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ON A CERTAIN ALGEBRA ASSOCIATED WITH A POLARIZED ALGEBRAIC VARIETY

Dedicated to Professor Minoru Kurita on the occasion of his 60th birthday.

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In the present note we associate a certain algebra of finite rank over \mathcal{Q} to each non-singular polarized algebraic variety defined over \mathcal{C} . For a surface the algebra is a Jordan algebra with identity, and for an abelian variety A the algebra is canonically isomorphic to the Jordan algebra of symmetric elements in $\text{End}_{\mathcal{Q}}(A)^{1)}$ with respect to the involution induced by the polarization. This algebra may be important for a polarized algebraic variety as much as $\text{End}_{\mathcal{Q}}(A)$ for an abelian variety A .

§ 1. Composition

1.1. Let V be a compact nonsingular algebraic variety of dimension n with a Hodge structure ω , a fundamental (1,1)-form on V , and let $H^{(\ell,\ell)}(V, \mathcal{C})$ be the space of harmonic (ℓ, ℓ) -forms on V with respect to the Hodge structure ω , ($0 \leq \ell \leq n$). Regarding $H^{(\ell,\ell)}(V, \mathcal{C})$ as a subgroup of the 2ℓ -th cohomology group $H^{2\ell}(V, \mathcal{C})$, we denote

$$\mathfrak{H}^{(\ell,\ell)}(V, \mathcal{Q}) = H^{(\ell,\ell)}(V, \mathcal{C}) \cap H^{2\ell}(V, \mathcal{Q})$$

and

$$\mathfrak{H}(V, \mathcal{Q}) = \bigoplus_{\ell=1}^n \mathfrak{H}^{(\ell,\ell)}(V, \mathcal{Q}).$$

Then $\mathfrak{H}(V, \mathcal{Q})$ is considered as a commutative \mathcal{Q} -algebra with the product given by

(1) $\xi \cdot \eta$ = the harmonic form cohomologous to the closed form $\xi \wedge \eta$.

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¹⁾ $\text{End}_{\mathcal{Q}}(A) = \text{End}(A) \otimes_{\mathcal{Z}} \mathcal{Q}$, where $\text{End}(A)$ means the ring of endomorphisms.

For each φ in $\mathfrak{S}^{(1,1)}(\mathcal{V}, \mathcal{Q})$ we mean by L_φ the operator

$$(2) \quad L_\varphi \xi = \varphi \cdot \xi$$

and denote

$$(3) \quad L = L_\omega, \Lambda = i(\omega),$$

where $i(\omega)\xi$ mean the inner product of the fundamental form ω with ξ with respect to the Hodge structure. These operators may be considered as operators on $\mathfrak{S}(\mathcal{V}, \mathcal{Q})$ and they satisfy the relations

$$(4) \quad [L, \Lambda] = H = \sum_{\ell=0}^n (2\ell - n)\pi^{(\ell, \ell)},$$

$$(5) \quad [H, L_\varphi] = 2L_\varphi, [H, L] = 2L,$$

$$(6) \quad [H, \Lambda] = -2\Lambda,$$

where $\pi^{(\ell, \ell)}$ is the projection $\mathfrak{S}(\mathcal{V}, \mathcal{Q}) \rightarrow \mathfrak{S}^{(\ell, \ell)}(\mathcal{V}, \mathcal{Q})$.

1.2. We define a binary composition \circ in $\mathfrak{S}^{(1,1)}(\mathcal{V}, \mathcal{Q})$ as follows

$$(7) \quad \varphi \circ \phi = \frac{1}{2}\{\Lambda\varphi \cdot \phi + \Lambda\phi \cdot \varphi - \Lambda(\varphi \cdot \phi)\} \quad (\varphi, \phi \in \mathfrak{S}^{(1,1)}(\mathcal{V}, \mathcal{Q})).$$

The composition is obviously commutative, i.e.,

$$(8) \quad \varphi \circ \phi = \phi \circ \varphi.$$

LEMMA 1.

$$(9) \quad \Lambda L\varphi = (n - 2)\varphi + \Lambda\varphi \cdot \omega \quad (\varphi \in \mathfrak{S}^{(1,1)}(\mathcal{V}, \mathcal{Q}))$$

$$(10) \quad \Lambda\omega = n = \dim \mathcal{V}.$$

Proof. From (4) we have

$$\Lambda L\varphi = (-H + \Lambda\Lambda)\varphi = (n - 2)\varphi + \Lambda\varphi \cdot \omega,$$

and

$$\Lambda\omega = \Lambda L1 = (-H + \Lambda\Lambda)1 = -H1 = n.$$

PROPOSITION 1.

$$(11) \quad \varphi \circ \omega = \varphi,$$

Proof. From (8) we have

$$\begin{aligned}
\varphi \circ \omega &= \frac{1}{2}\{\Lambda\omega \cdot \varphi + \Lambda\varphi \cdot \omega - \Lambda(\omega \cdot \varphi)\} \\
&= \frac{1}{2}\{n\varphi + \Lambda\varphi \cdot \omega - \Lambda L\varphi\} \\
&= \frac{1}{2}\{n\varphi + \Lambda\varphi \cdot \omega - (n-2)\varphi - \Lambda\varphi \cdot \omega\} \\
&= \varphi .
\end{aligned}$$

Let us give another expression of the composition \circ :

PROPOSITION 2.

$$(12) \quad \varphi \circ \phi = \frac{1}{2}[[L_\varphi, \Lambda], L_\phi]1$$

Proof. From (9), (10), (11) it follows that

$$\begin{aligned}
\frac{1}{2}[[L_\varphi, \Lambda], L_\phi]1 &= \frac{1}{2}\{L_\varphi \Lambda L_\phi + L_\phi \Lambda L_\varphi - \Lambda L_\varphi L_\phi - L_\varphi L_\phi \Lambda\}1 \\
&= \frac{1}{2}\{L_\varphi \Lambda \phi + L_\phi \Lambda \varphi - \Lambda(\varphi \cdot \phi)\} \\
&= \frac{1}{2}\{\Lambda \phi \cdot \varphi + \Lambda \varphi \cdot \phi - \Lambda(\varphi \cdot \phi)\} = \varphi \circ \phi .
\end{aligned}$$

PROPOSITION 3. Let ρ_φ be the linear endomorphism of $\mathfrak{S}^{(1,1)}(\mathcal{V}, \mathcal{Q})$ given by

$$(13) \quad \rho_\varphi \phi = \varphi \circ \phi .$$

Then

$$(14) \quad \rho_\varphi = \frac{1}{2}[L_\varphi, \Lambda] + \frac{1}{2}\Lambda\varphi \cdot \text{id} ,$$

as linear endomorphism on $\mathfrak{S}^{(1,1)}(\mathcal{V}, \mathcal{Q})$.

Proof. From the definitions it follows that

$$\begin{aligned}
(\frac{1}{2}[L_\varphi, \Lambda] + \frac{1}{2}\Lambda\varphi \cdot \text{id})\phi &= \frac{1}{2}\{[[L_\varphi, \Lambda], L_\phi]1 + L_\phi[L_\varphi, \Lambda]1 + \Lambda\varphi \cdot \phi\} \\
&= \varphi \circ \phi - \frac{1}{2}\Lambda\varphi \cdot \phi + \frac{1}{2}\Lambda\varphi \cdot \phi = \varphi \circ \phi = \rho_\varphi \phi .
\end{aligned}$$

PROPOSITION 4. The following two equalities are equivalent :

$$(15) \quad [[L_\varphi, \Lambda], [L_{\Lambda\varphi^2}, \Lambda]]\phi = 0$$

and

$$(16) \quad \varphi \circ ((\varphi \circ \varphi) \circ \phi) = (\varphi \circ \varphi) \circ (\varphi \circ \phi) \quad (\varphi, \phi \in \mathfrak{S}^{(1,1)}(\mathcal{V}, \mathcal{Q})) .$$

Proof. (16) is equivalent to

$$[\rho_\varphi, \rho_{\varphi \circ \varphi}]\phi = 0 .$$

On the other hand

$$\rho_\varphi \phi = \left\{ \frac{1}{2}[L_\varphi, A] + \frac{1}{2}A\varphi \cdot \text{id} \right\} \phi,$$

and thus (16) is equivalent to

$$[[L_\varphi, A], [L_{\varphi \circ \varphi}, A]]\phi = 0.$$

Since $\varphi \circ \varphi = \frac{1}{2}\{2A\varphi \cdot \varphi - A\varphi^2\}$, this equality is equivalent to (15).

1.3. A Jordan algebra is a commutative algebra satisfying (16), hence Proposition 4 may be stated as follows:

PROPOSITION 5. *The algebra $(\mathfrak{S}^{(1,1)}(V, \mathcal{Q}), \circ)$ is a Jordan algebra with identity ω , if and only if*

$$[[L_\varphi, A], [L_{A\varphi^2}, A]]\phi = 0 \quad (\varphi, \phi) \in \mathfrak{S}^{(1,1)}(V, \mathcal{Q}).$$

PROPOSITION 6. *If $\dim V = 2$, then $(\mathfrak{S}^{(1,1)}(V, \mathcal{Q}), \circ)$ is a Jordan algebra with identity ω .*

Proof. Since $\dim V = 2$, there exists a rational valued bilinear form $\beta_{\varphi, \phi}$ on $\mathfrak{S}^{(1,1)}(V, \mathcal{Q})$ such that $\varphi \cdot \phi = \beta_{\varphi, \phi} \omega^2$, and thus

$$L_{A\varphi^2} = \beta_{\varphi, \varphi} L_{A\omega^2} = \beta_{\varphi, \varphi} L_{AL\omega} = (2n - 2)\beta_{\varphi, \varphi} L_\omega.$$

Hence

$$\begin{aligned} [[L_\varphi, A], [L_{A\varphi^2}, A]] &= (2n - 2)\beta_{\varphi, \varphi} [[L_\varphi, A], H] \\ &= (2n - 2)\beta_{\varphi, \varphi} \{ [[L_\varphi, H], A] + [L_\varphi, [A, H]] \} = 0. \end{aligned}$$

§ 2. Abelian variety case.

2.1. We shall show another important example, an abelian variety, for which $(\mathfrak{S}^{(1,1)}(V, \mathcal{Q}), \circ)$ is a Jordan algebra.

Let A be an abelian variety of dimension n defined over \mathbf{C} , which is expressed as a quotient

$$A = \mathbf{C}^n / \Sigma$$

with a lattice Σ of rank $2n$. After a suitable choice of the coordinates z_1, \dots, z_n on \mathbf{C}^n , we may assume that

$$\omega = \sqrt{-1} dz_1 d^t \bar{z},$$

where $dz = (dz_1, \dots, dz_n)$.

We denote by $\text{End}_{\mathcal{Q}}(A)$ the \mathcal{Q} -algebra of $n \times n$ -matrices A such that

$$\sum A \subset \Sigma ,$$

and we mean by $\mathfrak{S}_Q(A, \omega)$ the subspace of $\text{End}_Q(A)$ consisting of symmetric matrices. Then two vector spaces $\mathfrak{S}_Q(A, \omega)$ and $\mathfrak{S}_Q^{(1,1)}(A, \mathcal{Q})$ are canonically isomorphic in the following correspondence

$$A \leftrightarrow \sqrt{-1}(dzA)_i^t d\bar{z} .$$

The space $\mathfrak{S}_Q(A)$ is a Jordan algebra with the composition

$$\alpha \circ \beta = \frac{1}{2}(\alpha\beta + \beta\alpha) .$$

PROPOSITION 7. *If A is an abelian variety, then $(\mathfrak{S}^{(1,1)}(V, \mathcal{Q}), \circ)$ is a Jordan algebra canonically isomorphic to the Jordan algebra $(\mathfrak{S}_Q(A, \omega), \circ)$.*

Proof. Since $A = -\sqrt{-1} \sum_{i=1}^n i(dz_{iA} d\bar{z}_i)$, using the above notations, we have

$$\begin{aligned} A\varphi &= -\sqrt{-1} \left(\sum_{i=1}^n i(dz_{iA} d\bar{z}_i) \right) \left(\sqrt{-1} \sum_{i,j=1}^n a_{ij}(\varphi) dz_{iA} d\bar{z}_j \right) \\ &= \text{tr } A_\varphi , \end{aligned}$$

$$\begin{aligned} A(\varphi \cdot \phi) &= \sqrt{-1} \left(\sum_{i=1}^n i(dz_{iA} d\bar{z}_i) \right) \left(\sum_{i,j,p,q} a_{i,p}(\varphi) a_{j,q}(\phi) dz_{iA} d\bar{z}_{pA} dz_{jA} d\bar{z}_q \right) \\ &= \text{tr } A_\varphi \phi + \text{tr } A_\phi \varphi - \sqrt{-1} (dz(A_\varphi A_\phi + A_\phi A_\varphi))_i^t d\bar{z} \\ &= A\phi \cdot \varphi + A\varphi \cdot \phi - \sqrt{-1} (dz(A_\varphi A_\phi + A_\phi A_\varphi))_i^t d\bar{z} . \end{aligned}$$

This shows that

$$A_{\varphi \circ \phi} = \frac{1}{2}(A_\varphi A_\phi + A_\phi A_\varphi) .$$

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