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ON THE EXISTENCE OF VARIOUS BOUNDED HARMONIC FUNCTIONS WITH GIVEN PERIODS, II

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1. Given a harmonic function u on a Riemann surface R, we define a *period function*

$$\Gamma_u(\gamma) = \int_\gamma *du$$

for every one-dimensional cycle γ of the Riemann surface R. $\Gamma_X(R)$ denote the totality of period functions Γ_u such that harmonic functions u satisfy a boundedness property X. As for X, we let B stand for boundedness, and D for the finiteness of the Dirichlet integral.

In our former paper [1] we showed that there exists a plane region Ω^* such that the inequality $\Gamma_B(\Omega^*) < \Gamma_D(\Omega^*)$ (strict inclusion) holds. On the contrary, we will show in the present paper that there exists a plane region Ω_* such that the inequality $\Gamma_D(\Omega_*) < \Gamma_B(\Omega_*)$ holds. Therefore we have the following

THEOREM. *There exist plane regions Ω^* and Ω_* such that $\Gamma_B(\Omega^*) < \Gamma_D(\Omega^*)$ and $\Gamma_D(\Omega_*) < \Gamma_B(\Omega_*)$.*

Let Ω denote the strip $\left\{z = x + yi;\, -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$ and 1_n denote the interval $[3n\ 3n + 1]$. We put

$$\Omega_* = \Omega - \bigcup_{n=-\infty}^{\infty} 1_n .$$

Let γ_n be a simple curve oriented clockwise enclosing 1_n so that γ_m and γ_n are disjoint if $m \neq n$. Then $\{\gamma_n\}_{n=-\infty}^{\infty}$ is a homology basis of the plane region Ω_*. A period function Γ_u is uniquely determined by values $\{\Gamma_u(\gamma_n)\}_{n=-\infty}^{\infty}$ and therefore, in order to study a period function Γ_u, it is sufficient that we pay attention only to values $\{\Gamma_u(\gamma_n)\}_{n=-\infty}^{\infty}$.

2. Let Δ_n denote the region $\Omega_* \cap \{z = x + yi; 3n - 1 < x < 3n + 2\}$

and ω_n denote the harmonic measure of the interval 1_n with respect to the region Δ_n.

LEMMA 1. *If a harmonic function u belongs to the class $HD(\Omega_*)$, then*

$$\int_{\gamma n} *du = (u, \omega_n); \tag{1}$$

$$\sum_{n=-\infty}^{\infty} \left| \int_{\gamma n} *du \right|^2 \leqslant D(\omega_0)D(u) . \tag{2}$$

Proof. Observe that $\omega_n(z) = \omega_0(z - 3n)$. Consider an exhaustion $\{\Delta_{n,m}\}_{m=1}^{\infty}$ of the region Δ_n. We may suppose that the region $\Delta_{n,m}$ are annuli, where we denote by $\alpha_{n,m}$ the boundary component of $\Delta_{n,m}$ the inside of which contains the interval 1_n and by $\beta_{n,m}$ the other boundary component. Let $\omega_{n,m}$ denote the harmonic measure of the curve $\alpha_{n,m}$ with respect to $\Delta_{n,m}$. Then

$$D(\omega_{n,m}) = \int_{\alpha_{n,m}} *d\omega_{n,m} \geq \int_{\alpha_{n,m}} *d\omega_{n,m+1} = \int_{\alpha_{n,m+1}} *d\omega_{n,m+1} = D(\omega_{n,m+1}) .$$

Hence the harmonic functions $\omega_{n,m}$ converge to the harmonic function ω_n in the CD-topology [2], and

$$\int_{\gamma n} *du = \int_{\alpha_{n,m}} *du = \int_{\alpha_{n,m}} \omega_{n,m} *du = (\omega_{n,m}, u) .$$

Therefore, letting $m \to \infty$, we obtain $\int_{n} *du = (u, \omega_n)$. Since

$$\left| \int_{\gamma n} *du \right|^2 = |(u, \omega_n)|^2 \leqslant D_{\Delta_n}(u)D(\omega_n) = D_{\Delta_n}(u)D(\omega_0) ,$$

we conclude that

$$\sum_{n=-\infty}^{\infty} \left| \int_{\gamma n} *du \right|^2 \leqslant \sum_{n=-\infty}^{\infty} D_{\Delta_n}(u)D(\omega_0) = D(u)D(\omega_0) .$$

3. Let b_n denote the harmonic measure of the interval 1_n with respect to the region Ω. The harmonic measure b_n has a property that $b_n(z) = b_0(z - 3n)$. We also consider the harmonic measure b_n as a potential, so that we have the following representation

$$b_n(z) = \int_{1_n} G(z, t)d\mu(t)$$

where the function $G(z, t) = \log\left|\frac{e^z + e^t}{e^z - e^t}\right|$ is the Green's function for the region Ω with pole at t.

LEMMA 2. *The function* $b(z) = \sum_{n=-\infty}^{\infty} b_n(z)$ *belongs to the class* $HB(\Omega_*)$.

Proof. Consider the function $b(z)$ on the region Ω. Then $b(z + 3n) = b(z)$. Therefore in order to prove the lemma it is sufficient to show that the function $b(z)$ is bounded on the region Δ_0. Since $b_0(z) = b_0(\frac{1}{2} - z)$, using the representation of the function b_n as a potential, we see, for $n \geq 1$, that

$$b_n(z) \leqslant b_n(3) = b_0(3 - 3n) = b_0(\tfrac{1}{2} - 3 + 3n) \leqslant b_0(3n - 3);$$
$$b_{-n}(z) \leqslant b_{-n}(-2) = b_0(-2 + 3n) \leqslant b_0(3n - 3) .$$

Hence

$$b(z) = \sum_{n=1}^{\infty} b_n(z) + \sum_{n=1}^{\infty} b_{-n}(z) + b_0(z) \leqslant 2 \sum_{n=1}^{\infty} b_0(3n - 3) + 1 .$$

We will show that $\sum_{n=1}^{\infty} b_0(3n - 3) < \infty$.

By the function $w = e^z$, the region is mapped onto the right half plane of the complex plane, and the interval 1_0 is mapped onto the interval $[1 \ e]$. The function $b_0(\log w)$ is the harmonic measure of the interval $[1 \ e]$ with respect to the right half plane. We put

$$u[1 \ e](w) = \int_1^e \log\left|\frac{w + t}{w - t}\right| dt .$$

Then by lemma 1 of [1], for any point x on the interval 1_0,

$$u[1 \ e](x) \geq (e - 1) \log (1 + e) \geq 1 .$$

Therefore, since

$$b_0(\log w) \leqslant [1 \ e](w) ,$$

we obtain

$$\sum_{n=1}^{\infty} b_0(3n - 3) \leqslant \sum_{n=2}^{\infty} u[1 \ e](e^{3n-3}) + 1$$
$$= \sum_{n=2}^{\infty} \int_1^e \log\left|\frac{e^{3n-3} + t}{e^{3n-3} - t}\right| dt + 1$$

$$= \sum_{n=2}^{\infty} \int_1^e \log \left| 1 + \frac{2t}{e^{3n-3} - t} \right| dt + 1$$

$$\leqslant \sum_{n=2}^{\infty} \int_1^e \frac{2e}{e^{3n-3} - e} dt + 1$$

$$= (e-1) \sum_{n=2}^{\infty} \frac{2e}{e^{3n-3} - e} + 1 < \infty .$$

4. Given a harmonic function u on the region Ω_*, we can construct a sequence $\{\Gamma_u(\gamma_n)\}_{n=-\infty}^{\infty}$. A period function Γ_u and the associated sequence can be considered identical. We will show that $\Gamma_B(\Omega_*) = \ell^\infty$ and $\Gamma_{BD}(\Omega_*) = \Gamma_D(\Omega_*) = \ell^2$. It follows that $\Gamma_D(\Omega_*) < \Gamma_B(\Omega_*)$.

LEMMA 3. $\Gamma_B(\Omega_*) = \ell^\infty$.

Proof. Put any function u belonging to the class $HB(\Omega_*)$. Suppose that $1 < u < M - 1$. Consider the set $\{z\,;\, tM\omega_n > u\}$. Then, for some $t, 0 < t < 1$, the set $\{z\,;\, tM\omega_n = u\}$ is a simple closed curve, which is denoted by δ_n and homologous to γ_n. Then

$$\int_{\gamma_n} *du = \int_{\delta_n} *du \leqslant \int_{\delta_n} *d(tM\omega_n) \leqslant M \int_{\gamma_n} *d\omega_n .$$

Also $1 < M - u < M - 1$. We obtain

$$|\Gamma_u(\gamma_n)| = \left| \int_{\gamma_n} *du \right| \leqslant M \int_{\gamma_n} *d\omega_n .$$

Hence $\Gamma_B(\Omega_*) \subset \ell^\infty$. Note that $\int_{\gamma_n} *db_n = \int_{\gamma_0} *db_0$, and we denote the common value $\int_{\gamma_0} *db_0$ by c.

Conversely, let $\{x_n\}$ be any sequence belonging to the space ℓ^∞. We consider

$$u = \sum_{n=-\infty}^{\infty} \frac{x_n}{c} b_n .$$

Then the function u belongs to the class $HB(\Omega_*)$ and

$$\int_{\gamma_n} *du = \frac{x_n}{c} \int_{\gamma_n} *db_n = x_n .$$

Hence $\ell^\infty \subset \Gamma_B(\Omega_*)$.

LEMMA 4. $\Gamma_{BD}(\Omega_*) = \Gamma_D(\Omega_*) = \ell^2$.

Rroof. It follows from Lemma 1 that $\Gamma_D(\Omega_*) \subset \ell^2$. Let $\{x_n\}_{n=-\infty}^{\infty}$ be any sequence belonging to the space ℓ^2. It follows from Lemma 3 that the function

$$u = \sum_{n=-\infty}^{\infty} \frac{x_n}{c} b_n$$

belongs to the class $HB(\Omega_*)$ and $\int_{\gamma n} * du = x_n$. Moreover

$$D\left(\sum_{n=-p}^{q} x_n b_n\right) = \sum_{-p \leq i,j,k \leq q} \int_{1_k} x_i b_i * d(x_j b_j) = \sum x_i x_j \int_{1_k} b_i * db_j \ .$$

Since $* db_j = 0$ on k_k if $k \neq j$, the last term of the above is equal to

$$\sum_{ij} x_i x_j \int_{1_j} b_i * db_j \ ,$$

$$\sum_{i,j=-\infty}^{\infty} |x_i||x_j| \int_{1_j} b_i * db_j$$
$$< \sum |x_i||x_j| \max_{t \in 1_j} b_i(t) \int_{1_j} * db_j$$
$$< c \sum_{k=0}^{\infty} \sum_{i=-\infty}^{\infty} |x_i||x_{i+k}| \max_{t \in 1_{i+k}} b_i(t) + |x_i||x_{i-k}| \max_{t \in 1_{i-k}} b_i(t)$$
$$< 2c \sum |x_i|^2 \sum_{k=0}^{\infty} \max_{t \in 1_k} b_0(t) + \max_{t \in 1_{-k}} b_0(t)$$
$$< 4c \sum |x_i|^2 \max b(z) \ .$$

Therefore the function u belongs to the class $HBD(\Omega_*)$, which proves the lemma.

REFERENCES

[1] Hara, M.: On the existence of various bounded harmonic functions with given periods, Nagoya Math. J. (to appear).
[2] Sario, L. and M. Nakai: Classification Theory of Riemann Surfaces. Springer (1970).

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