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ON THE DISTRIBUTION OF ZEROS OF A STRONGLY ANNULAR FUNCTION

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A function $f(z)$, regular in the unit disk D, is called annular ([1], p. 340) if there is a sequence of closed Jordan curves $J_n \subset D$ satisfying

(A_1) J_n is contained in the interior of J_{n+1} for every n,

(A_2) given $\varepsilon > 0$, there exists a positive number $n(\varepsilon)$ such that, for each $n > n(\varepsilon), J_n$ lies in the region $1 - \varepsilon < |z| < 1$ and

(A_3) $\lim_{n\to\infty} \min\{|f(z)|\,;\, z \in J_n\} = +\infty$.

One says that $f(z)$ is strongly annular if the J_n can be taken as circles concentric with the unit circle C. As for examples of annular functions, see ([4], p. 18).

Given a function $f(z)$ in D, denote by $Z(f)$ the set of zeros of $f(z)$ and $Z'(f)$ the set of limit points of $Z(f)$. If $f(z)$ is annular, $Z(f)$ is an infinite set of points of D ([1], p. 340) and clearly $Z'(f) \subset C$. In [1], Bagemihl and Erdös raised the following question: If $f(z)$ is annular, is $Z'(f) = C$? This question seems to be reasonable because many early examples of annular functions had this property. In [3], however, an example of an annular function $g(z)$ was constructed with $Z'(g) = \{1\}$. It is not known, regretfully, whether or not this example is strongly annular. Thus the problem of Bagemihl and Erdös remains open in the case where "annular" is replaced by "strongly annular" ([5], p. 141). In this note we shall give an example of a strongly annular function $f(z)$ with $Z'(f) = \{1\}$, modifying the technique for constructing the example of Barth and Schneider [3].

1. We shall first make some definitions. Given a, b and θ such that $0 < a < b < 1$ and $0 < \theta < \pi/2$, we consider the annular sector $D(a, b\,;\theta) = \{z \in D\,; a < |z| < b$ and $-\theta < \arg z < \theta\}$. Moreover, for c, θ_1 and θ_2 with $0 < c < 1$ and $-\pi/2 < \theta_2 < \theta_1 < \pi/2$, let $\sigma(c\,;\theta_2, \theta_1)$ denote the circular arc $\{z \in D\,; |z| = c$ and $\theta_2 \leqq \arg z \leqq \theta_1\}$. Now we are to state

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LEMMA. *Let* a_i $(i=1,2,3)$ *and* θ_j $(j=1,2)$ *satisfy*

(1) $0<a_1<a_2<a_3<1$ *and* $a_2^2>a_1a_3$ *and*

(2) $0<\theta_2<\theta_1<\pi/2$ *and* $\tan\theta_1/2<(a_3-a_2)/(a_3+a_2)$.

Then for any $\varepsilon>0$ *and any* $K>0$, *there exists a rational function* $p(z)$, *with its only pole in the open line segment* (a_2, a_3), *satisfying*

(3) $|p(z)|\geqq K$ *on* $\sigma(a_2;-\theta_2,\theta_2)$,

(4) $\mathrm{Re}\,p(z)\geqq 0$ *on* $\sigma(a_2;\theta_2,\theta_1)\cup\sigma(a_2;-\theta_1,-\theta_2)$ *and*

(5) $|p(z)|\leqq\varepsilon$ *on* $\Omega_z-D(a_1',a_3;\theta_1)$

where Ω_z *is the z-sphere and* $a_1'=a_2^2/a_3$.

Proof. First we note that $a_1<a_1'<a_2$. By the function $\zeta=i(a_2-z)/(a_2+z)$, we map the disk $|z|<a_2$ onto the upper half plane of the ζ-plane. Here simply put $\sigma(a_2;-\theta_2,\theta_2)=\sigma$, $\sigma(a_2;\theta_2,\theta_1)\cup\sigma(a_2;-\theta_1,-\theta_2)=\alpha$, $(a_{j+1}-a_j)/(a_{j+1}+a_j)=b_j$ and $\tan\theta_j/2=c_j$ $(j=1,2)$. Then the circular arc σ (or the union of two circular arcs α) is mapped onto the closed segment $[-c_2,c_2]$ (or the union of two closed segments $[-c_1,-c_2]\cup[c_2,c_1]$) respectively. Thus we have only to construct a rational function

$$q(\zeta)=k(\zeta+\rho i)^{-2m}$$

where $k(>0)$, an integer $m(>0)$ and ρ $(0<\rho<b_2)$ are chosen such that

(3)′ $|q(\zeta)|\geqq K$ on $[-c_2,c_2]$,

(4)′ $\mathrm{Re}\,q(\zeta)\geqq 0$ on $[-c_1,-c_2]\cup[c_2,c_1]$ and

(5)′ $|q(\zeta)|\leqq\varepsilon$ on $\Omega_\zeta-E$

where Ω_ζ is the ζ-sphere and E is the image of $D(a_1',a_3;\theta_1)$ by $\zeta=i(a_2-z)/(a_2+z)$. In order to see the existence of k,m and ρ satisfying (3)′, (4)′ and (5)′, using $c_1<b_2$ and geometrical properties of E, it is sufficient to show the existence of an integer m (>0) and ρ $(0<\rho<b_2)$ such that

(6) $(R_1-\sqrt{\rho^2+r_1^2})^{2m}(\rho^2+c_2^2)^{-m}\geqq K/\varepsilon$ where $R_1=\frac{1}{2}(1/c_1+c_1)$ and $r_1=\frac{1}{2}(1/c_1-c_1)$ and

(7) $\pi/4m\geqq\tan^{-1}\rho/c_2$.

By means of elementary calculations we can conclude that such m and ρ surely exist.

2. By virtue of the method used in [3] and our lemma, we shall construct a strongly annular function $f(z)$ with $Z'(f)=\{1\}$.

THEOREM. *Let $\Gamma_j = \{z; z = z_j(t), 0 \leqq t \leqq 1\}$ $(j = 1, 2)$ be two Jordan arcs such that*

(8) $z_1(0) = iy_1$ $(0 < y_1 < 1)$ *and* $z_2(0) = iy_2$ $(-1 < y_2 < 0)$,

(9) $z_j(1) = 1$ $(j = 1, 2)$ *and*

(10) *except for* $z_j(0)$ *and* $z_j(1)$ $(j = 1, 2)$, *we have* $\Gamma_1 \subset \{\operatorname{Re} z > 0\} \cap \{\operatorname{Im} z > 0\} \cap D$ *and* $\Gamma_2 \subset \{\operatorname{Re} z > 0\} \cap \{\operatorname{Im} z < 0\} \cap D$. *Further take any two sequences of real numbers* $\{a_n\}$ *and* $\{K_n\}$ *such that*

(11) $a_n^2 > a_{n-1}a_{n+1}$ *for all* $n \geqq 1$ *and* $0 < a_n \uparrow 1$ *and*

(12) $K_n \geqq 1$ *for each* $n \geqq 1$ *and* $\lim_{n\to\infty} K_n = +\infty$.

Then there exists a function $f(z)$, regular in D, satisfying

(13) $|f(z)| \geqq K_n$ *on the circle* $|z| = a_n$ *for every* $n \geqq 1$ *and*

(14) $Z(f) \subset R$

where R denotes the bounded region determined by Γ_1, Γ_2 and the line segment $\{z = x + iy; x = 0, y_2 \leqq y \leqq y_1\}$.

Proof. Set $(a_{n+1} - a_n)/(a_{n+1} + a_n) = b_n$ and then clearly $1 > b_n \downarrow 0$. Now by virtue of (8), (9) and (10), we can choose θ_n $(n = 0, 1, 2, \cdots)$ so small that the region R includes two line segments $\{z = re^{i\theta_n}; 0 \leqq r \leqq a_{n+2}\}$, $\{z = re^{-i\theta_n}; 0 \leqq r \leqq a_{n+2}\}$ and the circular arc $\sigma(a_{n+2}; -\theta_n, \theta_n)$. Needless to say, we may assume that θ_n satisfies

$$0 < \theta_{n+1} < \theta_n < \frac{\pi}{2} \quad \text{and} \quad \tan\frac{\theta_n}{2} < b_{n+1}\,.$$

Now consider, as before, the annular sector $D_n = D(a'_{n-1}, a_{n+1}; \theta_{n-1})$ where $a'_{n-1} = a_n^2/a_{n+1}$ for each $n \geqq 1$. Moreover simply set $\sigma(a_n; -\theta_n, \theta_n) = \sigma_n, \sigma(a_n; \theta_n, \theta_{n-1}) \cup \sigma(a_n; -\theta_{n-1}, -\theta_n) = \alpha_n$ and $\{|z| = a_n\} - \sigma_n = \gamma_n$. Then making a slight modification of a standard technique of Bagemihl and Seidel ([2], [3], p. 181) based on Mergelyan's approximation theorem, we can construct a function $g(z)$, regular in D, such that

(15) $g(z) \neq 0$ in D and $|g(z)| \geqq 2K_n$ on γ_n for every $n \geqq 1$.

Next we choose $\{\varepsilon_n\}$ such that $\varepsilon_n > 0$ and $\sum_{n=1}^{\infty} \varepsilon_n = \varepsilon < \frac{1}{4}$. Then by Lemma, there is a rational function $p_1(z)$, with its only pole in the open line segment (a_1, a_2), such that

(16) $|p_1(z)| \geqq 2/s_1^2$ on σ_1 where $s_1 = \min\left\{1/2K_1, \min_{z\in\sigma_1}|g(z)|\right\}$,

(17) $\operatorname{Re} p_1(z) \geqq 0$ on α_1 and

(18) $|p_1(z)| \leqq \varepsilon_1$ on $\Omega_z - D_1$.

Our desire is, now, to approximate $p_1(z)$ by a regular function in $D - D_1$

minus a certain narrow region including the segment $[a_2, 1)$, pointed at $z = 1$. Since $p_1(z)$ has, fortunately, its only pole in the open segment (a_1, a_2), we can sweep out, as is seen in ([6], [3], p. 182), the poles to the boundary point $z = 1$, and consequently obtain a function $h_1(z)$, regular in D, satisfying

(16)′ $|h_1(z)| \geqq 1/s_1^2$ on σ_1,

(17)′ $\operatorname{Re} h_1(z) \geqq -\varepsilon_1$ on α_1 and

(18)′ $|h_1(z)| \leqq 2\varepsilon_1$ on $D - D_1 - \bigcup_{k=2}^{\infty} D(a_k, a_{k+1}; \theta_{k+1}) - \bigcup_{k=2}^{\infty} \sigma(a_k; -\theta_{k+1}, \theta_{k+1})$.

Now we shall inductively construct rational functions $p_n(z)$ and regular functions $h_n(z)$ as follows. Let $t_n = \sum_{k=1}^{n-1} \max\{|h_k(z)|; z \in \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \cdots \cup \sigma_n\}$. Then using Lemma again, we get a rational function $p_n(z)$, with its only pole in the open segment (a_n, a_{n+1}), such that

(19) $|p_n(z)| \geqq 2/s_n^2 + 2t_n$ on σ_n where $s_n = \min\left\{1/2K_n, \min\limits_{z \in \sigma_n} |g(z)|\right\}$,

(20) $\operatorname{Re} p_n(z) \geqq 0$ on α_n and

(21) $|p_n(z)| \leqq \varepsilon_n$ on $\Omega_z - D_n$.

Then as in the first step, we can find a function $h_n(z)$, regular in D, such that

(19)′ $|h_n(z)| \geqq 1/s_n^2 + t_n$ on σ_n,

(20)′ $\operatorname{Re} h_n(z) \geqq -\varepsilon_n$ on α_n and

(21)′ $|h_n(z)| \leqq 2\varepsilon_n$ on $D - D_n - \bigcup_{k=n+1}^{\infty} D(a_k, a_{k+1}; \theta_{k+1}) - \bigcup_{k=n+1}^{\infty} \sigma(a_k; -\theta_{k+1}, \theta_{k+1})$.

By virtue of (21)′ the series $\sum_{n=1}^{\infty} h_n(z)$ uniformly converges on any compact subset of D and hence we obtain a function $h(z) = 1 + \sum_{n=1}^{\infty} h_n(z)$, regular in D. Now consider the function

$$f(z) = g(z)h(z) .$$

Then using almost the same technique as is seen in ([3], p. 182–183), we can find that

$$|f(z)| > \frac{1}{s_n} - 2s_n \text{ on } \sigma_n \text{ and } |f(z)| \geqq \tfrac{1}{2}|g(z)| \text{ on } \gamma_n .$$

Consequently, from (15) and the definition of s_n stated in (19), we get that

$$|f(z)| \geqq K_n \quad \text{on } |z| = a_n .$$

As for the distribution of zeros of $f(z)$, remember that $g(z) \neq 0$ in D

and note that $\bigcup_{n=1}^{\infty} D_n \subset R$. Further, by virtue of (21)′, we have

$$|h(z)| > \tfrac{1}{2} \quad \text{in } D - \bigcup_{n=1}^{\infty} D_n .$$

Thus we see that $f(z)$ does not vanish outside of R.

Remark. According to a theorem of Bonar and Carrol ([5], p. 143), there exist no strongly annular functions, all zeros of which lie on the radius $[0, 1)$. Our theorem, however, shows that zeros of strongly annular functions can be distributed arbitrarily near the radius $[0, 1)$.

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