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Nagoya Math. J.
Vol. 53 (1974), 127-135

# ON THE THEOREM OF KISHI FOR A CONTINUOUS FUNCTION-KERNEL

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## 1. Introduction

In the potential theory with respect to a non-symmetric function-kernel, the following theorem is obtained by M. Kishi ([3]).

Let $X$ be a locally compact Hausdorff space and $G$ be a lower semi-continuous function-kernel on $X$. Assume that $G(x, x)>0$ for any $x$ in $X$ and that $G$ and the adjoint kernel $\check{G}$ of $G$ satisfy "the continuity principle". Then the following four statements are equivalent.

(1) $G$ satisfies the domination principle.

(2) $\check{G}$ satisfies the domination principle.

(3) $G$ satisfies the balayage principle.

(4) $\check{G}$ satisfies the balayage principle.

In the class of lower semi-continuous function-kernels on $X$, the subclass of continuous function-kernels on $X$ is essential for the continuity principle. We remark that the continuity principle follows from a certain maximum principle.

A function $G$ defined in the product space $X \times X$ is called a continuous function-kernel if $G$ is non-negative, continuous in the extended sense and finite outside the diagonal set of $X \times X$.

In this paper, we shall prove the continuity principle for $\check{G}$ follows from the domination principle for $G$ under a certain additional condition. On the other hand, it is well-known that the domination principle for $G$ implies the continuity principle for $G$ itself. Using our result, we shall obtain that the above theorem is valid in the case of a continuous function-kernel on $X$ without the assumption that $G$ and $\check{G}$ satisfy the continuity principle. In the proof, we shall use a result of one of the authors (cf. [2]), and the following proposition will be essential in our proof.

Received August 8, 1973.

Let $G$ be a continuous function-kernel on $X$ satisfying the domination principle. Assume that every point of $X$ is non-isolated and that $G(x, x) > 0$ for any $x$ in $X$ and for any non-empty open set $\omega$ in $X$, the $G$-capacity of $\omega$ is positive. Then $G(x, y) > 0$ in $X \times X$.

We remark that M. Kishi first proved the above equivalence for a strictly positive function-kernel (cf. [3]).

## 2. Preliminaries

Let $X$ be a locally compact Hausdorff space and $\Delta$ denote the diagonal set of the product space $X \times X$. A function $G$ defined everywhere in $X \times X$ is called a function-kernel if $G$ is non-negative and Borel measurable ([2]). The function-kernel $\check{G}$ on $X$, defined by $\check{G}(x, y) = G(y, x)$ for any $(x, y)$ in $X \times X$, is called the adjoint kernel of $G$. For a positive Radon measure $\mu$ in $X$, the potential $G\mu$ and the adjoint potential $\check{G}\mu$ of $\mu$ are defined by

$$G\mu(x) = \int G(x, y)d\mu(y) \quad \text{and} \quad \check{G}\mu(x) = \int \check{G}(x, y)d\mu(y) .$$

These are Borel functions on $X$ and $0 \leqq G\mu(x) \leqq +\infty$, $0 \leqq \check{G}\mu(x) \leqq +\infty$ in $X$. The $G$-energy of $\mu$ is defined by $\int G\mu d\mu$. Evidently the $G$-energy of $\mu$ is equal to the $\check{G}$-energy of $\mu$.

We denote by $M_0$ the family of all positive Radon measures in $X$ with compact support and by $E_0 = E_0(G)$ the subfamily of $M_0$ constituted by positive Radon measures with finite $G$-energy. We have evidently $E_0(G) = E_0(\check{G})$.

For a compact set $K$ in $X$, we set

$$\operatorname{cap}_G(K) = \sup\left\{\mu(K) ; \mu \in E_0, S\mu \subset K, \int G\mu d\mu \leqq \int d\mu\right\} ,$$

where $S\mu$ denotes the support of $\mu$. For a subset $A$ of $X$, we denote by $\operatorname{cap}_G(A)$ the quantity $\sup\{\operatorname{cap}_G(K) ; K: \text{compact} \subset A\}$, and we call it the $G$-capacity of $A$[1]. Evidently $\operatorname{cap}_G(A) = \operatorname{cap}_{\check{G}}(A)$. For a subset $A$ of $X$, $\operatorname{cap}_G(A) = 0$ if and only if $\{\mu \in E_0 ; S\mu \subset A\} = \{0\}$. We say that a property holds $G$-*p.p.p.* on a subset $A$ of $X$ if $\operatorname{cap}_G(B) = 0$, where $B$ is the set of points in $A$ where the property fails to hold.

Let us define the potential theoretical principles for a function-kernel $G$ on $X$.

[1] This is usually called the inner $G$-capacity of $A$.

(I) $G$ satisfies the restrained domination principle if, for $\mu, \nu$ in $E_0$, $G\mu(x) \leqq G\nu(x)$ $G$-$p.p.p.$ in $X$ whenever $G\mu(x) \leqq G\nu(x)$ $G$-$p.p.p.$ on $S\mu$.

PROPOSITION 1 ([2]). *Let $G$ be a strictly positive function-kernel on $X$. Then $G$ satisfies the restrained domination principle if and only if $\check{G}$ satisfies it.*

(II) $G$ satisfies the domination principle if, for $\mu$ in $E_0$ and $\nu$ in $M_0$, an inequality $G\mu(x) \leqq G\nu(x)$ on $S\mu$ implies the same inequality in the whole space.

PROPOSITION 2. *If $G$ satisfies the domination principle, then $G$ satisfies the restrained domination principle.*

In fact, if for $\mu, \nu$ in $E_0, G\mu(x) \leqq G\nu(x)$ $G$-$p.p.p.$ on $S\mu$, then there exists an increasing sequence $(K_n)_{n=1}^{\infty}$ of compact sets contained in $S\mu$ such that $\mu(CK_n) \downarrow 0$ with $n \uparrow +\infty$ and $G\mu(x) \leqq G\nu(x)$ on $K_n$. Then, by the domination principle for $G$, $G\mu_n(x) \leqq G\nu(x)$ everywhere in $X$, where $\mu_n$ is the restriction of $\mu$ to $K_n$. Letting $n \uparrow +\infty$, we have $G\mu(x) \leqq G\nu(x)$ in $X$.

(III) $G$ satisfies the balayage principle if, for a given compact set $K$ in $X$ and a given $\mu$ in $M_0$, there exists a positive Radon measure $\mu'$ in $M_0$ supported by $K$ such that

$$G\mu'(x) \leqq G\mu(x) \quad \text{on } X \quad \text{and} \quad G\mu'(x) = G\mu(x) \quad G\text{-}p.p.p. \quad \text{on } K .$$

This measure $\mu'$ is called a $G$-balayaged measure of $\mu$ on $K$.

PROPOSITION 3. *If $G$ satisfies the balayage principle, then $\check{G}$ satisfies the domination principle.*

This can be proved in the same way as in [3].

(IV) $G$ satisfies the complete maximum principle if for $\mu$ in $E_0$ and $\nu$ in $M_0$, an inequality $G\mu(x) \leqq G\nu(x) + 1$ on $S\mu$ implies the same inequality in the whole space.

It is evident that the complete maximum principle for $G$ implies the domination principle for $G$.

(V) $G$ satisfies the classical maximum principle if for $\mu$ in $M_0$, an inequality $G\mu(x) \leqq 1$ on $S\mu$ implies the same inequality on $X$.

(IV) $G$ satisfies the continuity principle if for a $\mu$ in $M_0$, the finite continuity of the restriction of $G\mu$ to $S\mu$ implies that $G\mu$ is finite continuous in the whole space.

When we discuss the continuity principle, it is natural to assume that a function-kernel is lower semi-continuous or continuous in the extended sense. A function-kernel $G$ on $X$ is said to be lower semi-continuous if $G$ is a lower semi-continuous function in $X \times X$. $G$ is called a continuous function-kernel on $X$ if $G$ is continuous in the extended sense in $X \times X$ and $G(x, y) < +\infty$ for any $(x, y)$ in $X \times X - \Delta$. The following proposition is well-known and can be proved by the same way as in the classical case (i.e., the continuity principle for the Newton kernel).

PROPOSITION 4. *If a continuous function-kernel $G$ on $X$ satisfies the domination principle or the classical maximum principle, then $G$ satisfies the continuity principle.*

## 3. The positivity of a continuous function-kernel and the continuity principle

Throughout this section, $G$ is a continuous function-kernel on $X$. We say that $G$ satisfies the condition $(*)$ if:

$(*)$ For any non-empty open set $\omega$ in $X$, $\mathrm{cap}_G(\omega) > 0$ .

*Remark.* The condition $(*)$ is very natural in the potential theory. Let us observe it for a continuous composition kernel on a locally compact abelian group.

Let $G(x, y) = k(x - y)$ be a continuous composition kernel on a locally compact abelian group $X$, where $k$ is continuous in the extended sense and finite outside the origin. Suppose that $G$ satisfies the domination principle. Then $G$ satisfies the condition $(*)$ if and only if $k$ is $\xi$-summable in a certain neighborhood of 0, where $\xi$ is a Haar measure on $X$.

If $k$ is $\xi$-summable in a certain neighborhood of 0, then for any finite continuous function $f$ in $X$ with compact support, the convolution $k*f$ is defined in $X$ and finite continuous. Therefore we obtain that the "if" part is valid. We shall show that the "only if" part is valid. By the domination principle for $G, k$ is identically equal to 0 if $k(0) = 0$, and hence we may assume $k(0) > 0$. The condition $(*)$ implies $E_0 \neq \{0\}$. We can find a $\lambda$ $(\neq 0)$ in $E_0$ supported in $C\{0\} \cap \{x \in X; k(x) > 0\}$. We may assume that $G\lambda$ is bounded on $S\lambda$. By virtue of the domination principle for $G$, there exists a constant $c > 0$ such that $G\lambda(x) \leqq ck(x)$ on

$X$, and hence $G\lambda$ is locally bounded on $X$. Therefore, for any finite continuous function $f$ in $X$ with compact support,

$$+\infty > \int G\lambda(x)\,|f(x)|\,d\xi(x) = \int k(x)(\check{\lambda} * |f|)(x)d\xi(x) ,$$

where $\check{\lambda}$ is the measure defined by $\check{\lambda}(e) = \lambda(-e)$ for any Borel set $e$. Consequently $k$ is locally $\xi$-summable.

Let us consider our continuous function-kernel on $X$. Our first theorem is the following

THEOREM 1. *Let $G$ be a continuous funtion-kernel on $X$ satisfying the domination principle and the condition* $(*)$. *Assume that $G > 0$ on $\Delta$ and every point in $X$ is not isolated. Then $G(x, y) > 0$ on $X \times X$.*

*Proof.* Suppose that there exists a point $(x_0, y_0)$ in $X \times X$ where $G$ vanishes. Then $(x_0, y_0) \in X \times X - \Delta$. Put $g(y) = G(x_0, y)$. Then $g$ is defined in $X$ and continuous in the extended sense. Every point in $X$ being not-isolated, we can find $y_1$ in the support of $g$ satisfying $y_1 \neq x_0$ and $g(y_1) = 0$. By $G(y_1, y_1) > 0$, there exists an open neighborhood $V$ of $y_1$ such that $G(x, y) > 0$ in $V \times V$. Set $\omega = V \cap \{y \in X\,;\, g(y) > 0\}$. Then $\omega \neq \emptyset$, and there exists a positive Radon measure $\lambda$ $(\neq 0)$ contained in $E_0$ and supported in $\omega$. By virtue of the Lusin theorem and the continuity principle for $G$ (cf. Proposition 4), we may assume that $G\lambda$ is finite continuous in $X$. Then we can find a positive constant $a$ such that $G\lambda(x) \leqq aG\varepsilon_{y_1}(x)$ on $S\lambda$, where $\varepsilon_{y_1}$ is the unit measure at $y_1$, because $G\varepsilon_{y_1}(x) > 0$ on $S\lambda$. By the domination principle for $G$, we have $G\lambda(x) \leqq aG\varepsilon_{y_1}(x)$ on $X$. Hence

$$0 = aG(x_0, y_1) = aG\varepsilon_{y_1}(x_0) \geqq G\lambda(x_0) = \int g(y)d\lambda(y) > 0 .$$

This is a contradiction. Consequently $G(x, y) > 0$ in $X \times X$. This completes the proof.

We discuss the continuity principle for a continuous function-kernel on $X$. For a closed subset $X'$ of $X$, we denote by $G'$ the restriction of $G$ to $X' \times X'$. Then $G'$ is evidently a continuous function-kernel on $X'$.

LEMMA 1. *If $G$ satisfies the domination principle, then $G'$ satisfies it.*

This follows from the fact that for any positive Radon measure $\mu$ in $X'$, $G\mu(x) = G'\mu(x)$ on $X'$.

LEMMA 2. *Suppose that $G$ is strictly positive in $X \times X$. If $G$ satisfies the restrained domination principle and the condition $(*)$, then $G$ satisfies the continuity principle.*

This lemma can be shown by the same manner as in the usual case. Let us give the proof. Suppose that for a $\mu$ in $M_0$, the restriction of $G\mu$ to $S\mu$ is finite continuous. It sufficies to show $\lim_{x \to x_0} G\mu(x) = G\mu(x_0)$ for every boundary point $x_0$ of $S\mu$.

If $\mu(\{x_0\}) > 0, G(x_0, x_0) < +\infty$, because $G\mu(x_0) < +\infty$, and hence our desired equality follows immediately from the finite continuity of $G$ at $(x_0, x_0)$.

Suppose $\mu(\{x_0\}) = 0$. By $G\mu(x_0) < +\infty$, for a given positive number $\varepsilon$, there exists an open neighborhood $V$ of $x_0$ such that $\int_V G(x_0, y) d\mu(y) < \varepsilon$. The function $\int_V G(x, y) d\mu(y)$ of $x$ being finite continuous as a function on $S\mu \cap V$, we can choose another open neighborhood $W$ of $x_0$ which satisfies $W \subset V$ and

$$\int_W G(x, y) d\mu(y) \leqq \int_V G(x, y) d\mu(y) < \int_V G(x_0, y) d\mu(y) + \varepsilon$$

for any $x$ in $S\mu \cap W$. Denote by $\mu'$ the restriction of $\mu$ to $W$. Then $G\mu' < 2\varepsilon$ on $S\mu'$. We may assume $\bar{V} \neq X$. By the condition $(*)$ and $G(x, y) > 0$ in $X \times X$, there exists a $\nu$ in $E_0$ such that $S\nu \cap \bar{V} = \emptyset$ and $G\nu(x) > 1$ on $S\mu'$. By virtue of the restrained domination principle for $G$ and the continuity of $G\mu'$ in $CS\mu'$, we have

$$G\mu'(x) \leqq 2\varepsilon G\nu(x) \qquad \text{in } W .$$

On the other hand, $G(\mu - \mu')$ is finite and continuous at $x_0$, and hence

$$\overline{\lim_{x \to x_0}}\, G\mu(x) \leqq G\mu(x_0) + 2\varepsilon G\nu(x_0) .$$

$G\mu$ being lower semi-continuous and $\varepsilon$ being arbitrary, we have $\lim_{x \to x_0} G\mu(x) = G\mu(x_0)$. This completes the proof.

By Proposition 1, we have the following

LEMMA 3. *Under the same assumptions as in Lemma 2, $\check{G}$ satisfies the continuity principle.*

THEOREM 2. *Let $G$ be a continuous function-kernel on $X$ satisfying* *the condition* $(*)$. *If* $G>0$ *on* $\Delta$ *and* $G$ *satisfies the domination principle, then* $G$ *and* $\check{G}$ *satisfy the continuity principle.*

*Proof.* We denote by $X'$ the closed subset of $X$ constituted by all points which are not isolated and by $G'$ the restriction of $G$ to $X'\times X'$. By Lemma 1, $G'$ is a continuous function-kernel on $X'$ satisfying the domination principle. Evidently $G'$ satisfies the condition $(*)$ and every point of $X'$ is not isolated. Consequently $\check{G}'$ satisfies the continuity principle by Proposition 2 and Lemma 3. The continuity principle for $G$ is well-known, and we shall prove only the continuity principle for $\check{G}$. Suppose that for $\mu\in M_0$, the restriction of $G\mu$ to $S\mu$ is finite continuous. Denote by $\mu'$ the restriction of $\mu$ to $X'$. Then, as a function on $S\mu'$, $G'\mu'$ is finite continuous, because $G'\mu'=G\mu-G(\mu-\mu')$ on $X'$, and hence $G'\mu'$ is finite continuous in $X'$. By $G\mu'=G'\mu'$ on $X'$ and the fact that $X-X'$ is discrete, $G\mu'$ is finite continuous on $X$. On the other hand, $S(\mu-\mu')$ is discrete, and then $G(\mu-\mu')$ is finite continuous on $X$, i.e., $G\mu$ is finite continuous on $X$. This completes the proof.

### 4. Remarks on Kishi's theorem

Let us start the following theorem.

THEOREM 3. *Let* $G$ *be a continuous function-kernel on* $X$. *Assume* $G(x,x)>0$ *on* $\Delta$. *Then the following four statements are equivalent.*

(1) $G$ *satisfies the domination principle.*

(2) $\check{G}$ *satisfies the domination priciple.*

(3) $G$ *satisfies the balayage principle.*

(4) $\check{G}$ *satisfies the balayage principle.*

*Proof.* By Proposition 3, we know (3) ⇒ (2) and (4) ⇒ (1). The implication (2) ⇒ (4) is the dual form of (1) ⇒ (3). Therefore it suffices to show the implication (1) ⇒ (3). Suppose that (1) is valid. Put $\Omega=\bigcup\{\omega:\text{open}, \operatorname{cap}_G(\omega)=0\}$ and $X'=X-\Omega$. Let $G'$ the restriction of $G$ to $X'\times X'$. Then $G'$ is a continuous function-kernel on $X'$ satisfying the domination principle and the condition $(*)$. Therefore, by Theorem 2, $G'$ satisfies the continuity principle. Let us remember the existence theorem of Kishi ([4]).

PROPOSITION 5. *Let* $\overline{G}$ *be a lower semi-continuous function-kernel on a locally compact Hausdorff space* $Y$. *Assume that* $\overline{G}(x,x)>0$ *for any*

*$x$ in $Y$ and $\check{\overline{G}}$ satisfies continuity principle. For a given compact set $F$ in $Y$ and a given non-negative finite continuous function $u$ on $F$, there exists a positive Radon measure $\lambda$ in $Y$ supported by $F$ such that $\overline{G}\lambda(x) \geqq u(x)$ $G$-p.p.p. on $F$ and $\overline{G}\lambda(x) \leqq u(x)$ $\lambda$-a.e. on $Y$.*

We continue our proof. Let $K$ be a compact set in $X$ and $\mu$ be a positive Radon measure in $X$ with compact support. First we suppose $S\mu \cap K = \emptyset$. Let $\mu'$ be a positive Radon measure in $X$ obtained in Proposition 5 for the case of $\overline{G} = G', F = K \cap X'$ and $u = G\mu$. We have evidently $G\mu'(x) = G'\mu'(x)$ on $X'$ and $\mu' \in E_0(G)$, and "$G'$-p.p.p. on $K \cap X'$" is equal to "$G$-p.p.p. on $K \cap X'$". By virtue of the domination principle for $G$ and the inequality $G\mu'(x) \leqq G\mu(x)$ $\mu'$-a.e. in $X$, we obtain, by the usual way, the inequality $G\mu'(x) \leqq G\mu(x)$ everywhere in $X$. Consequently $\mu'$ is a $G$-balayaged measure of $\mu$ on $K \cap X'$. For an arbitrary $\nu$ in $E_0(G)$ supported by $K, \nu$ is supported by $K \cap X'$. In fact, if $\nu(\Omega) > 0$, there exists an open set $\omega \subset \Omega$ such that $\mathrm{cap}_G(\omega) = 0$ and $\nu(\omega) > 0$, which is a contradiction. Hence "$G$-p.p.p. on $K$" is equivalent to "$G$-p.p.p. on $K \cap X'$". Therefore $\mu'$ is a $G$-balayaged measure of $\mu$ on $K$.

We remark here that, to show immediately the existence of a $G$-balayaged measure of $\mu$ on $K$ in the case of $S\mu \cap K \neq \emptyset$, it is necessary that $\check{G}$ satisfies the continuity principle (cf. [3]). But, by the above discussion, we obtain that $\check{G}$ satisfies the domination principle. In fact, suppose that for a $\nu$ in $E_0(G)$ and a $\lambda$ in $M_0$, $\check{G}\nu(x) \leqq \check{G}\lambda(x)$ on $S\nu$. For an arbitrary $y$ in $CS\nu$, there exists a $G$-balayaged measure $\varepsilon'_y$ of $\varepsilon_y$ on $S\nu$, and hence we have

$$\begin{aligned}\check{G}\nu(y) &= \int G\varepsilon_y(x)d\nu(x) = \int G\varepsilon'_y(x)d\nu(x) = \int \check{G}\nu(x)d\varepsilon'_y(x)\\ &\leqq \int \check{G}\lambda(x)d\varepsilon'_y(x) = \int G\varepsilon'_y(x)d\lambda(x) \leqq \int G\varepsilon_y(x)d\lambda(x) = \check{G}\lambda(y)\ ,\end{aligned}$$

i.e., $\check{G}\nu \leqq \check{G}\lambda$ on $X$. Consequently $\check{G}$ satisfies the domination principle, and hence $\check{G}$ satisfies the continuity principle (cf. Proposition 4). In the same way as in [3], we obtain that $G$ satisfies the balayage principle. This completes the proof.

By the present theorem and Proposition 4, we have the following

COROLLARY. *Let $G$ be a continuous function-kernel on $X$ satisfying $G > 0$ on $\Delta$. If $G$ satisfies the domination principle, then $G$ and $\check{G}$ satisfy the continuity principle.*

M. Kishi discussed other potential theoretical properties for $G$ implied by the domination principle for $G$ under the condition that $\check{G}$ satisfies the continuity principle (cf. [3], [4], & c.). By the above corollary, in these cases, we can omit the continuity principle for $\check{G}$. In particular, the following theorem is fundamental.

THEOREM 4. *Let $G$ be a continuous function-kernel on $X$ satisfying $G > 0$ on $\Delta$. Then $G$ satisfies the complete maximum principle if and only if $G$ satisfies the domination principle and the classical maximum principle.*

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