H. Morikawa
Nagoya Math. J.
Vol. 54 (1974), 179–189

ON A CANONICAL LIE ALGEBRA sl$(2r+2)$-BUNDLE OVER GRASS(n, r)

HISASI MORIKAWA

In the present note we shall construct a Lie algebra sl $(2r+2)$-bundle over Grass (n, r), canonically, which may be useful for theory of holomorphic vector bundles probably.

1. We use the following notations:

$U(m)$: unitary group of degree m,
$u(m)$: unitary Lie algebra of degree m,
$U(r+1, n+1) = \{w \,|\, (r+1) \times (n+1)\text{-matrices such that } w^t\overline{w} = I\}$,
$u(r+1, n+1) = \{(A, B) \,|\, (r+1) \times (n+1)\text{-matrices such that } A + {}^t\overline{A} = 0\}$,
T^*M: the cotangent bundle of M,
$\overline{T}^*M$: the complex conjugate bundle of T^*M,
$E(M)$: the exterior algebra bundle over M generated by $T^*M \oplus \overline{T}^*M$ for a complex manifold M.

We mean by the upper $(r+1)$-part of an $(n+1) \times (n+1)$-matrix

$$X = \begin{pmatrix} Y \\ Z \end{pmatrix} \begin{matrix} \}r+1 \\ \}n-r \end{matrix}$$

the $(r+1) \times (n+1)$-matrix Y, then $U(r+1, n+1)$ and $u(r+1, n+1)$ are regarded as the upper $(r+1)$-parts of $U(n+1)$ and $u(n+1)$, respectively. There exist the natural bundle structures

$$\begin{array}{l} U(n+1) \\ \downarrow \tilde{\pi} \\ U(r+1, n+1) = U(n-r)\backslash U(n+1) \end{array}$$

Received September 14, 1973.

$$\downarrow \pi$$

$$\begin{aligned}\text{Grass}(n, r) &= U(r+1)\backslash U(r+1, n+1) \\ &= U(r+1) \times U(n-r)\backslash U(n+1)\ .\end{aligned}$$

The space $U(r+1, n+1)$ is the canonical $U(r+1)$-bundle over complex Grassmann manifold $\text{Grass}(n, r)$.

An exponential map

$$\exp : u(r+1, n+1) \longrightarrow U(r+1, n+1)$$

is defined by

$$\exp(A, B) = \text{the upper } (r+1)\text{-part of } \exp\begin{pmatrix} A & B \\ -{}^t\bar{B} & 0 \end{pmatrix}. \tag{1}$$

We choose a positive number κ such that the projection π induces a homeomorphism of the submanifold

$$\{\exp(0, B) \mid \|B\| < \kappa\}$$

in $U(r+1, n+1)$ onto an open neighbourhood W of $\pi(I, 0)$ on $\text{Grass}(n, r)$. Then we get a system of system of real analytic local cross sections σ_α ($\alpha \in U(n+1)$) of $\text{Grass}(n, r)$ into $U(r+1, n+1)$ given by

$$\sigma_\alpha(\pi(\exp(0, B)\cdot\alpha)) = \exp(0, B)\alpha \qquad (\|B\| < \kappa, \alpha \in U(n+1))\ . \tag{2}$$

The local cross section σ_α is defined on the image W_α of W by the action α:

$$\sigma_\alpha : W\alpha \longrightarrow \sigma_\alpha(W\alpha)\ .$$

There exist real analytic maps

$$\tau_{\beta,\alpha} : W\alpha \cap W\beta \longrightarrow U(r+1) \qquad (\alpha, \beta \in U(n+1))$$

such that

$$\tau_{\gamma,\beta}\tau_{\beta,\alpha} = \tau_{\gamma,\alpha}\ , \qquad (\alpha, \beta, \gamma \in U(n+1))\ . \tag{3}$$

$$\sigma_\beta = \tau_{\beta,\alpha}\sigma_\alpha \tag{4}$$

2. Denoting by $w = (w_{i\ell})$ the system of coordinates on $U(r+1, n+1)$, then

$$\theta = w d^t\bar{w} \tag{5}$$

is a $U(n+1)$-invariant connection form on $U(r+1, n+1)$, because

$$\theta + {}^t\bar{\theta} = wd^t\overline{w} = d(w^t\overline{w}) = dI = 0$$

and

$$(w\alpha)d^t(\overline{w}\overline{\alpha}) = w\alpha^t\overline{\alpha}d^t\overline{w} = wd^t\overline{w} \qquad (\alpha \in U(n+1)) .$$

We mean by ω the curvature form of θ, i.e.

$$\omega = d\theta + \theta \wedge \theta . \tag{6}$$

LEMMA 1. *Putting* $w = (\overbrace{w^{(1)}}^{r+1}, \overbrace{w^{(2)}}^{n-r})$, *we have*

$$\theta_{(I,0)} = -dw^{(1)} = d^t\overline{w}^{(1)} , \tag{7}$$

$$\omega_{(I,0)} = dw^{(2)} \wedge d^t\overline{w}^{(2)} . \tag{8}$$

Proof. From the definitions it follows;

$$\begin{aligned}
dw^{(1)}d^t\overline{w}^{(1)} &= d(w^{(1)t}\overline{w}^{(1)} + w^{(2)t}\overline{w}^{(2)})_{(I,0)} = 0 , \\
\theta_{(I,0)} &= (I, 0)\begin{pmatrix} d^t\overline{w}^{(1)} \\ d^t\overline{w}^{(2)} \end{pmatrix} = d^t\overline{w}^{(1)} = -dw^{(1)} , \\
\omega_{(I,0)} &= (d\theta)_{(I,0)} + \theta_{(I,0)} \wedge \theta_{(I,0)} \\
&= dw^{(1)} \wedge d^t\overline{w}^{(1)} + dw^{(2)} \wedge d^t\overline{w}^{(2)} - dw^{(1)} \wedge d^t\overline{w}^{(2)} \\
&= dw^{(2)} \wedge d^t\overline{w}^{(2)} .
\end{aligned}$$

LEMMA 2.

$$\tau^*_{\beta,\alpha}(\omega) = \tau_{\beta,\alpha}\omega\tau^{-1}_{\beta,\alpha} \qquad (\alpha, \beta \in U(n+1)) . \tag{9}$$

Proof. Since $\tau^*_{\beta,\alpha}(w) = \tau_{\beta,\alpha}w$, it follows:

$$\begin{aligned}
\tau^*_{\beta,\alpha}(\theta) &= \tau^*_{\beta,\alpha}(w)d^t\overline{(\tau^*_{\beta\alpha}(w))} = \tau_{\beta,\alpha}wd({}^t\overline{w}\tau^{-1}_{\beta,\alpha}) \\
&= \tau_{\beta,\alpha}wd^t\overline{w}\tau^{-1}_{\beta,\alpha} - \tau_{\beta,\alpha}w^t\overline{w}\tau^{-1}_{\beta,\alpha}d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} \\
&= \tau_{\beta,\alpha}\theta\tau^{-1}_{\beta,\alpha} - d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} ,
\end{aligned}$$

and

$$\begin{aligned}
\tau^*_{\beta,\alpha}(\omega) &= \tau^*_{\beta,\alpha}(d\theta) + \tau^*_{\beta,\alpha}(\theta) \wedge \tau^*_{\beta,\alpha}(\theta) \\
&= d\tau_{\beta,\alpha} \wedge \theta\tau^{-1}_{\beta,\alpha} + \tau_{\beta,\alpha}d\theta\tau^{-1}_{\beta,\alpha} \\
&\quad + \tau_{\beta,\alpha}\theta \wedge \tau^{-1}_{\beta,\alpha}d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} - d\tau_{\beta,\alpha} \wedge \tau^{-1}_{\beta,\alpha}d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} \\
&\quad + \tau_{\beta,\alpha}\theta \wedge \theta\tau^{-1}_{\beta,\alpha} - \tau_{\beta,\alpha}\theta\tau^{-1}_{\beta,\alpha} \wedge d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} \\
&\quad - d\tau_{\beta,\alpha} \wedge \theta\tau^{-1}_{\beta,\alpha} + d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} \wedge d\tau_{\beta,\alpha}\tau^{-1}_{\beta.\alpha} \\
&= \tau_{\beta,\alpha}(d\theta + \theta \wedge \theta)\tau^{-1}_{\beta,\alpha} .
\end{aligned}$$

PROPOSITION 1. *Let $\omega^{(\alpha)}$ be the restriction of the curvature form on the local cross section $\sigma_\alpha(W\alpha)$ of* Grass(n, r) *in* $U(r+1, n+1)$. *Then it follows*:

$$\tau^*_{\beta,\alpha}(\omega^{(\beta)}) = \tau_{\beta,\alpha}\omega^{(\alpha)}\tau^{-1}_{\beta,\alpha} \qquad (\alpha, \beta \in U(n+1)) . \tag{10}$$

Proof. From Lemma 2 it follows

$$\begin{aligned}\tau^*_{\beta,\alpha}(\omega^{(\beta)}) &= \tau^*_{\beta,\alpha}(\omega|_{\sigma_\beta(W\beta)}) = \tau^*_{\beta,\alpha}(\omega)|_{\sigma_\alpha(W\alpha)} \\ &= (\tau_{\beta,\alpha}\omega\tau^{-1}_{\beta,\alpha})|_{\sigma_\alpha(W\alpha)} = \tau_{\beta,\alpha}\omega|_{\sigma_\alpha(W\alpha)}\tau^{-1}_{\beta,\alpha} = \tau_{\beta,\alpha}\omega^{(\alpha)}\tau^{-1}_{\beta,\alpha} .\end{aligned}$$

3. Let E be the exterior algebra generated by dw_{ip}, $d\overline{w}_{ip}$ $(0 \le i \le r; r+1 \le p \le n)$. We mean by $e(\xi)\eta$ and $i(\xi)\eta$ respectively the order and inner product $\xi \wedge \eta$ and the inner product of ξ with η with respect to the metric

$$2\sum_{i=0}^{r}\sum_{p=r+1}^{n}(dw_{ip}, d\overline{w}_{ip}) .$$

Since w_{ip} $(0 \le i \le r; r+1 \le p \le n)$ are independent complex variables, the inner and outer products satisfy

$$\begin{aligned}e(dw_{jp})i(dw_{jp}) + i(dw_{jp})e(dw_{jp}) &= \mathrm{id} , \\ e(d\overline{w}_{jp})i(d\overline{w}_{jp}) + i(d\overline{w}_{jp})e(d\overline{w}_{jp}) &= \mathrm{id} .\end{aligned}$$

Except these two cases $e(dw_{jp}), i(dw_{jp}), e(d\overline{w}_{jp}), i(d\overline{w}_{jp})$ $(0 \le j \le r; r+1 \le p \le n)$ are auti-commutative each other.

LEMMA 3.

$$\begin{aligned}&\left[\sum_{p=r+1}^{n} e(dw_{jp})e(d\overline{w}_{ip}), \sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{\ell p})\right] \\ &\quad = -\delta_{k\ell}\sum_{p=r+1}^{n} i(d\overline{w}_{\ell p})e(d\overline{w}_{ip}) + \delta_{i\ell}\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{kp}) ,\end{aligned} \tag{11}$$

$$\begin{aligned}&\left[\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{ip}), \sum_{p=r+1}^{n} e(dw_{\ell p})e(d\overline{w}_{kp})\right] \\ &\quad = \delta_{\ell i}\sum_{p=r+1}^{n} e(w_{jp})e(d\overline{w}_{kp}) ,\end{aligned} \tag{12}$$

$$\begin{aligned}&\left[\sum_{p=r+1}^{n} i(d\overline{w}_{jp})e(d\overline{w}_{ip}), \sum_{p=r+1}^{n} e(dw_{ip})e(d\overline{w}_{kp})\right] \\ &\quad = -\delta_{jk}\sum_{p=r+1}^{n} e(dw_{\ell p})e(d\overline{w}_{ip}) ,\end{aligned} \tag{13}$$

(14)
$$\left[\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{ip}),\ \sum_{p=r+1}^{n} i(dw_{kp})i(\overline{w}_{ip})\right] = -\delta_{jk} \sum_{p=r+1}^{n} e(dw_{ip})i(d\overline{w}_{\ell p}) ,$$

(15)
$$\left[\sum_{p=r+1}^{n} i(d\overline{w}_{jp})e(d\overline{w}_{ip}),\ \sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{\ell p})\right] = \delta_{\ell i} \sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{jp}) .$$

Proof. From the above remark it follows

$$\begin{aligned}
&\sum_{p,q=r+1}^{n} \{e(dw_{jp})e(d\overline{w}_{ip})i(dw_{kq})i(d\overline{w}_{\ell q}) - i(dw_{kq})i(d\overline{w}_{\ell q})e(dw_{jp})e(d\overline{w}_{ip})\} \\
&= \sum_{p=r+1}^{n} \{-e(dw_{jp})i(dw_{kp})e(d\overline{w}_{ip})i(d\overline{w}_{\ell p}) + i(dw_{kp})e(dw_{jp})i(d\overline{w}_{\ell p})e(d\overline{w}_{ip})\} \\
&= -\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{kp})\{e(d\overline{w}_{ip})i(d\overline{w}_{\ell p}) + i(d\overline{w}_{\ell p})e(d\overline{w}_{ip})\} \\
&\quad + \delta_{kj} \sum_{p=r+1}^{n} i(d\overline{w}_{\ell p})e(d\overline{w}_{ip}) \\
&= \delta_{kj} \sum_{p=r+1}^{n} i(d\overline{w}_{\ell p})e(d\overline{w}_{ip}) - \delta_{i\ell} \sum_{p=r+1}^{n} e(dw_{jp})i(dw_{kp}) ,
\end{aligned}$$

$$\begin{aligned}
&\sum_{p,q=r+1}^{n} \{e(dw_{jp})i(dw_{ip})e(dw_{\ell p})e(d\overline{w}_{kq}) - e(dw_{\ell q})e(d\overline{w}_{kq})e(dw_{jp})i(dw_{ip})\} \\
&= \sum_{p=r+1}^{n} e(dw_{jp})\{i(dw_{ip})e(dw_{\ell p}) + e(dw_{\ell p})i(dw_{ip})\}e(d\overline{w}_{kp}) \\
&= \delta_{i\ell} \sum_{p=r+1}^{n} e(dw_{jp})e(d\overline{w}_{kp}) ,
\end{aligned}$$

$$\begin{aligned}
&\sum_{p,q=r+1}^{n} \{i(d\overline{w}_{jp})e(d\overline{w}_{ip})e(dw_{\ell q})e(d\overline{w}_{kq}) - e(dw_{\ell q})e(d\overline{w}_{kq})i(d\overline{w}_{jp})e(d\overline{w}_{ip})\} \\
&= -\sum_{p=r+1}^{n} e(dw_{\ell p})\{i(d\overline{w}_{jp})e(d\overline{w}_{kp}) + e(d\overline{w}_{kp})i(d\overline{w}_{jp})\}e(d\overline{w}_{ip}) \\
&= -\delta_{kj} \sum_{p=r+1}^{n} e(dw_{\ell p})e(d\overline{w}_{ip}) ,
\end{aligned}$$

$$\begin{aligned}
&\sum_{p,q=r+1}^{n} \{e(dw_{jp})i(dw_{ip})i(dw_{kq})i(d\overline{w}_{\ell q}) - i(dw_{kq})i(d\overline{w}_{\ell q})e(dw_{jp})i(dw_{jp})\} \\
&= -\sum_{p=r+1}^{n} \{e(dw_{jp})i(dw_{kp}) + i(dw_{kp})e(dw_{jp})\}i(dw_{ip})i(d\overline{w}_{\ell p}) \\
&= -\delta_{jk} \sum_{p=r+1}^{n} i(dw_{ip})i(d\overline{w}_{\ell p}) ,
\end{aligned}$$

$$\sum_{p,q=r+1}^{n} \{i(d\overline{w}_{jp})e(d\overline{w}_{ip})i(dw_{kq})i(d\overline{w}_{\ell q}) - i(dw_{kq})i(d\overline{w}_{\ell q})i(d\overline{w}_{jp})e(d\overline{w}_{ip})\}$$
$$= \sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{ip})\{e(d\overline{w}_{ip})i(d\overline{w}_{\ell p}) + i(d\overline{w}_{\ell p})e(d\overline{w}_{ip})\}$$
$$= \delta_{i\ell} \sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{jp}) \ .$$

THEOREM 1. *Let L and Λ be the $(r+1)\times(r+1)$-matrices whose (i, j)-th entries are given by*

$$L_{ij} = \sqrt{-1} \sum_{p=r+1}^{n} e(dw_{ip})e(d\overline{w}_{jp})$$

and

$$\Lambda_{ij} = -\sqrt{-1} \sum_{p=r+1}^{n} i(dw_{ip})i(d\overline{w}_{jp}) \ .$$

Then there exists a representation ρ of Lie algebra sl $(2r+2)$ *such that*

$$\rho\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \operatorname{tr}(AL) = \sum_{i,j=0}^{r} a_{ij}L_{ji} \tag{16}$$

$$\rho\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \operatorname{tr}({}^tB\Lambda) = \sum_{i,j=0}^{r} b_{ij}\Lambda_{ij} \ . \tag{17}$$

Proof. Let ε_{ij} be the $(r+1)\times(r+1)$-matrix whose only non-zero entry is the (i, j)-th entry 1. We denote by ρ the linear mapping given by

$$\rho\begin{pmatrix} 0 & 0 \\ \varepsilon_{ij} & 0 \end{pmatrix} = L_{ij} \ ,$$
$$\rho\begin{pmatrix} 0 & \varepsilon_{ij} \\ 0 & 0 \end{pmatrix} = \Lambda_{ij} \ ,$$
$$\rho\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{ij} \end{pmatrix} = -\sum_{p=r+1}^{n} i(d\overline{w}_{jp})e(d\overline{w}_{ip}) \ ,$$
$$\rho\begin{pmatrix} \varepsilon_{ij} & 0 \\ 0 & 0 \end{pmatrix} = -\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{ip}) \ .$$

Then by virtue of (11), (12), (13), (14), (15) it follows

$$\left[\rho\begin{pmatrix} 0 & 0 \\ \varepsilon_{ij} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_{k} \\ 0 & 0 \end{pmatrix}\right] = [L_{ji}, \Lambda_{k\ell}] = \delta_{kj}\rho\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{i\ell} \end{pmatrix} - \delta_{i\ell}\rho\begin{pmatrix} \varepsilon_{kj} & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \rho\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{ij}\varepsilon_{k\ell} \end{pmatrix} - \rho\begin{pmatrix} \varepsilon_{k\ell}\varepsilon_{ij} & 0 \\ 0 & 0 \end{pmatrix} = \rho\left[\begin{pmatrix} 0 & 0 \\ \varepsilon_{ij} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_{k\ell} \\ 0 & 0 \end{pmatrix}\right] ,$$

In the previous paper [ ] we have proved the essentially same result.

$$\left[\rho\begin{pmatrix}\varepsilon_{ij} & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ \varepsilon_{k\ell} & 0\end{pmatrix}\right] = \rho\begin{pmatrix}-\varepsilon_{k\ell}\varepsilon_{ij} & 0\\ 0 & 0\end{pmatrix} = \rho\left[\begin{pmatrix}\varepsilon_{ij} & \\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ \varepsilon_{k\ell} & 0\end{pmatrix}\right],$$

$$\left[\rho\begin{pmatrix}0 & 0\\ 0 & \varepsilon_{ij}\end{pmatrix}, \rho\begin{pmatrix}0 & 0\\ \varepsilon_{k\ell} & 0\end{pmatrix}\right] = \rho\begin{pmatrix}-\varepsilon_{ij}\varepsilon_{k\ell} & 0\\ 0 & 0\end{pmatrix} = \rho\left[\begin{pmatrix}0 & 0\\ 0 & \varepsilon_{ij}\end{pmatrix}, \begin{pmatrix}0 & 0\\ \varepsilon_{k\ell} & 0\end{pmatrix}\right],$$

$$\left[\rho\begin{pmatrix}\varepsilon_{ij} & 0\\ 0 & 0\end{pmatrix}, \rho\begin{pmatrix}0 & \varepsilon_{k\ell}\\ 0 & 0\end{pmatrix}\right] = \rho\begin{pmatrix}0 & \varepsilon_{ij}\varepsilon_{k\ell}\\ 0 & 0\end{pmatrix} = \rho\left[\begin{pmatrix}\varepsilon_{ij} & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & \varepsilon_{k\ell}\\ 0 & 0\end{pmatrix}\right],$$

$$\left[\rho\begin{pmatrix}0 & 0\\ 0 & \varepsilon_{ij}\end{pmatrix}, \rho\begin{pmatrix}0 & \varepsilon_{k\ell}\\ 0 & 0\end{pmatrix}\right] = \rho\begin{pmatrix}0 & -\varepsilon_{k\ell}\varepsilon_{ij}\\ 0 & 0\end{pmatrix} = \rho\left[\begin{pmatrix}0 & 0\\ 0 & \varepsilon_{ij}\end{pmatrix}, \begin{pmatrix}0 & \varepsilon_{k\ell}\\ 0 & 0\end{pmatrix}\right],$$

This means that ρ gives a Lie algebra homomorphism of $(2r+2)$. For any $(r+1)\times(r+1)$ matrices A and B

$$\rho\begin{pmatrix}0 & 0\\ A & 0\end{pmatrix} = \sum_{i,j=0}^{r} a_{ij}\rho\begin{pmatrix}0 & 0\\ \varepsilon_{ij} & 0\end{pmatrix} = \sum_{i,j=0}^{r} a_{ij}L_{ji} = \operatorname{tr}(AL),$$

$$\rho\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix} = \sum_{i,j=0}^{r} b_{ij}\rho\begin{pmatrix}0 & \varepsilon_{ij}\\ 0 & 0\end{pmatrix} = \sum_{i,j=0}^{r} b_{ij}\Lambda_{ij} = \operatorname{tr}(B\Lambda).$$

4. We mean by $E(\sigma_\alpha(W\alpha))$ the exterior algebra bundle generated by $T^*\sigma_\alpha(W\alpha)\oplus\bar{T}^*\sigma_\alpha(\alpha)$. We mean by $\omega^{(\alpha)}$ the restriction of the curvature form ω on the local cross section $\sigma_\alpha(W\alpha)$ of Grass(n, r), and we define linear operators acting on $E(\sigma_\alpha(W\alpha))$ as follows:

(18) $$L_{ij}^{(\alpha)} = \sqrt{-1}\, e(\omega_{ij}^{(\alpha)})$$

(19) $$\Lambda_{ij}^{(\alpha)} = -\sqrt{-1}\, i_\alpha(\omega_{ij}^{(\alpha)}),$$

where $\omega_{ij}^{(\alpha)} = (\omega_{ij}^{(\alpha)})$ and $i_\alpha(\ )$ means the inner product with respect to the metric corresponding to $\operatorname{tr}\omega^{(\alpha)} = \sum_{i=0}^{r}\omega_{ii}^{(\alpha)}$.

LEMMA 5. *We denote*

(20) $$\tau_{\beta,\alpha}^*(L_{ij}^{(\beta)}) = \sqrt{-1}\, e(\tau_{\beta,\alpha}^*(\omega_{ij}^{(\beta)})),$$

(20) $$\tau_{\beta,\alpha}^*(\Lambda_{ij}^{(\beta)}) = \sqrt{-1}\, i_\alpha(\tau_{\beta,\alpha}^*(\omega_{ij}^{(\beta)})).$$

Then we have

(22) $$\tau_{\beta,\alpha}^*(L^{(\beta)}) = \tau_{\beta,\alpha}L^{(\alpha)}\tau_{\beta,\alpha}^{-1},$$

(23) $$\tau_{\beta,\alpha}^*(\Lambda^{(\beta)}) = {}^t\tau_{\beta,\alpha}^{-1}\Lambda^{(\alpha)}{}^t\tau_{\beta,\alpha} \qquad (\alpha, \beta\in U(n+1)).$$

Proof. Since $e(dw_{i\ell})$ and $i(dw_{i\ell})$ depend on $dw_{i\ell}$ $(0\le i\le r;\ 0\le\ell\le n)$ respectively covariantly and contravariantly, hence we have

$$e(\tau_{\beta,\alpha}\omega\tau_{\beta,\alpha}^{-1}) = e(\tau_{\beta,\alpha}\omega^t\bar{\tau}_{\beta,\alpha}) = \tau_{\beta,\alpha}e(\omega)^t\bar{\tau}_{\alpha,\beta} = \tau_{\beta,\alpha}e(\omega)\tau_{\beta,\alpha}^{-1} ,$$
$$i(\tau_{\beta,\alpha}\omega\tau_{\beta,\alpha}^{-1}) = i(\tau_{\beta,\alpha}\omega^t\bar{\tau}_{\beta,\alpha}) = {}^t\tau_{\beta,\alpha}^{-1}i(\omega)^t\tau_{\beta,\alpha} .$$

Therefore by virtue of the definitions of L and Λ, we have (26) and (27).

PROPOSITION 2. *There exists a representation* $\rho^{(\alpha)}$ *of* $\mathrm{sl}\,(2r+2)$ *as linear operators on* $E(\sigma_\alpha(W\alpha))$ *such that*

$$\rho^{(\alpha)}\begin{pmatrix}0 & 0\\ A & 0\end{pmatrix} = \mathrm{tr}\,(AL^{(\alpha)}) , \tag{24}$$

$$\rho^{(\alpha)}\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix} = \mathrm{tr}\,({}^tB\Lambda^{(\alpha)}) , \tag{25}$$

$$\tau_{\beta,\alpha}^*(\rho^{(\beta)}(X)\xi_\beta) = \rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1} \times \begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}\right)\tau_{\beta,\alpha}^*(\xi_\beta) , \tag{26}$$
$$(\alpha, \beta \in U(n+1)\,;\ X \in \mathrm{sl}\,(2r+2)) .$$

Proof. Since $\omega_{(I,0)} = (\sum_{p=r+1}^n dw_{ip} \wedge d\bar{w}_{jp})$ and $dw_{ip}, d\bar{w}_{ip}$ $(0 \leq i \leq r;\ r+1 \leq p \leq n)$ form a linear base of the fibre $T_{(I,0)}^*\sigma_e(W) \oplus \bar{T}_{(I,0)}^*\sigma_e(W)$, the representation ρ of $\mathrm{sl}\,(2r+2)$ in Theorem 1 is the representation of $\mathrm{sl}\,(2r+2)$ acting on the fibre of $E(\sigma_\alpha(W))$ at $(I, 0)$ which satisfies (22) and (23). Since $\mathrm{Grass}\,(n, r)$ is homogeneous for $U(n+1)$ and ω is $U(n+1)$-invariant, translating the fibre by α in $U(n+1)$, we get a representation $\rho^{(\alpha)}$ of $\mathrm{sl}\,(2r+2)$ acting on the fibre $E(\sigma_\alpha(W\alpha))$ at $\tilde{\pi}\alpha$ satisfying (22) and and (23), where $\tilde{\pi}\alpha$ means the upper $(r+1)$-part of α. For each point Z_β on $\sigma_\beta(W\beta)$ there exists an element α in $(U(n+1)$ such that $\tau_{\beta,\alpha}(\tilde{\pi}\alpha) = Z_\beta$. Hence it is sufficient to prove that, if there exists a representation $\rho^{(\alpha)}$ of $\mathrm{sl}\,(2r+2)$ acting on the fibre of $E(\sigma_\alpha(W))$ at Z_α satisfying (22) and (23), then for a point $Z_\beta = \tau_{\beta,\alpha}Z_\alpha$ on $\sigma_\beta(W\beta)$ there exists a representation $\rho^{(\beta)}$ of $\mathrm{sl}\,(2r+2)$ acting on the fibre $E(\sigma_\beta(W\beta))$ at Z_β satisfying (22), (23), (24). By virtue of (20) and (21) we have

$$\tau_{\beta,\alpha}^*(L^{(\beta)}) = \tau_{\beta,\alpha}L^{(\alpha)}\tau_{\beta,\alpha}^{-1}$$

and

$$\tau_{\beta,\alpha}^*(\Lambda^{(\beta)}) = {}^t\tau_{\beta,\alpha}^{-1}\Lambda^{(\alpha)t}\tau_{\beta,\alpha} ,$$

hence it follows:

$$\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ A & 0\end{pmatrix}\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}\right)$$
$$= \mathrm{tr}\,(\tau_{\beta,\alpha}^{-1}A\tau_{\beta,\alpha}L^{(\alpha)}) = \mathrm{tr}\,(A\tau_{\beta,\alpha}L^{(\alpha)}\tau_{\beta,\alpha}^{-1}) = \tau_{\beta,\alpha}^*\,\mathrm{tr}\,(AL^{(\beta)}) ,$$

$$\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1}\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}\right)$$
$$= \mathrm{tr}\,({}^t(\tau_{\beta,\alpha}^{-1}\beta\tau_{\beta,\alpha})A^{(\alpha)}) = \mathrm{tr}\,({}^tB\,{}^t\tau_{\beta,\alpha}^{-1}A^{(\alpha)}\,\tau_{\beta,\alpha}) = \tau_{\beta,\alpha}^*\,\mathrm{tr}\,(AL^{(\beta)})\ .$$

Since $\mathrm{sl}\,(2r+2)$ is generated by the elements

$$\left\{\begin{pmatrix}0 & 0\\ A & 0\end{pmatrix}, \begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\middle| A, B \text{ are } (r+1)\times(r+1) \text{ matrices}\right\}$$

we get a representation $\rho^{(\beta)}$ of $\mathrm{sl}\,(2r+2)$ acting on the fibre $E(\sigma_\beta(W\beta))$ at Z_β such that

$$\rho^{(\beta)}(X)\xi_\beta = \tau_{\beta,\alpha}^*\left(\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}\right)\tau_{\beta,\alpha}^*\xi_\beta\right)\ .$$

This representation $\rho^{0\beta)}$ satisfies, (22), (23) and (24). Morever the representation depends only on Z_β, which does not depend on the choise of α and Z_α, because by virtue of (8) and (03)

$$\tau_{\beta\gamma}\tau_{\gamma\alpha} = \tau_{\beta\alpha} \qquad (\alpha, \beta, \gamma \in U(n+1))$$

and

$$\tau_{\beta\alpha}\left\{\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}\right)\tau_{\beta,\alpha}^*\xi_\beta\right\}$$
$$= \tau_{\beta,\alpha}^*\tau_{\alpha,\gamma}^*\left\{\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta(\alpha}\end{pmatrix}^{-1}\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\gamma} & 0\\ 0 & \pi_{\beta,\gamma}\end{pmatrix}\right)\tau_{\beta,\alpha}^*(\tau_{\beta,\gamma}^*\xi_\beta)\right\}$$
$$= \tau_{\gamma,\beta}^*\left\{\rho^{(\gamma)}\begin{pmatrix}\tau_{\beta,\gamma} & 0\\ 0 & \tau_{\beta,\gamma}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\gamma} & 0\\ 0 & \tau_{\beta,\gamma}\end{pmatrix}\tau_{\beta,\gamma}^*\xi_\beta\right\}\ .$$

1.5. We can now construct a canonical $\mathrm{sl}\,(2r+2)$-Lie algebra bundle $L(n,r)$ which acts naturally on $E\,(\mathrm{Grass}\,(n,r))$.

We define an equivalence relation $\sim$ in

$$\bigcup_{\alpha\in U(n+1)} \sigma_\alpha(W\alpha)\times \mathrm{sl}\,(2r+2)$$

such that $(Z_\alpha, X_\alpha) \sim (Z_\beta, X_\beta)$ if and only if

$$\pi Z_\alpha = \pi Z_\beta$$

and

$$X_\beta = \begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix} \qquad (\alpha, \beta \in U(n+1))\ .$$

Since $\tau_{\gamma,\alpha}\tau_{\beta,\alpha} = \tau_{\gamma,\alpha}$ $(\alpha, \beta, \gamma \in U(n+1))$, the equivalence classes form a real analytic vector bundle

$$L(n,r) = \bigcup_{\alpha \in U(n+1)} \sigma_\alpha(W\alpha) \times \mathrm{sl}\,(2r+2)/\sim$$

over Grass(n,r). Let us show $L(n,r)$ acts naturally on $E\,(\mathrm{Grass}\,(n,r))$. The bundle $E\,(\mathrm{Grass}\,(n,r))$ may be expressed

$$E\,(\mathrm{Grass}\,(n,r)) = \bigcup_{\alpha \in U(n+1)} E(\sigma_\alpha(W\alpha))/\sim \ .$$

where $\xi_\alpha \sim \xi_\beta$ if and only if $\xi_\alpha = \tau_{\beta,\alpha}\xi_\beta$. Lie algebra sl$(2r+2)$ acts on $E(\sigma_\alpha W\alpha))$ as follows:

$$(X, \xi_\alpha) \longrightarrow \rho^{(\alpha)}(X)\xi_\alpha \ .$$

By virtue of (30) in Proposition 2 it follows

$$\tau^*_{\beta,\alpha}(\rho^{(\beta)}(X)\xi_\beta) = \rho^{(\alpha)}\left(\begin{pmatrix} \tau_{\beta,\alpha} & 0 \\ 0 & \tau_{\beta,\alpha} \end{pmatrix}^{-1} X \begin{pmatrix} \tau_{\beta,\alpha} & 0 \\ 0 & \tau_{\beta,\alpha} \end{pmatrix}\right)\tau^*_{\beta,\alpha}(\xi_\beta)$$
$$(\alpha, \beta \in U(n+1), \xi_\beta \in E(\sigma_\beta(W\beta)) \ .$$

This means that $L(n,r)$ acts on $E\,(\mathrm{Grass}\,(n,r))$.

We now concluded that:

THEOREM 2. *Let* $U(n-r)\backslash U(n+1) \xrightarrow{\pi} \mathrm{Grass}\,(n,r)$ *be the canonical* $U(r+1)$*-bundle over* Grass(n,r), *and let* ω *be the curvature form of the* $U(n+1)$*-invariant connections*

$$\theta = wd^t\overline{w} \ ,$$

where $U(n-r)\backslash U(n+1)$ *is regarded the space of* $(r+1)\times(n+1)$*-matrices* w *satisfying* $w\,^t\overline{w} = I$. *Let* $E\,(\mathrm{Grass}\,(n,r))$ *be the exterior algebra bundle over* Grass(n,r) *generated by*

$$T^*\,\mathrm{Grass}\,(n,r) \oplus \overline{T}^*\,\mathrm{Grass}\,(n,r) \ .$$

Then there exists a sl$(2r+2)$-Lie algebra bundle $L(n,r)$ over Grass(n,r) with the following properties:

i) The action of $U(n+1)$ on Grass(n,r) is lifted on $L(n,r)$.

ii) The $U(r+1)$-bundle $U(n-r)\backslash U(n+1)$ acts on $L(n,r)$ as follows

$$(\tau, \rho(X)) \longrightarrow \rho\left(\begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}^{-1} X \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}\right)$$

for local sections τ and ρ into $U(n-r)\backslash U(n+1)$ and $L(n,r)$, respectively.

iii) $L(n,r)$ acts on $E(\mathrm{Grass}(n,r))$ as follows:
Let E be the exterior algebra generated over the dual space of the horizontal space at a point Z on $U(n-r)\backslash U(n+1)$ with respect to the connection $wd^t\bar{w}$, and let L_{ij} and Λ_{ij} $(0 \leq i, j \leq r)$ be the linear operators actings on E given by

$$L_{ij} = \sqrt{-1}\, e(\tilde{\omega}_{ij}) \;, \qquad \Lambda_{ij} = -\sqrt{-1}\, i(\tilde{\omega}_{ij}) \;,$$

where $\tilde{\omega} = (\tilde{\omega}_{ij})$ is the restriction of the curvature form ω on the horizontal space and the inner product $i(\ )$ corresponds to $\mathrm{tr}\,\tilde{\omega}$. Then, identified E with the fibre of $E(\mathrm{Grass}(n,r))$ at the base point, of Z, the action of $L(n,r)$ on E is given by

$$\rho\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \mathrm{tr}\,(AL) = \sum_{i,j=0}^{r} a_{ij} L_{ji} \;,$$

$$\rho\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \mathrm{tr}\,({}^tB\Lambda) = \sum_{i,j=0}^{r} b_{ij} \Lambda_{ij} \;.$$

References

[1] H. Morikawa, On a canonical operation of sl(2m) on the exterior algebra of the vector space of complex $m \times n$-matrices, J. Math. Kyoto Univ. **B-1** (1973) 191–201.

Nagoya University