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ON THE ZEROS OF A CONFORMAL VECTOR FIELD

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1. Introduction

In [1] S. Kobayashi showed that the connected components of the set of zeros of a Killing vector field on a Riemannian manifold (M^n, g) are totally geodesic submanifolds of (M^n, g) of even codimension including the case of isolated singular points. The purpose of this short note is to give a simple proof of the corresponding result for conformal vector fields on compact Riemannian manifolds. In particular we prove the following

THEOREM. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 2$. Let F be the set of zeros of a conformal vector field ξ and let $F = \bigcup V_i$ where the V_i's are the connected components of F. Then each V_i is either an umbilical submanifold of (M^n, g) of even codimension including the case of isolated singular points or an isolated singular point of a conformal non-Killing vector field on a Euclidean sphere.*

The idea of our proof is to reduce the problem to Kobayashi's case by a simple application of a theorem of M. Obata characterizing a sphere to conformality. In Section 2 we discuss Obata's result and then prove our theorem in Section 3.

2. Preliminaries

A Riemannian metric $\bar{g}$ is said to be *conformal* to g if there exists a smooth function ρ on M^n such that $\bar{g} = e^{2\rho}g$. Let $f: M^n \to M^n$ be a diffeomorphism of M^n onto itself; we say f is a *conformal diffeomorphism* if f^*g is conformal to g.

Let $C(M^n, g)$ denote the Lie group of all conformal diffeomorphisms of (M^n, g) and $C_0(M^n, g)$ the connected component of the identity. A

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subgroup G of $C(M^n, g)$ is said to be *essential* if it does not become a group of isometries under any conformal change of metric, and a conformal vector field is said to be *essential* if its one-parameter group is essential. In [2] and [3] M. Obata obtained the following results.

THEOREM (Obata [2]). *Let (M^n, g) be a compact Riemannian manifold of dimension $n > 2$. Then $C_0(M^n, g)$ is essential if and only if (M^n, g) is conformally diffeomorphic to a Euclidean sphere.*

THEOREM (Obata [3]). *Let ξ be an essential conformal vector field on a Euclidean sphere. Then ξ has either exactly one or exactly two singular points.*

3. Proof of the Theorem

The proof of the theorem is, for n > 2, by cases using Obata's result. If $C_0(M^n, g)$ is inessential, then there exists a conformal change of metric, say $\bar{g} = e^{2\rho}g$, such that $C_0(M^n, g)$ is a group of isometries with respect to $\bar{g}$. Thus given a conformal vector field ξ on (M^n, g), ξ is Killing on $(M^n, \bar{g})$ and hence by Kobayashi's Theorem each V_i is a totally geodesic submanifold of $(M^n, \bar{g})$ of even codimension. Thus it remains only to show that V_i is umbilical in (M^n, g). To this end let ∇ and $\bar{\nabla}$ be the Riemannian connexions of g and $\bar{g}$ respectively and let $P = \operatorname{grad} \rho$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + (X\rho)Y + (Y\rho)X - g(X, Y)P . \tag{3.1}$$

Now let ι denote the imbedding of V_i in M^n. Considering V_i as a submanifold of (M^n, g) with g' and ∇' denoting the induced Riemannian metric and connexion, choose a local orthonormal frame $\eta_1, \cdots, \eta_k$ of normal vector fields on V_i, $k = \operatorname{codim} V_i$, and let h^α denote the corresponding second fundamental forms. Then the Gauss equation is

$$\nabla_{\iota_* X}\iota_* Y = \iota_*\nabla'_X Y + h^\alpha(X, Y)\eta_\alpha \tag{3.2}$$

summed on α. Considering V_i as a submanifold of $(M^n, \bar{g})$ with $\bar{\nabla}'$ denoting the induced Riemannian connexion, the Gauss equation is

$$\bar{\nabla}_{\iota_* X}\iota_* Y = \iota_*\bar{\nabla}'_X Y . \tag{3.3}$$

Thus using (3.1), (3.2) and (3.3) we have

$$\iota_*\nabla'_X Y + h^\alpha(X, Y)\eta_\alpha = \iota_*\bar{\nabla}'_X Y - (\iota_* X\rho)\iota_* Y - (\iota_* Y\rho)\iota_* X + g(\iota_* X, \iota_* Y)P .$$

Now taking the g inner product with η_β we have

$$h^\beta(X, Y) = (\eta_\beta \rho) g'(X, Y)$$

and hence that V_i is umbilical in (M^n, g).

If on the other hand $C_0(M^n, g)$ is essential then (M^n, g) is conformally diffeomorphic to a Euclidean sphere, but a conformal vector field remains conformal under a conformal change of metric. Thus if a conformal vector field ξ is essential, its zeros are isolated. If ξ is inessential then again there exists a conformal change of metric with respect to which ξ becomes a Killing vector field.

Finally if $n = 2$, there exist local isothermal parameters with respect to which (M^2, g) becomes a Hermitian (complex) manifold. If now ξ is a conformal vector field on (M^2, g), the conformal transformations of its oneparameter group are given by analytic functions. Thus by the identity theorem for analytic functions, their fixed points are isolated and hence the zeros of ξ are isolated

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