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A CHARACTERIZATION OF UNITARY OPERATORS INDUCED BY NONSINGULAR TRANSFORMATIONS AND ITS APPLICATIONS

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In this paper we give a necessary and sufficient condition for a unitary operator on an L^2-space to be induced by a nonsingular transformation and its applications.

§ 1. Introduction

Let (X, m) be a σ-finite measure space, $L^2(m)$ be the complex Hilbert space of all square summable functions and $L^\infty(m)$ be the algebra of all bounded measurable functions on (X, m). Then for every α in $L^\infty(m)$ we associate a bounded linear operator $T[\alpha]$ on $L^2(m)$ by

$$T[\alpha] : \xi(x) \to \alpha(x)\xi(x) , \qquad \xi \in L^2(m) .$$

As is well-known, the correspondence between $L^\infty(m)$ and

$$\Lambda(m) = \{T[\alpha] ; \alpha \in L^\infty(m)\}$$

is isomorphic and $\Lambda(m)$ is a commutative von Neumann algebra.

A transformation f of (X, m) is a *nonsingular transformation* if it satisfies the following conditions:

(N.1) There exists a null set N such that f is a bijective transformation of $X - N$ onto itself.
(N.2) f is bimeasurable.
(N.3) $m(E) = 0$ if and only if $m \circ f(E) = m(f(E)) = 0$.

We say $[f ; q]$ is a *nonsingular pair* if f is a nonsingular transformation and $q = q(x)$ is a complex measurable function such that

$$|q(x)|^2 = \frac{dm \circ f}{dm}(x) , \qquad \text{a.e.}(m) .$$

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For every nonsingular pair $[f; q]$ we define a unitary operator $U[f; q]$ on $L^2(m)$ by

$$U[f; q] : \xi(x) \to q(x)\xi(f(x)) , \qquad \xi \in L^2(m) .$$

In Section 2 we prove the following theorem:

THEOREM 1. *Let (X, m) be a σ-finite abstract Lebesgue space and U be a unitary operator on $L^2(m)$. Then there exists a nonsingular pair $[f; q]$ such that*

$$U = U[f; q]$$

if and only if

$$U^{-1}\Lambda(m)U \subset \Lambda(m) .$$

As applications of Theorem 1, we prove the following theorems in Section 3 and 4:

THEOREM 3. *Let (X, m) be a non-atomic σ-finite abstract Lebesgue space and U be a unitary operator on $L^2(m)$ such that $U - I$ is compact. Then U is induced by a nonsingular pair if and only if U is the identity operator.*

This theorem implies that any non-trivial finite dimensional unitary operator on such an L^2-space is never induced by a nonsingular pair.

THEOREM 5. *Let $[f; q]$ be a nonsingular pair of $(\boldsymbol{R}^1, dx)$ and $\mathfrak{F}$ be the Fourier transform. Furthermore, assume that $U[f; q]$ is a rotation of $\mathscr{S}$, the nuclear space of all rapidly decreasing functions on the real line. Then $\mathfrak{F}^{-1}U[f; q]\mathfrak{F}$ is again induced by a nonsingular pair if and only if $[f; q]$ is given by*

$$f(x) = \alpha x + \beta ,$$
$$q(x) = \sqrt{|\alpha|}\, e^{i(\theta x + \tau)} ,$$

where $\alpha\,(\neq 0)$, β, θ and τ are real constants.

This theorem enables us to construct two one-parameter unitary groups, one of which is induced by nonsingular pairs and the another is not, and still both of them have the same spectral type.

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§ 2. Unitary operators induced by nonsingular pairs

In this section we prove the following theorem:

THEOREM 1. *Let (X, m) be a σ-finite abstract Lebesgue space and U be a unitary operator on $L^2(m)$. Then there exists a nonsingular pair $[f\,;q]$ such that*

$$U = U[f\,;q]$$

if and only if

$$U^{-1}A(m)U \subset A(m)\ .$$

Let (X, m) be a σ-finite abstract Lebesgue space and $[f\,;q]$ be a nonsingular pair. Put $q^{\#}(x) = q(f^{-1}(x))^{-1}$. Then it is easy to show that $[f^{-1}; q^{\#}]$ is also a nonsingular pair and

$$U[f\,;q]^{-1} = U[f^{-1}; q^{\#}]\ .$$

Furthermore for every α in $L^\infty(m)$ we have

$$U[f\,;q]^{-1}T[\alpha]U[f\,;q] = U[f^{-1}; q^{\#}]T[\alpha]U[f\,;q] = T[\alpha \circ f^{-1}]$$

and this proves the necessity of Theorem 1.

To prove the sufficiency we make use of the following theorem owed to J. Dixmier.

THEOREM 2. *Let (X, m) be a σ-finite abstract Lebesgue space and Φ be an automorphism of the commutative von Neumann algebra $A(m)$. Then there exists a nonsingular transformation f of (X, m) such that*

$$\Phi(T[\alpha]) = T[\alpha \circ f^{-1}]$$

for every α in $L^\infty(m)$.

In J. Dixmier [1], Appendix IV, the above theorem is proved under the different assumption that X is a locally compact topological space with a countable basis and m is a positive Radon measure. On the other hand every abstract Lebesgue space is considered, measure-theoretically, as a pair of such a topological space and a Radon measure (H. Totoki [2]) and we have Theorem 2.

Proof of the sufficiency of Theorem 1. Let (X, m) be a σ-finite abstract Lebesgue space and U be a unitary operator on $L^2(m)$ such that

$$U^{-1}A(m)U \subset A(m) ,$$

that is, for every α in $L^\infty(m)$ we have

$$U^{-1}T[\alpha]U = T[\beta]$$

for some β in $L^\infty(m)$. Then it is obvious that

$$\Phi(T[\alpha]) = U^{-1}T[\alpha]U$$

is an automorphism of $A(m)$ and by Theorem 2 there exists a nonsingular transformation f of (X, m) such that

$$\Phi(T[\alpha]) = T[\alpha \circ f^{-1}]$$

for every α in $L^\infty(m)$.

Meanwhile let V be a unitary operator on $L^2(m)$ which is induced by a nonsingular pair $\left[f ; \sqrt{\frac{dm \circ f}{dm}}\right]$ of (X, m). Then by a simple estimation we have for every α in $L^\infty(m)$

$$V^{-1}T[\alpha]V = T[\alpha \circ f^{-1}] .$$

Consequently we have

$$V^{-1}T[\alpha]V = U^{-1}T[\alpha]U$$

and

$$UV^{-1}T[\alpha] = T[\alpha]UV^{-1}$$

for every α in $L^\infty(m)$. On the other hand, it is well-known that $A(m)$ is a maximal abelian von Neumann algebra, that is, the von Neumann algebra of all bounded operators on $L^2(m)$ that commute with every operator in $A(m)$. Therefore UV^{-1} belongs to $A(m)$ and there exists γ in $L^\infty(m)$ such that

$$UV^{-1} = T[\gamma] .$$

Since UV^{-1} is a unitary operator, we have $|\gamma(x)| \equiv 1$ almost everywhere. Define

$$q(x) = \gamma(x)\sqrt{\frac{dm \circ f}{dm}(x)} .$$

Then it is easy to verify that $[f ; q]$ is a nonsingular pair of (X, m) and we have

$$U = T[\gamma]V = T[\gamma]U\left[f\,;\ \sqrt{\frac{dm\circ f}{dm}}\right] = U[f\,;\,q]\ .$$

This completes the proof of Theorem 1.

NOTE. In Theorem 1, the criterion is substituted by

$$U^{-1}\mathfrak{A}U \subset A(m)\ ,$$

where $\mathfrak{A}$ is a set of unitary operators on $L^2(m)$ which generates $A(m)$ as a von Neumann algebra.

In particular, if (X, m) is the real line $(\boldsymbol{R}^1, dx)$ or the unit interval $([0,1], dx)$, then it is obvious that $A(dx)$ is generated from a one-parameter unitary group $\{E[t]\,;\, t\in\boldsymbol{R}^1\}$ defined by

$$E[t]\xi(x) = e^{itx}\xi(x)\ , \qquad \xi\in L^2(\boldsymbol{R}^1, dx)\ ,$$

or from a single operator $E[2\pi]$ defined by

$$E[2\pi]\xi(x) = e^{2\pi ix}\xi(x)\ , \qquad \xi\in L^2([0,1], dx)\ ,$$

respectively. Hence we have the following corollaries.

COROLLARY 1. *Let U be a unitary operator on $L^2(\boldsymbol{R}^1, dx)$. Then U is induced by a nonsingular pair of $(\boldsymbol{R}^1, dx)$ if and only if*

$$U^{-1}E[t]U \in A(\boldsymbol{R}^1, dx)$$

for every t in $\boldsymbol{R}^1$.

COROLLARY 2. *Let U be a unitary operator on $L^2([0,1], dx)$. Then U is induced by a nonsingular pair of $([0,1], dx)$ if and only if*

$$U^{-1}E[2\pi]U \in A([0,1], dx)\ .$$

For example, let $\mathfrak{F}$ be the Fourier transform of $L^2(\boldsymbol{R}^1, dx)$ defined by

$$(\mathfrak{F}\xi)(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\lambda x}\xi(x)dx\ , \qquad \xi\in L^2(\boldsymbol{R}^1, dx)\ .$$

Then $\mathfrak{F}$ is not induced by any nonsingular pair of $(\boldsymbol{R}^1, dx)$. In fact, assume that $\mathfrak{F}$ is induced by a nonsingular pair. Then by Corollary 1 there exists a real measurable function $h(x)$ such that

$$(\mathfrak{F}^{-1}E[t]\mathfrak{F}\xi)(x) = e^{ith(x)}\xi(x)\ , \qquad \xi\in L^2(\boldsymbol{R}^1, dx)$$

for every t in $\boldsymbol{R}^1$ and almost all x in $\boldsymbol{R}^1$. On the other hand, it is well-known that $\mathfrak{F}^{-1}E[t]\mathfrak{F}$ is a shift transformation. Thus we have

$$|\xi(x+t)| = |\xi(x)|$$

for every t in $\boldsymbol{R}^1$ and $\xi \in L^2(\boldsymbol{R}^1, dx)$, but this is impossible.

§ 3. In case $U-I$ is a compact operator

In this section, we prove the following theorem:

THEOREM 3. *Let (X, m) be a non-atomic σ-finite abstract Lebesgue space and U be a unitary operator on $L^2(m)$ such that $U-I$ is a compact operator. Then U is induced by a nonsingular pair if and only if U is the identity operator I.*

Proof. Since the sufficiency is trivial, we prove the necessity. Let $U = U[f; q]$ be a unitary operator on $L^2(m)$ which is induced by a nonsingular pair $[f; q]$ and $U-I$ be a compact operator. Then, by Theorem 1, it follows that for every α in $L^\infty(m)$, there exists β in $L^\infty(m)$ such that

$$U^*T[\alpha]U = T[\beta] .$$

Therefore we have

$$T[\beta-\alpha] = (U^*-I)T[\alpha] + T[\alpha](U-I) + (U^*-I)T[\alpha](U-I)$$

and consequently $T[\beta-\alpha]$ is a compact operator. Since m is non-atomic, the multiplication operator $T[\beta-\alpha]$ is compact if and only if $\beta-\alpha=0$. This implies that

$$U^*T[\alpha]U = T[\alpha]$$

for every α in $L^\infty(m)$ and consequently U commutes to all multiplication operators $T[\alpha]$, $\alpha \in L^\infty(m)$. Since a bounded operator which commutes to all multiplication operators $T[\alpha]$, $\alpha \in L^\infty(m)$, is itself a multiplication operator, there exists γ in $L^\infty(m)$ such that

$$U = T[\gamma] .$$

By the assumption, the operator $T[\gamma-1] = U-I$ is compact and this implies that U is the identity operator I.

This completes the proof.

We say a unitary operator U on a Hilbert space $\mathscr{H}$ is *finite dimensional* if there exists a finite dimensional subspace $\mathscr{K}$ of $\mathscr{H}$ such that $U\mathscr{K} = \mathscr{K}$ and $U = I$ on $\mathscr{H} \ominus \mathscr{K}$.

Since it is clear that if U is a finite dimensional unitary operator, then $U - I$ is a compact operator, we have the following corollary of Theorem 3.

COROLLARY 3. *Let (X, m) be a non-atomic σ-finite abstract Lebesgue space and U be a finite dimensional unitary operator on $L^2(m)$. Then U is induced by a nonsingular pair if and only if U is the identity operator I.*

Let U be a unitary operator on a separable Hilbert space $\mathscr{H}$ and $\{\xi_i\}_{i=1,2,\dots}$ be an orthonormal basis in $\mathscr{H}$. Representing U as an infinite dimensional unitary matrix (α_{ij}) by the orthonormal basis $\{\xi_i\}$, where $\alpha_{ij} = (U\xi_i, \xi_j)$, $i, j = 1, 2, \cdots$, it is clear that $U - I$ is a Hilbert-Schmidt operator if and only if

$$\sum_{i,j=1}^{\infty} |\alpha_{ij} - \delta_{ij}|^2 < +\infty \ .$$

Therefore we have the following corollary.

COROLLARY 4. *Let (X, m) be a non-atomic σ-finite abstract Lebesgue space and U be a unitary operator on $L^2(m)$ which is represented by a unitary matrix (α_{ij}) such that*

$$\sum_{i,j=1}^{\infty} |\alpha_{ij} - \delta_{ij}|^2 < +\infty \ .$$

Then U is not induced by a nonsingular pair unless U is the identity operator I.

§ 4. Conjugation by Fourier transform

We say a unitary operator U of $L^2(\boldsymbol{R}^1, dx)$ is a *rotation* of $\mathscr{S}$, the nuclear space of all rapidly decreasing functions on the real line, if the restriction of U to $\mathscr{S}$ is a homeomorphism of $\mathscr{S}$ and denote the group of all rotations of $\mathscr{S}$ by $U(\mathscr{S})$. It is well-known that the Fourier transform $\mathfrak{F}$ is a rotation of $\mathscr{S}$.

We say a nonsingular pair $[f; q]$ of $(\boldsymbol{R}^1, dx)$ is *admissible* if $U[f; q]$ is a rotation of $\mathscr{S}$. Then we have the following theorem:

THEOREM 4. (*H. Sato* [3], *Theorem* 2) *A nonsingular pair* $[f; q]$ *of* $(\boldsymbol{R}^1, dx)$ *is admissible if and only if it satisfies the following three conditions*:

(R.1) $q(x)$ *is a slowly increasing function.*

(R.2) *There exists a positive number* γ *such that*

$$\inf_x (1+|x|^\gamma)\,|q(x)| > 0.$$

(R.3) $f(x)$ *is a continuous function and there exists a positive number* α *such that*

$$\lim_{|x|\to\infty} \frac{|f(x)|}{|x|^\alpha} = +\infty .$$

We say a function on the real line is *slowly increasing* if it is infinitely differentiable and each of the derivatives is slowly increasing. (R.1) and (R.3) implies that $f(x)$ is a slowly increasing function.

In [3], we consider the following two subgroups of $U(\mathscr{S})$. One is

$$\mathfrak{G} = \{U[f; q];\ [f; q] \text{ is admissible}\}$$

and another is

$$\tilde{\mathfrak{G}} = \{\tilde{U} = \mathfrak{F}^{-1}U\mathfrak{F};\ U \in \mathfrak{G}\} .$$

The intersection of them is not empty (Example 1~4 of [3]) and in the following theorem, we determine the intersection explicitly.

THEOREM 5. *Let* $[f; q]$ *be an admissible nonsingular pair of* $(\boldsymbol{R}^1, dx)$ *and* $\mathfrak{F}$ *be the Fourier transform. Then* $\tilde{U}[f; q] = \mathfrak{F}^{-1}U[f; q]\mathfrak{F}$ *is again induced by a nonsingular pair if and only if* $[f; q]$ *is given by*

$$f(x) = \alpha x + \beta ,$$
$$q(x) = \sqrt{|\alpha|}\, e^{i(\theta x+\tau)} ,$$

where $\alpha(\neq 0)$, β, θ *and* τ *are real constants.*

Proof. If a nonsingular pair $[f; q]$ is given as above, then we have

$$\begin{aligned}\tilde{U}[f; q]\xi(x) &= \mathfrak{F}^{-1}\sqrt{|\alpha|}\, e^{i(\theta\lambda+\tau)}(\mathfrak{F}\xi)(\alpha\lambda + \beta)\\ &= \frac{\sqrt{|\alpha|}\, e^{i\tau}}{\sqrt{2\pi}} \int e^{i\lambda(x+\theta)}(\mathfrak{F}\xi)(\alpha\lambda + \beta)d\lambda\\ &= \frac{\sqrt{|\alpha|}\, e^{i\tau}}{\sqrt{2\pi}} \int \exp\left[i\frac{\lambda-\beta}{\alpha}(x+\theta)\right](\mathfrak{F}\xi)(\lambda)\frac{d\lambda}{\alpha}\end{aligned}$$

$$= \frac{\sqrt{|\alpha|}}{\alpha} \exp\left[-i\frac{\beta}{\alpha}x + i\left(\tau - \frac{\beta\theta}{\alpha}\right)\right]\xi\left(\frac{x+\theta}{\alpha}\right)$$
$$= U[f_1\,;\, q_1]\xi(x)\ ,$$

where $[f_1\,;\, q_1]$ is a nonsingular pair defined by

$$f_1(x) = \frac{x+\theta}{\alpha}$$
$$q_1(x) = \frac{\sqrt{|\alpha|}}{\alpha} \exp\left[-i\frac{\beta}{\alpha}x + i\left(\tau - \frac{\beta\theta}{\alpha}\right)\right]$$

and this proves the sufficiency of the theorem.

To prove the necessity, assume that $\tilde{U} = \tilde{U}[f\,;\, q]$ is induced by a nonsingular pair. Then by Corollary 1, there exists a real measurable function $H = H(x)$ such that

$$\tilde{U}^{-1}E[t]\tilde{U}\xi(x) = \mathfrak{F}^{-1}U[f^{-1}\,;\, q^{\#}]\mathfrak{F}E[t]\mathfrak{F}^{-1}U[f\,;\, q]\mathfrak{F}\xi(x) = e^{itH(x)}\xi(x)$$

for every ξ in $\mathscr{S}$. Since $[f\,;\, q]$ is admissible, all these operations are homeomorphisms of $\mathscr{S}$ and by Theorem 4, $H(x)$ is infinitely differentiable function with slowly increasing derivatives. Considering $\mathfrak{F}E[t]\mathfrak{F}^{-1}$ is the shift $S_t : \xi(x) \to \xi(x - t)$, we have

$$\frac{q(f^{-1}(x) - t)}{q(f^{-1}(x))}\tilde{\xi}(f(f^{-1}(x) - t)) = \mathfrak{F}(e^{itH}\xi)$$

for every t in $\boldsymbol{R}^1$, where $\tilde{\xi} = \mathfrak{F}\xi$. Differentiating both sides at $t = 0$, we have

$$-\frac{q'(f^{-1}(x))}{q(f^{-1}(x))}\tilde{\xi}(x) - f'(x)\tilde{\xi}'(x) = \mathfrak{F}(iH\xi)$$
$$= (T_H * \tilde{\xi})(x) = T_H(\tilde{\xi}(x - \cdot))\ , \qquad \xi \text{ in } \mathscr{S}\ ,$$

where T_H is the Fourier transform of $\frac{1}{\sqrt{2\pi}}iH(x)$ in the distribution sense. Since the left side is a value of a distribution supported by a single point x and of order 1, T_H is a distribution supported by the origin of order 1 and independent of the choice of x. Therefore we have

$$T_H = -\alpha\delta' - \gamma\delta\ ,$$

where δ is the Dirac measure and α and γ are complex constants. Consequently we have

$$f'(x) \equiv \alpha \, , \qquad \frac{q'(f^{-1}(x))}{q(f^{-1}(x))} \equiv \gamma$$

and therefore

$$f(x) = \alpha x + \beta \, , \qquad q(x) = \rho e^{\gamma x} \, .$$

Since $[f\,;q]$ is a nonsingular pair, we have

$$|q(x)|^2 = |\rho|^2 |e^{\gamma x}|^2 = |f'(x)| \equiv |\alpha| \neq 0 \, ,$$

and γ is a pure imaginary number, say,

$$\gamma = i\theta \, , \qquad \theta \in \boldsymbol{R}^1$$

and

$$\rho = \sqrt{|\alpha|}\, e^{i\tau} \, , \qquad \tau \in \boldsymbol{R}^1 \, .$$

Thus we have

$$f(x) = \alpha x + \beta \, , \qquad q(x) = \sqrt{|\alpha|}\, e^{i(\theta x + \tau)}$$

and this completes the proof.

Theorem 5 enables us to construct two one-parameter unitary groups, one of which is induced by nonsingular pairs and the another is not, and still both of them have the same spectral type. For example, one-parameter unitary groups $\{U[x\,; e^{itx^3}]\}$ and $\{\tilde{U}[x\,; e^{itx^3}]\}$ are of simple Lebesgue spectrum. But the former is induced by nonsingular pairs and the latter is not.

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