E. Bannai
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NORMAL SUBGROUPS OF FINITE MULTIPLY TRANSITIVE PERMUTATION GROUPS

EIICHI BANNAI*)

Introduction

Wagner [5] and Ito [2] proved the following theorems respectively.

THEOREM OF WAGNER. *Let G be a triply transitive permutation group on a set $\Omega = \{1, 2, \cdots, n\}$, and let n be odd and $n > 4$. If H is a normal subgroup $(\neq 1)$ of G, then H is also triply transitive on Ω.*

THEOREM OF ITO[1]. *Let G be a 4-ply transitive permutation group on a set $\Omega = \{1, 2, \cdots, n\}$. Let $n \not\equiv 0 \pmod 3$ and $n > 5$. If H is a normal subgroup $(\neq 1)$ of G, then H is also 4-ply transitive on Ω.*

The purpose of this note is to give some generalization of the results of Wagner and Ito. Namely, we will prove the following theorem and several corollaries of it.

THEOREM 1. *Let G be a t-ply transitive $(t \geqslant 4)$ permutation group on a set $\Omega = \{1, 2, \cdots, n\}$ with $n > t + 1$. Let q be an odd prime which divides $t - 1$. Let H be a normal subgroup $(\neq 1)$ of G, and let us assume that the following two conditions are satisfied:*

(i) *q does not divide n,*

(ii) *the index of H in G is equal to the number s of orbits of $H_{1,2,\cdots,t-1}$ (the pointwise stabilizer of $t - 1$ points of Ω in G) on $\Omega - \{1, 2, \cdots, t - 1\}$.*

Then $s = 1$, i.e., $G = H$.

COROLLARY 1. *Let G be a t-ply transitive $(t \geqslant 4)$ permutation group on a set $\Omega = \{1, 2, \cdots, n\}$. Let q be a prime which divides $t - 1$ and*

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1) This result was also obtained by J. Saxl independently (Ph.D Thesis, 1973, June, Oxford).

let $n \not\equiv 0 \pmod{q}$ *and* $n > t+1$. *If* H *is a normal subgroup* $(\neq 1)$ *of* G *and* G/H *is solvable, then* H *is also t-ply transitive on* Ω.

COROLLARY 2. *Let* G *be a* 6*-ply transitive permutation group on a set* $\Omega = \{1, 2, \cdots, n\}$. *Let* $n \not\equiv 0 \pmod{5}$ *and* $n > 7$. *If* H *is a normal subgroup* $(\neq 1)$ *of* G, *then* H *is also* 6*-ply transitive on* Ω.

COROLLARY 3. *Let* G *be a* 8*-ply transitive permutation group on a set* $\Omega = \{1, 2, \cdots, n\}$. *Let* $n \not\equiv 0 \pmod{7}$ *and* $n > 9$. *If* H *is a normal subgroup* $(\neq 1)$ *of* G, *then* H *is also* 8*-ply transitive on* Ω.

It is possible to obtain a similar kind of result for some other values of t. But we will not go further on this problem, and instead, we will prove a result on permutation groups of prime degree.

COROLLARY 4. *Let* p *be a prime and let* G *be a t-ply* $(t \geqslant 1)$ *transitive permutation group of degree* p. *If* G *is nonsolvable and not* S^{Ω} *(the symmetric group on* Ω*), then any normal subgroup* $(\neq 1)$ *of* G *is also t-ply transitive on* Ω.

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1. Proof of Theorem 1

Let G and H satisfy the assumptions of Theorem 1 and let us assume that $s \geqslant 2$. And we will drive a contradiction. Let $\Delta_1, \Delta_2, \cdots, \Delta_s$ be the orbits of $H_{1,2,\cdots,t-1}$ on $\Omega - \{1, 2, \cdots, t-1\}$. (Here note that H is $(t-1)$-ply transitive on Ω and that $|\Delta_1| = \cdots = |\Delta_s|$ with $s \geqslant 2$. cf. [5].) By the assumption of Theorem 1 that q does not divide n, we obtain that s is prime to q. Now we use the following lemma due essentially to Frobenius. (For a proof, see Nagao [3], Proposition 14.5.)

LEMMA OF FROBENIUS[2]. *Let* G *be a permutation group on a set* $\Omega = \{1, 2, \cdots, n\}$. *For an element* x *in* G, *let* $\alpha_i(x)$ *denote the number of cycles of length* i *in the cycle structure of* x. *Then we obtain that*

$$\sum_{x \in G} \binom{\alpha_1(x)}{\kappa} \binom{\alpha_2(x)}{\lambda} \cdots = \frac{m \cdot |G|}{1^{\kappa} \cdot \kappa ! \cdot 2^{\lambda} \cdot \lambda ! \cdots}.$$

[2] This lemma was communicated to the author by Professor Nagao.

Here m is an integer obtained in the following way. Let

$$\Omega^{(u)} = \{(i_1, \cdots, i_\kappa, j_1, j_1', \cdots, j_\lambda, j_\lambda', \cdots)\}$$

be the family of ordered sets consisting of $u\,(= \kappa + \lambda + \cdots)$ points of Ω such that there is at least one element x of G of the form

$$x = (i_1)(i_2) \cdots (i_\kappa)(j_1 j_1') \cdots (j_\lambda j_\lambda') \cdots .$$

When G is regarded as a permutation group on $\Omega^{(u)}$ by setting

$$(a_1, \cdots, a_u)^x = (a_1^x, \cdots, a_u^x)$$

for $x \in G$ and $(a_1, \cdots, a_u) \in \Omega^{(u)}$, m is the number of orbits of G on $\Omega^{(u)}$.

Now we will continue the proof of Theorem 1. Since for any subset Δ of t points in Ω, $H_{(\Delta)}^{\Delta} = S^{\Delta}$ (here $H_{(\Delta)}$ denotes the global stabilizer of Δ in H), by the argument given in [1] (or in [5]), we obtain the following assertion for the permutation group on Ω: if $u\,(= \kappa + \lambda + \cdots) = t$, then the number m (given in Lemma of Frobenius) is equal to s; of course, if $u \leqslant t-1$ then $m = 1$, because H is $(t-1)$-ply transitive. Now let S be a Sylow q-subgroup of $H_{1,2,\cdots,t-1}$. Now we will show that $S = 1$. By Lemma of Frobenius, $G_{1,2,\cdots,t} = H_{1,2,\cdots,t}$, because if an element $x \in G$ has a partial sum of cycle lengths of some of cycles of x which is equal to t, then we obtain that $x \in H$. Since S is a Sylow q-subgroup of $G_{1,2,\cdots,t-1}$ and $H_{1,2,\cdots,t-1}$, we obtain that $N_G(S) \not\subseteq H$. If $S \neq 1$, then by induction, we have either $H \cap N_G(S)$ is t-ply transitive on $I(S)$ (= the set of the points of Ω which are fixed by any element of S), or $|I(S)| = t+1$. If the first case holds, then $N_G(S)_{1,2,\cdots,t} \supsetneqq H \cap N_G(S)_{1,2,\cdots,t}$, and this is a contradiction. If the second case holds, then $H \cap N_G(S)$ cannot contain a t-cycle (if t is odd), or an element consisting of one 2-cycle and one $(t-2)$-cycle on $I(S)$ (if t is even), and this contradicts Lemma of Frobenius. Therefore, $S = 1$. Now, also by Lemma of Frobenius, $G - H$ contains an element x such that $x = (1)(2) \cdots (t-1-q)(t-q, \cdots, t-1)$ on $\{1, 2, \cdots, t-1\}$. Moreover, since q and s are relatively prime, and since $S = 1$, the sum of the lengths of cycles (on $\Omega - \{1, 2, \cdots, t-1\}$) whose lengths are prime to q of the element x is no more than $q-1$ and more than 2, because $s \geqslant 2$. Now, since q and s are relatively prime, x^q also lies in $G - H$. While, since x^q has certain partial sum of cycle lengths which is equal to t, we have $x \in H$, by Lemma of Frobenius. But this is a contradiction, and we have completed the proof of Theorem 1.

2. Proof of Corollaries

Proof of Corollary 1. By induction of $|G:H|$, we may assume that G/H is of prime order. Thus G and H satisfy the assumptions of Theorem 1 provided that H is not t-ply transitive. Thus we have the assertion.

Proof of Corollary 2[3]. By the proof of Lemma 4 and Remark (iii) of [5], we obtain that $s \leqslant 4$. Since any subgroup of S_4 (the symmetric group of degree 4) is solvable we have that G/H is solvable by induction. Thus by Corollary 1, we have the assertion.

Proof of Corollary 3. By the proof of Lemma 4 and Remark (iii) of [5], we obtain that $s \leqslant 6$. Thus we may assume by induction that one of the following cases holds: (1) G/H is solvable, (2) $G/H = A_5$ and $s = 5$, (3) $G/H = A_5$ and $s = 6$, and (4) $G/H = A_6$ and $s = 6$. If the first case holds, then we have nothing to prove because of Corollary 1. In every remaining three cases, the group G/H acting transitively on s symbols contains a proper subgroup which is transitive on the s symbols. Therefore, $G_{1,2,\cdots,t-1}$ has a proper subgroup which contains $H_{1,2,\cdots,t-1}$ and which acts transitively on the set $\{\Delta_i\ (i = 1, 2, \cdots, s)\}$. Now, there exists a proper subgroup of G which contains H and is 8-ply transitive on Ω. Thus, by induction, we have completed the proof of Corollary 3.

Proof of Corollary 4. By a result of Burnside (cf. [6], Theorem 11.7), we have $t \geqslant 2$. If $t = 2$, then we obtain the assertion also by a result of Burnside, because for any normal subgroup $H\,(\neq 1)$ of $G, G/H$ is solvable by the theorem of Sylow (Frattini argument) and so H is nonsolvable. If $t = 3$, then the assertion is true by Wagner [5]. Let $t \geqslant 4$. Then for any normal subgroup $H\,(\neq 1)$ of $G, G/H$ is solvable by the theorem of Sylow. Thus, we obtain the assertion by Corollary 1.

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[3] The proof of this result was first completed by Tsuzuku [4] by an analogous (but more character theoretic, i.e., using a result of [1]) strategy as that of the present paper. Some idea of the proof of Theorem 1 is indebted to him.

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Department of Mathematics
Tokyo University