformally equivalent to the unit sphere S^{2n-1}.*

This is clear from the above theorem and Proposition E.

Remark 7.6. In the case of second largest dimension ($r \geqq 1$), the homogeneity assumption is indispensable. In fact $Q_r \backslash \{\tilde{o}\} = Q_r^* \cup R_r^2(0)$ ($r \geqq 1$) is a connected (inhomogeneous) hypersurface of $P^n(\boldsymbol{C})$ for which $G^*(r)$ is the identity component of $A(Q_r \backslash \{\tilde{o}\})$. We will treat the inhomogeneous second largest dimension case in a forthcoming paper.

REFERENCES

[1] E. Artin, Geometric Algebra, Interscience Tracts #3. Interscience, New York, 1957.

[2] E. Cartan, Sur la geometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, I, Ann. Mat. Pura Appl., **11** (1932), 17–90. II, Ann. Scuola. Norm. Sup. Pisa **1** (1932), 333–354.

[3] S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.

[4] S. Kobayashi, Transformation Groups in Differential Geometry. Springer, Berlin-Heidelberg-New York, 1972.

[5] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry. John Wiley & Sons, New-York, Vol. 1, 1963, Vol. 2, 1969.

[6] N. Tanaka, On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables, J. Math. Soc. Japan **14** (1962), 397–429.
[7] N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japanese J. Math. vol. 2, No. 1 (1976).

Department of Mathematics, Kyoto Univ.