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A NOTE ON A RESULT OF A. J. LOHWATER AND GEORGE PIRANIAN

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1. Introduction.

Let I denote the set of all inner functions in H^∞, where H^∞ is the Banach algebra of all bounded analytic functions on the open unit disk D. Let I^* denote the set of all functions $f(z)$ in H^∞ for which the cluster set $C(f,\alpha)$ at any point α on the circumference $C = \{\alpha \mid |\alpha| = 1\}$ is either the closed unit disk $|w| \leq 1$ or else a single point of modulus one. Clearly, I is a subset of I^*. In [3] the author has proved that I is properly contained in I^*. Recently, Lohwater and Piranian [7] have shown that there is an outer function in I^*. The purpose of this note is to point out some applications of this result. In particular we shall show in Theorem 2.3 that there exist outer functions whose boundary behavior is similar to that of inner functions.

2. Applications.

Let Γ denote the Silov boundary of H^∞ and let $\Gamma_\alpha = \Gamma \cap M_\alpha$ where M_α is the fiber lying above the point α on C. (For the terminology used here we refer the reader to Hoffmann [5].) We begin with a theorem which establishes a connection between the boundary values of an arbitrary function $f(z)$ in H^∞ and the set of values which f assumes on Γ_α. (See [4] for the proof.)

THEOREM 2.1. *Let $f(z)$ be a function in H^∞ and let $C_\rho(f,\alpha)$ denote the radial cluster set of f at the point α on D. Then, the set*

$$E = \{\alpha \in C \mid C_\rho(f,\alpha) \cap f(\Gamma_\alpha) = \emptyset\}$$

is of measure zero.

As an immediate consequence of this theorem and a result of

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Hoffman ([5], p. 179) we obtain

COROLLARY 2.2. *A function $f(z)$ in H^∞ is an inner function if and only if the image of the Silov boundary, $f(\Gamma)$ is a subset of $|z| = 1$.*

Ohtsuka [8] has constructed an inner function which admits every value of modulus less than one as a radical limit. Thus, Theorem 2.1 is best possible in the sense that E, in general, need not be a countable set.

In view of Theorem 2.1 it is natural to inquire whether a point which belongs to $f(\Gamma_\alpha) - FrC(f, \alpha)$, where $FrC(f, \alpha)$ denotes the frontier of $C(f, \alpha)$, for a "large" set of points α on C is necessarily a radial limit value of f. The solution to this problem is contained in the following.

THEOREM 2.3. *Let f be an outer function in I^*. Suppose that for each point α on C the cluster set $C(f, \alpha)$ is the closed unit disk $|w| \leq 1$ and suppose that zero is not a radial limit value of f. Then, $0 \in f(\Gamma_\alpha)$ for each α on C.*

In ([7], Theorem 3) Lohwater and Piranian have constructed a function in I^* which satisfies the hypotheses of this theorm. The proof of this theorem involves the use of a result of Weiss ([9], Theorem, 5.2) and the extended form of the Gross-Iversen theorem (see for example [2, Theorem 5.10]).

Remark. It is known ([1], p. 112) that if g is meromorphic in D, then

$$FrC(g) \cup FrR(g) \subseteq \overline{\Omega(g)}\,, \tag{1}$$

where $C(g)$ denotes the global cluster set of $g, R(g)$ the range of values of g and $\overline{\Omega(g)}$ the closure of the set of radial limit values of g. Collingwood and Cartwright ([1], p. 113) have shown by an example that this result is best possible in the sense that $\overline{\Omega(g)}$ cannot be replaced in (1) by $\Omega(g)$. The function f of Theorem 2.3 provides an additional example. To see this we need merely to observe that

$$0 \in \mathscr{C}f(D) \cap C(f) \cap \overline{\Omega(f)} \cap \mathscr{C}\Omega(f) \cap f(\Gamma)\,, \tag{2}$$

where $\mathscr{C}S$ denotes the complement of the set S. The difference between the behavior of inner functions and functions in $I^* - I$ is further illuminated by relation (2). We recall that if g is an inner function, then every omitted value $w, |w| < 1$, is a radial limit value of g. On the other hand relation (2) shows that in general an omitted value of a function f in $I^* - I$ need not be a radial limit value.

Our next application is in real-variable theory. Recently Lohwater [6] has pointed out that the derivative of an absolutely continuous function may be more irregular in its behaviour than the derivative of a singular function. This observation is further substantiated by the following

THEOREM 2.4. *There exists an absolutely continuous nonincreasing function $\mu(\theta)$ on the interval $0 \leq \theta \leq 2\pi$ with the following property: for any neighborhood, $N(\theta_0)$, of an arbitrary point θ_0 in $0 \leq \theta \leq 2\pi$ and for any $K < 0$, no matter how large $|K|$ is, the set*

$$\{\theta \in N(\theta_0) \mid \mu'(\theta) < K\}$$

has positive Lebesgue measure while $\mu'(\theta) \neq -\infty$ for all θ in $0 \leq \theta \leq 2\pi$.

We will present an outline of the proof. Let f be a function in I^* which satisfies the hypothesis of Theorem 2.3. Next consider the Poisson-Stieltjes integral representation of f:

$$f(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right\}.$$

Then $\mu(\theta)$ is an absolutely continuous, nonincreasing function on $0 \leq \theta \leq 2\pi$. Since $0 \notin \Omega(f)$ $\mu'(\theta) \neq -\infty$ for all θ. It follows now from Theorem 2.3 and a result of Hoffman ([5], p. 171) that μ satisfies the required property.

REFERENCES

[1] E. F. Collingwood and M. L. Cartwright, Boundary theorems for a function meromorphic in the unit circle, Acta Math. **87** (1952), 83–146.

[2] E. F. Collingwood and A. J. Lohwater, The Theory of Cluster Sets, Cambridge University Press, Cambridge, 1966.

[3] G. Csordas, The Šilov boundary and a class of functions in H^∞, Ph.D. Dissertation, Case Western Reserve University, Cleveland, Ohio, 1969.

[4] G. Csordas, A note on the Šilov boundary and the cluster sets of a class of functions in H^∞, Acta Math. Acad. Sci. Hungar., **24** (1973), 5–11.

[5] K. Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, N. J., 1962.

[6] A. J. Lohwater, Some function-theoretic results involving Baire category, Proc. Internat. Conf. Analysis, Jyväskylä Finland, 1970, Springer-Verlag, 1972.

[7] A. J. Lohwater and George Piranian, Bounded analytic functions with large cluster sets, Ann. Acad. Sci. Fenn. Ser. AI, no. **499** (1971), 1–7.
[8] M. Ohtsuka, On the asymptotic values of functions analytic in a circle, Trans. Amer. Math. Soc. **78** (1955), 294–304.
[9] M. L. Weiss, Cluster sets of bounded analytic functions from a Banach algebraic viewpoint, Ann. Acad. Sci. Fenn. Ser. AI, no. **367** (1965), pp. 1–14.

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