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Nagoya Math. J.
Vol. 56 (1974), 19–27

SOLUTIONS TO EXTREMAL PROBLEMS IN E^p SPACE

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1. Introduction.

Let Ω be a bounded domain (in the complex plane) whose boundary, C, consists of finitely many disjoint, rectifiable, closed Jordan curves.

By definition, $F \in E^p(\Omega)$ $(p \in (0, \infty))$ if F is holomorphic on Ω and if there exists a sequence, $\{\Omega_j\}_{j=1}^{\infty}$, of domains such that $\bar{\Omega}_j \subset \Omega_{j+1} \subset \Omega, \bigcup_{j=1}^{\infty} \Omega_j = \Omega$, $\partial\Omega_j$ consists of rectifiable curves homologous to C, and $\sup_j \int_{\partial\Omega_j} |F(z)|^p |dz| < \infty$.

If $F \in E^p(\Omega)$, then F has boundary values for nontangential approach at almost every point of C. We denote the boundary function of F by F^+, and the collection of all such boundary functions by $E^p_+(C)$. $E^p_+(C)$ is a subspace of $L^p(C)$ (the p^{th} Lebesgue space with respect to arc length). (For proofs of the above assertions, see [9] and [2], Chapter 10.)

The following theorem is the basis of much of our work.

THEOREM 1.1. *Let* $p \in (1, \infty), q = p/(p-1), f \in L^p(C), g \in L^\infty(C), \dfrac{1}{g} \in L^\infty(C)$. *Then*:

i) *There exists a unique* $H_0^+ \in E^p_+(C)$ *for which*

$$\|f - gH_0^+\|_p = \inf\{\|f - gF^+\|_p : F^+ \in E^p_+(C)\} = d\ .$$

ii) $$d = \sup\left\{\mathrm{Re}\left(\int_C \frac{f(\zeta)}{g(\zeta)} G^+(\zeta) d\zeta\right) : G^+ \in E^q_+(C) \text{ and } \left\|\frac{G^+}{g}\right\|_q \le 1\right\}.$$

iii) *If* $d \neq 0$, *then there exists a unique* $G_0^+ \in E^q_+(C)$ *for which*

$$\left\|\frac{G_0^+}{g}\right\|_q \le 1 \quad \text{and} \quad d = \mathrm{Re}\int_C \frac{f(\zeta)}{g(\zeta)} G_0^+(\zeta) d\zeta\ .$$

Received August 17, 1972.

* These results were obtained in the author's doctoral dissertation written at the University of California, San Diego under the direction of Stefan E. Warschawski. The research was sponsored in part by the U.S. Air Force (under USAF Grant AFOSR-68-1514) and the National Aeronautics and Space Administration.

iv) *There is a unique* $H^+ \in E_+^p(C)$ *and a unique* $R^+ \in E_+^q(C)$ *such that*

$$f = gH^+ + \left|\frac{\zeta'}{g}R^+\right|^q \Big/ \left(\frac{\zeta'}{g}R^+\right).$$

(ζ' denotes the derivative of any arc length parametrization of C which leaves Ω to the left of C).

v) $H^+ = H_0^+$ *and (if* $d \neq 0$*)* $[R^+/\|R^+/g\|_q] = G_0^+$.

Proof. See Tumarkin and Havinson [8], pp. 209, 210. (The present formulation of the result is taken from [7].)

In this paper we assume ζ' is Hölder continuous in order to derive an operator equation which the extremal difference $f - gH^+$ satisfies. For $p = 2$, the operator equation is used to obtain a sequence of $L^2(C)$ functions converging at a geometrical rate in the $L^2(C)$ norm to H^+. (The Rayleigh-Ritz method may also be used to compute H^+, but the rate of convergence is not necessarily geometrical unless C is analytic, [7].) For the case that $p = 2$ and g is Hölder continuous, we transform the operator equation into a Fredholm integral equation in order to obtain a sequence of functions coverging uniformly to H^+.

2. The Operator Equation.

We say $\varphi \in \mathrm{Lip}\,(C, \beta)$ if φ is a (complex-valued) Hölder continuous function on C, whose exponent of Hölder continuity is β $(\in (0, 1])$. Similarly, $\psi \in \mathrm{Lip}\,(C \times C)$ if ψ is Hölder continuous on $C \times C$. (Whenever convenient, the exponent of Hölder continuity will be suppressed.)

LEMMA 2.1. *Let* $\zeta' \in \mathrm{Lip}\,(C)$, *and* $p \in (1, \infty)$. *Then for* $k \in L^p(C)$ *(and* $x \in C$*)*

$$\int_C' \frac{k(\zeta)}{\zeta - x} d\zeta$$

defines a bounded linear operator from $L^p(C)$ *to* $L^p(C)$. *(The symbol* $\int'$ *denotes the Cauchy-Lebesgue principal value integral.)*

Proof. See [4], pp. 19–21.

THEOREM 2.1. *Let the conditions and notation be as in Theorem* 1.1 *with the further assumption that* $\zeta' \in \mathrm{Lip}\,(C)$. *Then for almost every* $x \in C$

$$
\begin{aligned}
&f(x) - g(x)H^+(x) \\
(1) \qquad &+ \frac{1}{\pi i}\int_C' \left\{ \frac{\left| \frac{1}{\pi i}\int_C' \left(\frac{|f(\xi) - g(\xi)H^+(\xi)|^p}{f(\xi) - g(\xi)H^+(\xi)} \right) \frac{g(\xi)}{g(\zeta)} \frac{|d\xi|}{(\zeta - \xi)} \right|^q}{\frac{1}{\pi i}\int_C' \left(\frac{|f(\eta) - g(\eta)H^+(\eta)|^p}{f(\eta) - g(\eta)H^+(\eta)} \right) \frac{g(\eta)}{g(\zeta)} \frac{|d\eta|}{(\zeta - \eta)}} \right\} \frac{g(x)}{g(\zeta)} \frac{|d\zeta|}{(\zeta - x)} \\
&= f(x) - g(x)\frac{1}{\pi i}\int_C' \frac{f(\zeta)}{g(\zeta)} \frac{d\zeta}{(\zeta - x)} .
\end{aligned}
$$

Proof. From Theorem 1.1 (iv) it is clear that

$$
(2) \qquad R^+ = \left(\frac{|f - gH^+|^p}{f - gH^+} \right) \frac{g}{\zeta'}
$$

and

$$
(3) \qquad H^+ = \frac{f}{g} - \frac{1}{g}\left(\left| \frac{\zeta' R^+}{g} \right|^q \Big/ \left(\frac{\zeta' R^+}{g} \right) \right) .
$$

Since $R^+ \in E^q_+(C)$ and $q > 1$, the values of R may be recovered by applying the Cauchy integral formula to R^+ (see [2], Chapter 10). Hence it is clear from the Plemelj-Privalov formulas ([3], p. 431) that for almost every $x \in C$

$$
(4) \qquad R^+(x) = \frac{1}{\pi i}\int_C' \frac{R^+(\zeta)}{\zeta - x} d\zeta .
$$

Similarly,

$$
(5) \qquad H^+(x) = \frac{1}{\pi i}\int_C' \frac{H^+(\zeta)}{\zeta - x} d\zeta .
$$

Formally Theorem 2.1 may be obtained as follows: Substitute the right side of (2) for R^+ in the right side of (4). Substitute the resulting expression for R^+ in the right side of (3). Substitute this new expression for H^+ in the right side of (5). Routine manipulation then produces the desired conclusion. The application of Lemma 2.1 makes this argument rigorous.

3. The Solution when $p = 2$.

DEFINITION 3.1. Let $\zeta' \in \mathrm{Lip}\,(C)$ and let both g and $1/g$ be in $L^\infty(C)$. We then say that:

i) $I: L^2(C) \to L^2(C)$ is the identity operator.

ii) $T: L^2(C) \to L^2(C)$ is defined for each $h \in L^2(C)$ by

$$T(h)(x) = \frac{1}{\pi i} \int_C' h(\zeta) \frac{g(x)}{g(\zeta)} \frac{|d\zeta|}{(\zeta - x)} .$$

(From Lemma 2.1, we see that T is a bounded linear operator.)

iii) $\tilde{T}: L^2(C) \to L^2(C)$ is defined for each $h \in L^2(C)$ by

$$\tilde{T}(h)(x) = -\frac{1}{\pi i} \int_C' h(\zeta) \frac{\overline{g(\zeta)}}{\overline{g(x)}} \frac{|d\zeta|}{(\bar{x} - \bar{\zeta})} .$$

($\tilde{T}$ is also a bounded linear operator.)

If $p = 2$, then (1) is a linear operator equation, from which we obtain

$$(6) \qquad (I + T\tilde{T})(gH^+) = u$$

where $u(x) = g(x) \frac{1}{\pi i} \int_C' \frac{f(\zeta)}{g(\zeta)} \frac{d\zeta}{(\zeta - x)} + T\tilde{T}(f)(x)$, a known $L^2(C)$ function.

Finding H^+ (when $p = 2$) is now reduced to the problem of inverting the bounded linear operator $I + T\tilde{T}$.

LEMMA 3.1. *Let* $\zeta' \in \operatorname{Lip}(C)$, $p \in (1, \infty)$, $q = p/(p-1)$, $h \in L^p(C)$, $k \in L^q(C)$. *Then*

$$\int_C \left(\int_C' h(\zeta) k(\xi) \frac{d\zeta}{\zeta - \xi} \right) d\xi = \int_C \left(\int_C' h(\zeta) k(\xi) \frac{d\xi}{\zeta - \xi} \right) d\zeta .$$

Proof. See [4], p. 27.

LEMMA 3.2. *T and $\tilde{T}$ are adjoint operators.*

Proof. Let h and k be $L^2(C)$ functions. Formally then:

$$\langle Th, k \rangle = \int_C \left(\frac{1}{\pi i} \int_C' h(\zeta) \frac{g(\xi)}{g(\zeta)} \frac{|d\zeta|}{(\zeta - \xi)} \right) \overline{k(\xi)} \, |d\xi|$$

$$= \int_C h(\zeta) \overline{\left(-\frac{1}{\pi i} \int_C' k(\xi) \frac{\overline{g(\xi)}}{\overline{g(\zeta)}} \frac{|d\xi|}{(\bar{\zeta} - \bar{\xi})} \right)} |d\zeta| = \langle h, \tilde{T}k \rangle .$$

Lemma 3.1 justifies this formal manipulation.

THEOREM 3.1. *Let the conditions and notation be as in Theorem 1.1 with the further assumptions that* $p = 2$ *and* $\zeta' \in \operatorname{Lip}(C)$. *Let* $c \in \left(0, \frac{1}{\|I + T\tilde{T}\|}\right)$. *Then:*

i) $\|I - c(I + T\tilde{T})\| \leq 1 - c < 1$.

ii) $c \sum_{j=0}^{\infty} (I - c(I + T\tilde{T}))^j = (I + T\tilde{T})^{-1}$ (*convergence in the operator norm.*)

iii) $\frac{1}{g}\left(c \sum_{j=0}^{m} (I - c(I + T\tilde{T}))^j u\right)$ *is a sequence of* $L^2(C)$ *functions converging to* H^+ *in the* $L^2(C)$ *norm as* $m \to \infty$.

Proof. Since T is adjoint to $\tilde{T}$ we have that $I + T\tilde{T}$ is a self-adjoint operator. Thus, if $\|h\|_2 = 1$,

$$(7) \quad \langle (I - c(I + T\tilde{T}))h, h\rangle = 1 - c\langle (I + T\tilde{T})h, h\rangle \geq 1 - c\|I + T\tilde{T}\| > 0 .$$

Furthermore,

$$(8) \quad \begin{aligned} \langle (I - c(I + T\tilde{T}))h, h\rangle &= 1 - c\langle (I + T\tilde{T})h, h\rangle \\ &= 1 - c(1 + \|\tilde{T}h\|_2^2) \leq 1 - c < 1 . \end{aligned}$$

Since $I - c(I + T\tilde{T})$ is also self-adjoint, assertion (i) follows from (7) and (8). Assertion (ii) is an immediate consequence of (i), while (iii) may be obtained by applying (ii) to equation (6).

4. The Solution when $p = 2$ and $g \in \mathrm{Lip}\,(C)$.

LEMMA 4.1. *Let* ζ' *be continuous and* $\varphi \in \mathrm{Lip}\,(C \times C, \beta)$. *Then*

$$\omega(\xi, x) = \int_C' \frac{\varphi(\xi, \zeta)}{\zeta - x} d\zeta$$

is in $\mathrm{Lip}\,(C \times C, \delta)$, *where* δ *is any number on* $(0, \beta)$.

Proof. See [5], pp. 45–51.

Throughout the rest of this section we take the conditions and notation to be as in Theorem 1.1, with the further assumptions that $p = 2, \zeta' \in \mathrm{Lip}\,(C, \beta)$, and $g \in \mathrm{Lip}\,(C, \beta)$.

LEMMA 4.2. *For* $h \in L^2(C)$

i) $K(h)(x) = \int_C \left(\frac{1}{2\pi^2} \int_C' \frac{g(x)\overline{g(\xi)}}{|g(\zeta)|^2 (\zeta - x)(\bar{\xi} - \bar{\zeta})} |d\zeta|\right) h(\xi)\, |d\xi|$ *determines a bounded linear operator,* K, *from* $L^2(C)$ *to* $L^2(C)$.

ii) $K = \frac{1}{2}(I - T\tilde{T})$. (*See Definition* 3.1.)

Proof. From [5], p. 19, it may be seen that $\left(\frac{\xi - \zeta}{\bar{\xi} - \bar{\zeta}}\right)$ is in

Lip $(C \times C, \beta)$ (if the ratio is defined to be $(\zeta')^2$ when $\xi = \zeta$). Thus, as a function of ξ and ζ,

$$\varphi(\xi, \zeta) = \frac{1}{2\pi^2} \frac{\overline{g(\xi)}}{|g(\zeta)|^2} \left(\frac{\xi - \zeta}{\bar{\xi} - \bar{\zeta}} \right) \frac{1}{\zeta'} \tag{9}$$

is in Lip $(C \times C, \beta)$. If we define

$$\kappa(x, \xi) = \frac{1}{2\pi^2} \int_C' \frac{g(x)\overline{g(\xi)}}{|g(\zeta)|^2(\zeta - x)(\bar{\xi} - \bar{\zeta})} |d\zeta| ,$$

routine manipulation shows that

$$\kappa(x, \xi) = \left(\frac{\omega(\xi, x) - \omega(\xi, \xi)}{\xi - x} \right) g(x)$$

where ω is as in Lemma 4.1, and φ is defined by (9). Clearly, $\omega \in$ Lip $(C \times C, \delta)$ (for every $\delta \in (0, \beta)$) so that for $x \neq \xi$, κ is continuous and

$$|\kappa(x, \xi)| \leq \frac{M_\delta}{|\xi - x|^{1-\delta}}$$

(M_δ a positive constant independent of x and ξ). Thus κ is a Fredholm kernel with a weak singularity, and since

$$K(h)(x) = \int_C \kappa(x, \xi) h(\xi) |d\xi| ,$$

K must be a bounded linear operator from $L^2(C)$ to $L^2(C)$ (see, for example, [4], pp. 13–14). This proves (i).

If $h \in$ Lip (C), then $Kh = \frac{1}{2}(I - T\tilde{T})h$ follows from the Poincaré-Bertrand formula ([5], p. 57). But Lip (C) is dense in $L^2(C)$, and K and $\frac{1}{2}(I - T\tilde{T})$ are bounded linear operators, so that assertion (ii) must be true.

From Lemma 4.2 and (6) we have that

$$(I - K)(gH^+) = u_1 \tag{10}$$

where $u_1 = \dfrac{u}{2} \in L^2(C)$. (An integral equation similar to (10) was presented without proof and without solution in the paper of Rosenbloom and Warschawski [7].) Hence

$$(I - K^N)(gH^+) = u_N \tag{11}$$

where

$$u_N = \left(\sum_{\ell=0}^{N-1} K^\ell\right) u_1 \qquad (N = 1, 2, 3, \cdots) . \tag{12}$$

LEMMA 4.3. *Let v be continuous on C. Let W be a Fredholm integral operator (on $L^2(C)$) with a continuous kernel. Suppose there is a number c such that for every eigenvalue, λ, of W, $\left|(1-c) + \frac{c}{\lambda}\right| < 1$ and $|1 - c| < 1$. Then:*

The integral equation $(I - W)\varphi = v$ has exactly one solution in $L^2(C)$, and

$$c \sum_{j=0}^{m} (I - c(I - W))^j v$$

is a sequence of continuous functions converging uniformly to that solution as $m \to \infty$.

Proof. See Bückner [1], pp. 63–65. (Bückner states his result in terms of an iteration scheme, from which the above sequence may be easily obtained.)

THEOREM 4.1. *Let u_N be defined by* (12). *Let N be an odd integer greater than $\frac{1}{\beta}$ and let $c \in \left(0, \frac{2}{1 + \|K^N\|}\right)$. Then:*

i) $\frac{c}{g} \sum_{j=0}^{m} (I - c(I - K^N))^j (K^N u_N)$ *is a sequence of continuous functions converging uniformly to $H^+ - (u_N/g)$ as $m \to \infty$.*

ii) *If $f \in \mathrm{Lip}\,(C)$, $\frac{c}{g} \sum_{j=0}^{m} (I - c(I - K^N))^j (u_N)$ is a sequence of continuous functions converging uniformly to H^+ as $m \to \infty$.*

Proof. We know $\kappa(x, \xi)$ is continuous except when $x = \xi$, and has a weak singularity of order $1 - \delta$, where δ is any number on $(0, \beta)$. Thus if $N > \frac{1}{\beta}$, K^N has a continuous kernel (see, for example, [6], pp. 29–38). Since $K = \frac{1}{2}(I - T\tilde{T})$ is self-adjoint, any eigenvalue of K must be real. Furthermore, K has no eigenvalues on $[0, 2)$. (If λ is an eigenvalue with eigenfunction h, then

$$\frac{1}{\lambda} = \frac{\langle Kh, h\rangle}{\langle h, h\rangle} = \frac{\langle \frac{1}{2}(I - T\tilde{T})h, h\rangle}{\langle h, h\rangle} = \frac{1}{2} - \frac{1}{2}\frac{\langle T\tilde{T}h, h\rangle}{\langle h, h\rangle}$$

$$= \frac{1}{2} - \frac{1}{2}\frac{\langle \tilde{T}h, \tilde{T}h\rangle}{\langle h, h\rangle} \leq \frac{1}{2}. \quad \text{Thus when } \lambda \text{ is positive, } \lambda \geq 2.\Big)$$

Hence, the eigenvalues of K^N are real, and since N is odd, no eigenvalue of K_N lies on $[0, 2^N)$.

If λ is a negative eigenvalue of $K^N, 1 > (1 - c) + \frac{c}{\lambda} \geq 1 - c(1 + \|K^N\|) > -1$. If λ is a positive eigenvalue of $K^N, -1 < (1 - c) + \frac{c}{\lambda} \leq 1 - c + \frac{c}{2^N} < 1$. Hence for every eigenvalue, λ, of K^N, $\left|(1 - c) + \frac{c}{\lambda}\right| < 1$. From our choice of c, it is obvious that $|1 - c| < 1$.

Suppose $f \in \mathrm{Lip}\,(C)$. Then Lemma 4.1 may be used to show that $u_N \in \mathrm{Lip}\,(C)$. Hence assertion (ii) follows from Lemma 4.3 and (11) if we take W to be K^N and v to be u_N.

Lemma 4.3 also yields (i), if we take W to be K^N and v to be $K^N u_N$. ($K^N u_N$ is continuous since $u_N \in L^2(C)$ and K^N has a continuous kernel.)

Given the conditions in § 3 and § 4 it is clear that the results of these sections may be used to find the extremal function R^+ (which is expressed in terms of H^+, f, g in Theorem 1.1 (iv) and (v)).

BIBLIOGRAPHY

[1] H. Bückner, Die Praktische Behandlung von Integral-Gleichungen, Springer-Verlag, Berlin, 1952.

[2] P. L. Duren, Theory of H^p Spaces, Academic Press, New York, 1970.

[3] G. M. Goluzin, Geometric Theory of Functions of a Complex Variable. American Mathematical Society, Providence, 1969.

[4] B. V. Hvedelidze, "Linear Discontinuous Boundary Problems in the Theory of Functions, Singular Integral Equations and Some of Their Applications," Trudy Tbiliss. Mat. Inst. Razmadze **23** (1956), 3–158 (in Russian).

[5] N. I. Mushelišvili, Singular Integral Equations, P. Noordhoff, Groningen, 1953.

[6] I. G. Petrovskiĭ, Lectures on the Theory of Integral Equations, Graylock Press, Rochester, 1957.

[7] P. C. Rosenbloom and S. E. Warschawski, "A Division Problem for Analytic Functions," Seminars on Analytic Functions, Institute for Advanced Study, **1** (1958), 323–339.

[8] G. C. Tumarkin and S. Ja. Havison, "Investigation of the Properties of Extremal Functions by Means of Duality Relations in Extremal Problems for Classes of Analytic Functions in Multiply Connected Domains," Mat. Sb. **46** (1958), 195–228 (in Russian).

[9] G. C. Tumarkin and S. Ja. Havinson, "On the Definition of Analytic Functions of Class E_p in Multiply Connected Domains," Uspehi Mat. Nauk **13** (1958), No. 1, 201–206 (in Russian).

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