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Nagoya Math. J.
Vol. 53 (1974), 71-82

LIPEOMORPHISMS CLOSE TO AN ANOSOV DIFFEOMORPHISM

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§ 0. Introduction

It is well-known that an Anosov diffeomorphism f on a compact manifold is structurally stable in the space of all C^1-diffeomorphisms, with the C^1-topology (Anosov [1]). In this paper we show that f is also structurally stable in the space of all lipeomorphisms, with a lipschitz topology. The proof is similar to that of the C^1-case by J. Moser [4]. If a C^1-diffeomorphism g is sufficiently close to f in the C^1-sense g is also sufficiently close to f in the lipschitz sense by the mean value theorem. Hence our result is somewhat stronger than that of Anosov.

In the following let M be a compact connected boundaryless C^∞-manifold of dimension n with a Riemannian metric $\|\cdot\|$, d the distance function induced by $\|\cdot\|$, and $\{(U_\alpha, \alpha)\}$ a covering of M by finite charts $M = \bigcup_\alpha U_\alpha$, where each local diffeomorphism α onto an open subset of $\boldsymbol{R}^n$ is defined on an open subset of M which contains the closure of U_α: $\mathscr{D}(\alpha) \supset \bar{U}_\alpha (\mathscr{D}(\alpha)$ denotes the domain of α.). Let $|\cdot|$ be the standard norm in $\boldsymbol{R}^n$.

§ 1. Lipschitz maps on M.

Let $C^0(M)$ be the set of all continuous maps of M into itself and d_0 the distance function on $C^0(M)$ induced by the distance function d on M: $d_0(f, g) = \mathrm{Sup}_{x \in M}\, d(f(x), g(x))$ for $f, g \in C^0(M)$. $L(M)$ denotes the set of all lipschitz maps of M into itself. It is clear that $L(M)$ is contained in $C^0(M)$. We may choose a positive number λ_1 such that for any α $f(\bar{U}_\alpha) \subset \mathscr{D}(\alpha)$ holds for $f \in C^0(M)$ with $d_0(f, 1_M) < \lambda_1$, 1_M denoting the identity map of M. For any $f \in C^0(M)$ with $d_0(f, 1_M) < \lambda_1$, f is lipschitz if and only if for any α the map $\alpha \circ f \circ \alpha^{-1}$ of $\alpha(U_\alpha)$ into $\boldsymbol{R}^n$ is lipschitz i.e. the lipschitz constant of $\alpha \circ f \circ \alpha^{-1}: \alpha(U_\alpha) \to \boldsymbol{R}^n$, which is denoted by

Received June 29, 1973.

$L(\alpha\circ f\circ\alpha^{-1}$ on $\alpha(U_\alpha))$ or simply by $L(\alpha\circ f\circ\alpha^{-1})$, is finite. This follows from the facts that we can choose a positive number ρ_1 such that for each x the closed ρ_1-ball $B(x:\rho_1)=\{y\in M\,|\,d(x,y)\leqq\rho_1\}$ around x is contained in some U_α and that for any chart (V,γ) for M, and for each compact subset X of M contained in V the map $\gamma:(X,d)\to(\gamma(X),|\cdot|)$ is a lipeomorphism. We have the following

PROPOSITION 1-1. *There exists a positive number C_1 with the following property: For each α and each $x,y\in U_\alpha$ we have $C_1^{-1}|\alpha(x)-\alpha(y)|\leqq d(x,y)\leqq C_1|\alpha(x)-\alpha(y)|$.*

For each $f\in L(M)$ with $d_0(f,1_M)<\lambda_1$ we define $d_\ell(f,1_M)$ by $d_\ell(f,1_M)=d_0(f,1_M)+\mathrm{Sup}_\alpha L(\alpha\circ f\circ\alpha^{-1}-1$ on $\alpha(U_\alpha))$.

PROPOSITION 1-2. *Let f be any element in $L(M)$ with $d_0(f,1_M)<\lambda_1$. If $d_\ell(f,1_M)$ is sufficiently small f is a lipeomorphism.*

Proof. We use the following

LEMMA (*Lipschitz Inverse Function Theorem* [3]). *Let E,F be Banach space, $U\subset E$ and $V\subset F$ non-empty open sets and $g:U\to V$ a homeomorphism such that g^{-1} is lipschitz. Then for each $h:U\to F$ with $L(h-g)\cdot L(g^{-1})<1$, $h(U)=V'$ is an open set of F, $h:U\to V'$ is a homeomorphism and $h^{-1}:V'\to U$ is lipschitz.*

Let f be an element of $L(M)$ such that $d_0(f,1_M)<\lambda_1$ and $d_\ell(f,1_M)<\mathrm{Min}\{1,\rho_1/2\}$. By the above lemma and Prop 1–1 $f(U_\alpha)$ is an open set of M and $f:U_\alpha\to f(U_\alpha)$ is a lipeomorphism. In particular $f(M)$ is open. Since M is compact connected $f(M)=M$. We can complete the proof by proving that f is injective. To do this, take x, $y\in M$ with $f(x)=f(y)$. Then, $d(f(x),x)\leqq d_0(f,1_M)\leqq d_\ell(f,1_M)<\rho_1/2$. Similarly $d(f(y),y)<\rho_1/2$. Hence y is contained in $B(x:\rho_1)$ which is contained in some U_α. As $f:U_\alpha\to f(U_\alpha)$ is injective we have $x=y$. q.e.d.

§ 2. Lipschitz vector fields on M.

Let $X^0(M)$ denote the set of all continuous vector fields on M and $\|\cdot\|$ be the norm on $X^0(M)$ induced by the Riemannian metric $\|\cdot\|:\|u\|=\mathrm{Sup}_{x\in M}\|u_x\|$ for any $u=(u_x)_{x\in M}\in X^0(M)$. $(X^0(M),\|\cdot\|)$ is a Banach space. For each (U_α,α) put $U'_\alpha=\alpha(U_\alpha)$ and let $T_\alpha:TM|U_\alpha\to U'_\alpha\times\boldsymbol{R}^n$ be the isomorphism induced by α. Let $D\alpha:TM|U_\alpha\to\boldsymbol{R}^n$ be the composite of

$T_\alpha: TM|U_\alpha \to U'_\alpha \times \boldsymbol{R}^n$ and the projection $U'_\alpha \times \boldsymbol{R}^n \to \boldsymbol{R}^n$. $D\alpha$ is considered as the differential of α. Then for each $v \in X^0(M)$ we define v_α by $v_\alpha = D\alpha \circ v: U_\alpha \to \boldsymbol{R}^n$, and define $|v|$ by $|v| = \mathrm{Sup}_\alpha \mathrm{Sup}_{x \in U_\alpha} |v_\alpha(x)|$. Then $|\cdot|: X^0(M) \to \boldsymbol{R}^+ = \{a \in \boldsymbol{R} | a \geqq 0\}$ is a norm on $X^0(M)$ and it is equivalent to $\|\cdot\|$. The equivalence of $|\cdot|$ and $\|\cdot\|$ follows from the following.

PROPOSITION 2-1. *There exists a positive number C_2 such that for any α and any $v \in TM|U_\alpha$ we have $C_2^{-1}\|v\| \leqq |D\alpha(v)| \leqq C_2\|v\|$.*

An element $v \in X^0(M)$ is called a lipschitz vector field on M if and only if for each α, $v_\alpha: U_\alpha \to \boldsymbol{R}^n$ is lipschitz i.e. $v_\alpha \circ \alpha^{-1}: U'_\alpha \to \boldsymbol{R}^n$ is lipschitz. Denote the set of all lipschitz vector fields by $X_\ell(M)$. We define a norm $|\cdot|_\ell$ on $X_\ell(M)$ by $|v|_\ell = |v| + \mathrm{Sup}_\alpha\{L(v_\alpha \circ \alpha^{-1})\}$ for any $v \in X_\ell(M)$. Then $(X_\ell(M), |\cdot|_\ell)$ is a Banach space.

Let $\exp = (\exp_x)_{x \in M}$ be the exponential map induced by the Riemannian metric $\|\cdot\|$. In a normed space $(E, \|\cdot\|)$ we denote the closed λ-ball around the origin by $(E, \|\cdot\|)_\lambda$ and the open λ-ball around the origin by $(E, \|\cdot\|)_\lambda^\circ$. We can choose a positive number λ_2 such that for each $x \in M$ $\exp_x$ is a diffeomorphism of $(T_x(M), \|\cdot\|)_{\lambda_2}^\circ$ onto the open λ_2-ball around x in (M, d). Hence for this λ_2 $\exp: (X^0(M), \|\cdot\|)_{\lambda_2}^\circ \ni v \to \exp v = \exp \circ v \in \{f \in C^0(M) | d_0(f, 1_M) < \lambda_2\}$ is a bijective map. And for each $v \in (X^0(M), \|\cdot\|)_{\lambda_1}^\circ$ we have $d_0(\exp v, 1_M) = \|v\|$. For the convenience assume $\lambda_2 \leqq \lambda_1$. By the equivalence of $|\cdot|$ and $\|\cdot\|$ we can choose a positive number ε_1 such that $(X^0(M), |\cdot|)_{\varepsilon_1}^\circ$ is contained in $(X^0(M), \|\cdot\|)_{\lambda_2}^\circ$.

PROPOSITION 2-2. *We can choose a positive number $\varepsilon_2: 0 < \varepsilon_2 \leqq \varepsilon_1$ such that*
(i) *for each $v \in (M, |\cdot|)_{\varepsilon_2}^\circ$ $\exp v$ is contained in $L(M)$ if and only if v is contained in $X_\ell(M)$ and that*
(ii) *for each sequence $\{v^{(i)}\}_{i=1}^\infty \subset X_\ell(M) \cap (X^0(M), |\cdot|)_{\varepsilon_2}^\circ$*

$$d_\ell(\exp v^{(i)}, 1_M) \to 0 \quad as \quad i \to \infty ,$$

iff

$$|v^{(i)}|_\ell \to 0 \quad as \quad i \to \infty .$$

Proof. We take any (U_α, α) and fix it. For each $(x', \xi) \in U'_\alpha \times \boldsymbol{R}^n$ with $|\xi| < \varepsilon_1$ we define $e(x', \xi)$ by $e(x', \xi) = \alpha \circ \exp \circ T\alpha^{-1}(x', \xi)$. By the choice of ε_1 this is well-defined and e is of class C^∞. Since $e(x', 0) = x'$

and $(De)_{2(x',0)} = 1_{R^n}$, if we represent $e(x',\xi)$ by $e(x',\xi) = x' + \xi + r(x'\xi)$, then r is of class C^∞ and $(Dr)_{(x',0)} = 0$ as $(Dr)_{1(x',0)} = (Dr)_{2(x',0)} = 0$ for any $x' \in U'_\alpha$. Recalling that $\mathscr{D}(\alpha) \supset \overline{U}_\alpha$, by the mean value theorem, we have the following

(A): There exist a positive number $\varepsilon_2^{(\alpha)}: 0 < \varepsilon_2^{(\alpha)} \leqq \varepsilon_1$ and a function $L^{(\alpha)}: (0, \varepsilon_2^{(\alpha)}) \to [0, 1)$ satisfying the following properties.

(iii) For each $x', y' \in U'_\alpha$, $0 < \varepsilon < \varepsilon_2^{(\alpha)}$ and ξ, $\eta \in \boldsymbol{R}^n$ with $|\xi|, |\eta| \leqq \varepsilon$ we have $|r(x',\xi) - r(y',\eta)| \leqq L^{(\alpha)}(\varepsilon)\{|x' - y'| + |\xi - \eta|\}$.

(iv) $L^{(\alpha)}(\varepsilon)$ 0 as $\varepsilon \to 0$

Now, take $\varepsilon: 0 < \varepsilon < \varepsilon_2^{(\alpha)}$ and $v \in (X^0(M), |\cdot|)_\varepsilon$ and put $v_\alpha = D\alpha \circ v: U_\alpha \to \boldsymbol{R}^n$ and $h = \exp v \in C^0(M)$. We have $h(\overline{U}_\alpha) \subset \mathscr{D}(\alpha)$ since $d_0(h, 1_M) = \|v\| < \lambda_2 \leqq \lambda_1$. For each $x' \in U'_\alpha$ put $x = \alpha^{-1}(x')$. Then, we have

$$\begin{aligned}(x', v_\alpha \circ \alpha^{-1}(x')) &= T\alpha(v_x) = T\alpha \circ \exp_x^{-1}(h(x)) \\ &= T\alpha \circ \exp_x^{-1} \circ \alpha^{-1}(\alpha \circ h \circ \alpha^{-1}(x')) ,\end{aligned}$$

which implies

$$\begin{aligned}\alpha \circ h \circ \alpha^{-1}(x') &= e(x', v_\alpha \circ \alpha^{-1}(x')) \\ &= x' + v_\alpha \circ \alpha^{-1}(x') + r(x', v_\alpha \circ \alpha^{-1}(x')) ,\end{aligned}$$

from which we get

$$(\alpha \circ h \circ \alpha^{-1} - 1)(x') = v_\alpha \circ \alpha^{-1}(x') + r(x', v_\alpha \circ \alpha^{-1}(x')) .$$

Hence for each $x', y' \in U'_\alpha$ we have

$$\begin{aligned}&(\alpha \circ h \circ \alpha^{-1} - 1)(x') - (\alpha \circ h \circ \alpha^{-1} - 1)(y') \\ &\quad = \{v_\alpha \circ \alpha^{-1}(x') - v_\alpha \circ \alpha^{-1}(y')\} + \{r(x', v_\alpha \circ \alpha^{-1}(x')) - r(y', v_\alpha \circ \alpha^{-1}(y'))\} .\end{aligned}$$

By this equality we have the followings:

(v) If v is lipschitz then we have

$$\begin{aligned}&|(\alpha \circ h \circ \alpha^{-1} - 1)(x') - (\alpha \circ h \circ \alpha^{-1} - 1)(y')| \\ &\quad \leqq L(v_\alpha \circ \alpha^{-1})|x' - y'| + L^{(\alpha)}(\varepsilon)\{|x' - y'| + L(v_\alpha \circ \alpha^{-1})|x' - y'|\} \\ &\quad \leqq \{L^{(\alpha)}(\varepsilon) + |v|_\ell + L^{(\alpha)}(\varepsilon)|v|_\ell\}|x' - y'| .\end{aligned}$$

(vi) If $h = \exp v$ is lipschitz then we have

$$\begin{aligned}d_\ell(h, 1_M)\cdot|x' - y'| &\geqq L(\alpha \circ h \circ \alpha^{-1} - 1)|x' - y'| \\ &\geqq |(\alpha \circ h \circ \alpha^{-1} - 1)(x') - (\alpha \circ h \circ \alpha^{-1} - 1)(y')| \\ &\geqq |v_\alpha \circ \alpha^{-1}(x') - v_\alpha \circ \alpha^{-1}(y')|\end{aligned}$$

$$
\begin{aligned}
&- |r(x', v_\alpha \circ \alpha^{-1}(x')) - r(y', v_\alpha \circ \alpha^{-1}(y'))| \\
&\geqq |v_\alpha \circ \alpha^{-1}(x') - v_\alpha \circ \alpha^{-1}(y')| \\
&\quad - L^{(\alpha)}(\varepsilon)\{|x' - y'| + |v_\alpha \circ \alpha^{-1}(x') - v_\alpha \circ \alpha^{-1}(y')|\}
\end{aligned}
$$

As $0 \leqq L^{(\alpha)}(\varepsilon) < 1$ we have by this inequality

$$
\begin{aligned}
&|v_\alpha \circ \alpha^{-1}(x) - v_\alpha \circ \alpha^{-1}(y)| \\
&\quad \leqq [\{d_\ell(h, 1_M) + L^{(\alpha)}(\varepsilon)\}/(1 - L^{(\alpha)}(\varepsilon))] \cdot |x' - y'|
\end{aligned}
$$

The proof is complete by using (iv), (v) and (v). q.e.d.

§ 3. Lipeomorphisms close to an Anosov diffeomorphism on M.

LEMMA 3-1. *There exist positive numbers* $\varepsilon_3 : 0 < \varepsilon_3 \leqq \varepsilon_1$ *and* C_3 *with the following property. For any* $x \in U_\alpha$ *and* $\xi, \eta \in \boldsymbol{R}^n$ *with* $|\xi|, |\eta| < \varepsilon_3$ *we have*

$$
C_3^{-1} |\xi - \eta| \leqq |y' - z'| \leqq C_3 |\xi - \eta|
$$

where $y' = \alpha \circ \exp_x \circ (D\alpha)_x^{-1}(\xi)$ *and* $z' = \alpha \circ \exp_x \circ (D\alpha)_x^{-1}(\eta)$.

Proof. Take α and fix it. In the proof of Prop. 2–2 we defined e and r. By (A) we can choose a positive number $\varepsilon_3^{(\alpha)} : 0 < \varepsilon_3^{(\alpha)} \leqq \varepsilon_1$ such that for any $x', y' \in U'_\alpha$ and any $\xi, \eta \in \boldsymbol{R}^n$ with $|\xi|, |\eta| < \varepsilon_3^{(\alpha)}$ we have

$$
|r(x', \xi) - r(y', \eta)| \leqq 1/2(|x' - y'| + |\xi - \eta|)
$$

For any $x \in U_\alpha$ and $\xi, \eta \in \boldsymbol{R}^n$ with $|\xi|, |\eta| < \varepsilon_3^{(\alpha)}$ putting $y' = \alpha \circ \exp_x \circ (D\alpha)_x^{-1}(\xi)$, $z' = \alpha \circ \exp_x \circ (D\alpha)_x^{-1}(\eta)$ and $x' = \alpha(x)$, we have $y' = e(x', \xi)$ and $z' = e(x', \eta)$. Hence

$$
\begin{aligned}
|y' - z'| &\leqq |\xi - \eta| + |r(x', \xi) - r(x', \eta)| \leqq |\xi - \eta| + 1/2\,|\xi - \eta| \\
&\leqq C_3 |\xi - \eta|
\end{aligned}
$$

and

$$
\begin{aligned}
|y' - z'| &\geqq |\xi - \eta| - |r(x', \xi) - r(x', \eta)| \geqq |\xi - \eta| - 1/2\,|\xi - \eta| \\
&\geqq C_3^{-1} |\xi - \eta|
\end{aligned}
$$

Hence we can take $C_3 = 2$ and $\varepsilon_3 = \operatorname{Inf}_\alpha \{\varepsilon_3^{(\alpha)}\}$ q.e.d.

COROLLARY. *We can take positive numbers* λ *and* C *such that for any* $x \in M$ *and* $u, v \in T_x M$ *with* $\|u\|, \|v\| < \lambda$ *we have*

$$
C^{-1} \|u - v\| \leqq d(\exp_x u, \exp_x v) \leqq C \|u - v\| .
$$

Proof. This follows from Lemma 3–1, Prop. 1–1 and Prop. 2–1. q.e.d.

LEMMA 3-2. *There exist positive numbers* $\delta_1, \varepsilon_4 : 0 < \varepsilon_4 \leqq \varepsilon_3$, *a function* $L_1 : (0, \delta_1) \times (0, \varepsilon_4) \to \boldsymbol{R}^+$ *and a continuous map* $r : (X_\ell(M), |\cdot|_\ell)^\circ_{\delta_1} \times (X^0(M), |\cdot|)^\circ_{\varepsilon_4} \to X^0(M)$ *with the following properties*:

(i) *It holds that* $\exp w \circ \exp v = \exp(w + v + r(w, v))$ *for each* $w \in (X_\ell(M), |\cdot|_\ell)^\circ_{\delta_1}$ *and* $v \in (X^0(M), |\cdot|)^\circ_{\varepsilon_4}$

(ii) *For each* $\delta : 0 < \delta < \delta_1$, $\varepsilon : 0 < \varepsilon < \varepsilon_4$, $w \in (X_\ell(M), |\cdot|_\ell)_\delta$ *and* $v, v' \in (X^0(M), |\cdot|)_\varepsilon$ *we have* $|r(w, v) - r(w, v')| \leqq L_1(\delta, \varepsilon)|v - v'|$ *and* $r(w, 0) = r(0, v) = 0$.

(iii) $L_1(\delta, \varepsilon) \to 0$ *as* $\delta, \varepsilon \to 0$.

Proof. Choose open subsets V_α of M for each α such that $V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$ and $\bigcup_\alpha V_\alpha = M$. We define a norm $|\cdot|'$ on $X^0(M)$ with respect to the covering by finite charts, $\{(V_\alpha, \alpha)\}_\alpha$, in the same way as we defined $|\cdot|$: For each $v \in X^0(M)$ we define $|v|'$ by $|v|' = \mathrm{Sup}_\alpha \mathrm{Sup}_{x \in V_\alpha} |v_\alpha(x)|$, where $v_\alpha = D\alpha \circ v$. As $|\cdot|'$ and $\|\cdot\|$ are equivalent $|\cdot|'$ and $|\cdot|$ are equivalent. We can choose a positive number $\varepsilon_4' : 0 < \varepsilon_4' \leqq \varepsilon_3$ such that for any w, $v \in X^0(M)$ with $|w|, |v|' < \varepsilon_4'$ we have $\exp v(\overline{V}_\alpha) \subset U_\alpha$ for any α and $d_0(\exp w \circ \exp v, 1_M) < \lambda_2$. Then for each $w, v \in X^0(M)$ with $|w|, |v|' < \varepsilon_4'$ there exists a unique $r(w, v) \in X^0(M)$ such that $\exp w \circ \exp v = \exp(w + v + r(w, v))$ and $d_0(\exp w \circ \exp v, 1_M) = \|w + v + r(w, v)\|$. It is clear that r is continuous and $r(w, 0) = (0, v) = 0$. Take any α and fix it. Put $V_\alpha' = \alpha(V_\alpha)$. For each $(x', \xi, \eta) \in V_\alpha' \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ with $|\xi|, |\eta| < \varepsilon_4'$ we define $P_\alpha(x', \xi, \eta)$ by $P_\alpha(x', \xi, \eta) = D_\alpha \circ \exp_x^{-1} \circ \exp_y \circ (D\alpha)_y^{-1}(\xi)$, where $x = \alpha^{-1}(x')$ and $y = \exp_x \circ (D\alpha)_x^{-1}(\eta)$. By the choice of ε_4' this is well-defined and P_α is of class C^∞. It is clear that $P_\alpha(x', 0, 0) = 0$, $P_\alpha(x', \xi, 0) = \xi$ and $P_\alpha(x', 0, \eta) = \eta$. Hence if we express $P_\alpha(x', \xi, \eta)$ by $P_\alpha(x', \xi, \eta) = \xi + \eta + r^{(\alpha)}(x', \xi, \eta)$ then $r^{(\alpha)}$ is of class C^∞, $(Dr^{(\alpha)})_{1(x', \xi, 0)} = (Dr^{(\alpha)})_{1(x', 0, \eta)} = 0$, $(Dr^{(\alpha)})_{2(x', \xi, 0)} = 0$, $(Dr^{(\alpha)})_{3(x', 0, \eta)} = 0$ and so in particular $(Dr^{(\alpha)})_{(x', 0, 0)} = 0$. Noting that $\mathscr{D}(\alpha) \supset \overline{U}_\alpha \supset U_\alpha \supset \overline{V}_\alpha \supset V_\alpha$, we can conclude the following by the mean value theorem.

(B) There exist two positive numbers $\delta_1' : 0 < \delta_1' \leqq \varepsilon_4'$ and $\varepsilon_4'' : 0 < \varepsilon_4'' \leqq \varepsilon_4'$ and a function $L_1^{(\alpha)} : (0, \delta_1') \times (0, \varepsilon_4'') \to \boldsymbol{R}^+$ with the following properties:

(iv) For each $\delta : 0 < \delta < \delta_1'$, $\varepsilon : 0 < \varepsilon < \varepsilon_4''$, $x', y' \in V_\alpha'$ and $\xi, \eta, \zeta, \theta \in \boldsymbol{R}^n$ with $|\xi|, |\zeta| \leqq \delta$ and $|\eta|, |\theta| \leqq \varepsilon$ we have

$$|r^{(\alpha)}(x', \xi, \eta) - r^{(\alpha)}(y', \zeta, \theta)| \leqq L_1^{(\alpha)}(\delta, \varepsilon) \cdot \{|x' - y'| + |\xi - \zeta| + |\eta - \theta|\},$$

(v) $L_1^{(\alpha)}(\delta, \varepsilon) \to 0$ as $\delta, \varepsilon \to 0$.

Take any positive numbers δ, ε with $0<\delta<\delta_1'$ and $0<\varepsilon<\varepsilon_4''$ and fix them. For each $w, v, v' \in X^0(M)$ with $|w| \leqq \delta$ and $|v|', |v'|' \leqq \varepsilon$ we define w_α, v_α, v'_α, $r(w,v)_\alpha$ and $r(w,v')_\alpha$ as before. Then for each $x' \in V'_\alpha$ we have

$$r(w,v)_\alpha \circ \alpha^{-1}(x') = P_\alpha(x', w_\alpha \circ \alpha^{-1}(y'), v_\alpha \circ \alpha^{-1}(x')) - \{w_\alpha \circ \alpha^{-1}(x') + v_\alpha \circ \alpha^{-1}(x')\}$$

and

$$\begin{aligned} r(w,v')_\alpha \circ \alpha^{-1}(x') = P_\alpha(x', w_\alpha \circ \alpha^{-1}(z'), v'_\alpha \circ \alpha^{-1}(x')) \\ - \{w_\alpha \circ \alpha^{-1}(x') + v'_\alpha \circ \alpha^{-1}(x')\}, \end{aligned}$$

where

$$x = \alpha^{-1}(x'), \qquad y' = \alpha \circ \exp_x \circ (D\alpha)_x^{-1}(v_\alpha \circ \alpha^{-1}(x))$$

and

$$z' = \alpha \circ \exp_x \circ (D\alpha)_x^{-1}(v'_\alpha \circ \alpha^{-1}(x')) .$$

Hence we get

$$\begin{aligned} &|r(w,v)_\alpha \circ \alpha^{-1}(x') - r(w,v')_\alpha \circ \alpha^{-1}(x')| \\ &\quad \leqq |w_\alpha \circ \alpha^{-1}(y') - w_\alpha \circ \alpha^{-1}(z')| + |r^{(\alpha)}(x', w_\alpha \circ \alpha^{-1}(y'), v_\alpha \circ \alpha^{-1}(x')) \\ &\qquad - r^{(\alpha)}(x', w_\alpha \circ \alpha^{-1}(z'), v'_\alpha \circ \alpha^{-1}(x'))| \\ &\quad \leqq \{1 + L_1^{(\alpha)}(\delta,\varepsilon)\} \cdot |w_\alpha \circ \alpha^{-1}(y') - w_\alpha \circ \alpha^{-1}(z')| \\ &\qquad + L_1^{(\alpha)}(\delta,\varepsilon)|v_\alpha \circ \alpha^{-1}(x') - v'_\alpha \circ \alpha^{-1}(x')| . \end{aligned}$$

If we assume that w is contained in $L(M)$, then we have by Lemma 3–1

$$\begin{aligned} &|r(w,v)_\alpha \circ \alpha^{-1}(x') - r(w,v')_\alpha \circ \alpha^{-1}(x')| \\ &\quad \leqq \{1 + L_1^{(\alpha)}(\delta,\varepsilon)\} \cdot |w|_\ell \cdot |y' - z'| + L_1^{(\alpha)}(\delta,\varepsilon)|v_\alpha \circ \alpha^{-1}(x') - v'_\alpha \circ \alpha^{-1}(x')| \\ &\quad \leqq \{L_1^{(\alpha)}(\delta,\varepsilon) + C_3|w|_\ell \cdot (1 + L_1^{(\alpha)}(\delta,\varepsilon))\} \cdot |v_\alpha \circ \alpha^{-1}(x') - v'_\alpha \circ \alpha^{-1}(x')| . \end{aligned}$$

From this inequality, the equivalence of $|\cdot|$ and $|\cdot|'$ and (v) the proof of Lemma 3–2 is complete. q.e.d.

In the followings we assume that $f: M \to M$ is a C^1-diffeomorphism. For this f we define a linear automorphism f_* of $X^0(M)$ by

$$f_*(v) = df \circ v \circ f^{-1} \quad \text{for any} \quad v \in X^0(M) ,$$

where df is the differential of f.

LEMMA 3-3. *There exist a positive number* ε_5, *a bounded function* $L_2: (0, \varepsilon_5) \to \boldsymbol{R}^+$ *and a continuous map* $s: (X^0(M), |\cdot|)^\circ_{\varepsilon_5} \to X^0(M)$ *with the* *following properties.*

(i) $f \circ \exp v \circ f^{-1} = \exp(f_*(v) + s(v))$ *for any* $v \in (X^0(M), |\cdot|)^\circ_{\varepsilon_5}$,

(ii) $s(0) = 0$ *and for each* $\varepsilon: 0 < \varepsilon < \varepsilon_5$ *and* $v, v' \in (X^0(M), |\cdot|)_\varepsilon$ *we have*

$$|s(v) - s(v')| \leqq L_2(\varepsilon)|v - v'|\ ,$$

(iii) $L_2(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. (cf. [4]) We can define a map F of a neighborhood of the origin in $X^0(M)$ into $X^0(M)$ such that $\exp(F(v)) = f \circ \exp v \circ f^{-1}$ for each $v \in X^0(M)$ with $|v|$ sufficiently small. It is clear that $F(0) = 0$. Since f is of class C^1, F is so and in fact, the differential of F at the origin is f_*. Hence the proof is easy by using the mean value theorem for $s = F - f_*$.

For the convenience we may assume $\varepsilon_5 \leqq \varepsilon_4$.

Let $X_b(M)$ be the set of all bounded vector fields on M. A complete norm $\|\cdot\|_b$ on $X_b(M)$ is defined by

$$\|v\|_b = \operatorname{Sup}_{x \in M} \|v_x\| \quad \text{for any} \quad v \in X_b(M)\ .$$

Lemma 3–3 is also true for $(X_b(M), \|\cdot\|_b)$. We make use of the same notations as those in Lemma 3–3 for $(X_b, \|\cdot\|_b)$, f_*, ε_5, L_2, s. If f is an Anosov deffeomorphism $1 - f_*$ is a linear automorphism of $X^0(M)$ and also of $X_b(M)$, where 1 is the identity map (cf. [4]).

We will prove the following well known fact.

LEMMA 3-4. *If f is an Anosov diffeomorphism then f is expansive i.e. there exists a positive number λ_0 such that* $\operatorname{Sup}_{n \in Z} d(f^n(x), f^n(y)) > \lambda_0$ *for any $x, y \in M$ with $x \neq y$.*

Proof. (cf. [5]) By the above remark there exists a positive number $\lambda_0: 0 < 2\lambda_0 < \lambda_2$ such that for each $v, v' \in (X_b(M), \|\cdot\|_b)_{2\lambda_0}$ we have

$$\|s(v) - s(v')\|_b \leqq 1/2 \cdot \|(1 - f_*)^{-1}\|_b^{-1} \cdot \|v - v'\|_b\ .$$

We assert the following.

(C) Let u be a map of M into itself such that $f \circ u = u \circ f$ and $u \neq 1_M$. Then $d_0(u, 1_M) = \operatorname{Sup}_{x \in M} d(u(x), x) > 2 \cdot \lambda_0$.

Choose any map $u: M \to M$ with $f \circ u = u \circ f$ and $d_0(u, 1_M) \leqq 2 \cdot \lambda_0$. For this u there exists a unique element $v \in X_b(M)$ such that $u = \exp v$ and

$$d_0(u, 1_M) = \|v\|_b .$$

Then we have

$$f \circ \exp v \circ f^{-1} = f \circ u \circ f^{-1} = u = \exp v ,$$

and hence $f_*(v) + s(v) = v$, or $v = (1 - f_*)^{-1}(s(v))$.
By the choice of λ_0, $(1 - f_*)^{-1} \circ s$ is a lipschitz map of $(X_b(M), \|\cdot\|_b)_{2\lambda_0}$ into itself with the lipschitz constant $L((1 - f_*)^{-1} \circ s) \leqq 1/2$. Hence by the contraction principle v must be 0 i.e. u must be the identity map of M. Now, take any $x, y \in M$ with $x \neq y$. Put $\mathrm{Per}(f) = \{x \in M \mid x$ is a periodic point of $f\}$.
Case 1: the case of $x \notin \mathrm{Per}(f)$ or $y \notin \mathrm{Per}(f)$. Suppose $x \notin \mathrm{Per}(f)$.

We can define a map $u: M \to M$ as following:
For any $z \in M$

$$u(z) = \begin{cases} f^n(y) & \text{if } \exists n \text{ with } z = f^n(x) \\ z & \text{otherwise.} \end{cases}$$

Then it is clear that $f \circ u = u \circ f$ and that $u \neq 1_M$. By (c) we have $d_0(u, 1_M) > 2 \cdot \lambda_0$. Hence there exists an integer n with $d(f^n(x), f^n(y)) > \lambda_0$. The case of $y \notin \mathrm{Per}(f)$ is similar.
Case II: the case of $x \in \mathrm{Per}(f)$ and $y \in \mathrm{Per}(f)$. Let r and s be the smallest periods of x and y respectively. Suppose $r = s$. We can define a map $u: M \to M$ as following:
For any $z \in M$

$$u(z) = \begin{cases} f^n(y) & \text{if } \exists n \text{ with } z = f^n(x) \\ z & \text{otherwise.} \end{cases}$$

It is clear that $f \circ u = u \circ f$ and $u \neq 1_M$. By (c) we have $d_0(u, 1_M) > 2\lambda_0$. By the definition of u we conclude that there exists an integer n with $d(f^n(x), f^n(y)) = d_0(u, 1_M) > 2 \cdot \lambda_0 > \lambda_0$. Suppose $r > s$. We can define a map $u: M \to M$ as follows:
For any $z \in M$

$$u(z) = \begin{cases} f^{s+n}(x) & \text{if } \exists n \text{ with } z = f^n(x) \\ z & \text{otherwise.} \end{cases}$$

It is clear that $f \circ u = u \circ f$. Since $x \neq f^s(x)$, $u \neq 1_M$. Hence we have $d_0(u, 1_M) > 2 \cdot \lambda_0$. By the definition of u there exists an integer n with

$d(f^n(x), f^{s+n}(x)) > 2\cdot\lambda_0$. As $f^n(y) = f^{s+n}(y)$ we have

$$d(f^n(x), f^n(y)) + d(f^{s+n}(y), f^{s+n}(x)) \geqq d(f^n(x), f^{s+n}(x)) > 2\cdot\lambda_0 .$$

Hence $d(f^n(x), f^n(y)) > \lambda_0$ or $d(f^{n+s}(x), f^{n+s}(y)) > \lambda_0$.
The case of $r < s$ is similar. q.e.d.

For each $g \in L(M)$ with $d_0(g\circ f^{-1}, 1_M) < \lambda_1$ we define $d_\ell(g, f)$ by $d_\ell(g, f) = d_\ell(g\circ f^{-1}, 1_M)$. (Note that C^1-diffeomorphism on M is a lipeomorphism on M.)

THEOREM. *Assume that f is an Anosov diffeomorphism. Then there exists a positive number ε_0 satisfying the followihg condition. For any $\varepsilon: 0 < \varepsilon < \varepsilon_0$ there exists a positive number $\delta = \delta(\varepsilon)$ with the property that for each $g \in L(M)$ with $d_\ell(g, f) < \delta$ there exists a unique homeomorphism $u: M \to M$ such that $g\circ u = u\circ f$ and $d_0(u, 1_M) < \varepsilon$.*

Proof. Put $K = |f^*| + \mathrm{Sup}_{0<\varepsilon<\varepsilon_5} L_2(\varepsilon)$. K is finite by Lemma 3–3. For each $v \in (X^0(M), |\cdot|)^\circ_{\varepsilon_5}$ we have

$$|f_*(v) + s(v)| \leqq |f_*|\,|v| + L_2(|v|)\,|v| \leqq K\,|v| .$$

Choose a positive number ε_6 with $\varepsilon_6 \leqq \mathrm{Min}\{\varepsilon_5, \varepsilon_4/K\}$. From Lemma 3–2 and 3–3 we have

$$\exp w\circ f\circ \exp v\circ f^{-1} = \exp\{w + f_*(v) + s(v) + r(w: f_*(v) + s(v))\}$$

for any $w \in (X_\ell(M), |\cdot|_\ell)^\circ_{\delta_1}$ and $v \in (X^0(M), |\cdot|)^\circ_{\varepsilon_6}$. We may assume that $\|w + f_*(v) + s(v) + r(w: f_*(v) + s(v))\| < \lambda_2$ by making δ_1 and ε_6 sufficiently small. From the above expression we see that

$$\exp w\circ f\circ \exp v\circ f^{-1} = \exp v$$

holds if and only if

$$w + f_*(v) + s(v) + r(w: f_*(v) + s(v)) = v .$$

As f is Anosov, $1 - f_*$ is a linear automorphism. Hence the above equality is equivalent to

$$(1 - f_*)^{-1}(w + s(v) + r(w: f_*(v) + s(v))) = v .$$

Put $F(v) = f_*(v) + s(v)$ and $G_w(v) = (1 - f_*)^{-1}(w + s(v) + r(w: f_*(v) + s(v)))$. By (ii) in Lemma 3–2 and by (ii) in Lemma 3–3 we have

$$|r(w: F(v))| \leqq L_1(|w|_\ell, K\,|v|)K\,|v|$$

and $|s(v)| \leqq L_2(|v|)|v|$. Hence by (iii) in Lemma 3–2 and by (iii) in Lemma 3–3 we can choose positive numbers $\delta_2 : 0 < \delta_2 \leqq \delta_1$ and $\varepsilon_7 : 0 < \varepsilon_7 \leqq \varepsilon_6$ with the property that for each $w \in (X_\ell(M), |\cdot|_\ell)^\circ_{\delta_2}$ and $v \in (X^0(M), |\cdot|_{\varepsilon_7}$ we have

$$|(1 - f_*)^{-1}(r(w : F(v)))| \leqq 1/3\, |v|$$

and

$$|(1 - f_*)^{-1}(s(v))| \leqq 1/3\, |v|$$

On the other hand for each $w \in (X_\ell(M), |\cdot|_\ell)^\circ_{\delta_1}$ and $v,\ v' \in (X_0(M), |\cdot|)^\circ_{\varepsilon_6}$, putting $\delta = |w|_\ell$ and $\varepsilon = \text{Max}\,\{|v|, |v'|\}$, we have

$$\begin{aligned} |G_w(v) - G_w(v')| &\leqq |(1 - f_*)^{-1}|\,\{|s(v) - s(v')| + |r(w : F(v)) \\ &\quad - r(w : F(v'))|\} \\ &\leqq |(1 - f_*)^{-1}|\,\{L_2(\varepsilon)|v - v'| + L_1(\delta, K\varepsilon)(|f_*|\cdot|v - v'| \\ &\quad + L_2(\varepsilon)|v - v'|)\} \\ &\leqq |(1 - f_*)^{-1}|\,\{L_2(\varepsilon) + KL_1(\delta, K\varepsilon)\}|v - v'|\ . \end{aligned}$$

Hence by (ii) in Lemma 3–2 and by (iii) in Lemma 3–3 we can choose positive numbers $\delta_3 : 0 < \delta_3 \leqq \delta_1$ and $\varepsilon_8 : 0 < \varepsilon_8 \leqq \varepsilon_6$ such that for each $w \in (X_\ell(M), |\cdot|_\ell)^\circ_{\delta_3}$ and $v, v' \in (X^0(M), |\cdot|)^\circ_{\varepsilon_8}$ we have

$$|G_w(v) - G_w(v')| \leqq 1/2\, |v - v'|\ .$$

For the convenience we may assume that $\delta_3 \leqq \delta_2$ and $\varepsilon_8 \leqq \varepsilon_7$. Now, take any positive number ε with $0 < \varepsilon < \varepsilon_8$. For this ε we can choose a positive number δ' such that for each $w \in (X_\ell(M), |\cdot|_\ell)^\circ_{\delta'}$ we have

$$|(1 - f_*)^{-1}(w)| < 1/3\varepsilon\ .$$

Hence, putting $\delta = \text{Min}\,\{\delta', \delta_3\}$, we have the following

(i) $|G_w(v)| < \varepsilon$ for any $w \in (X_\ell(M), |\cdot|_\ell)^\circ_\delta$ and $v \in (X^0(M), |\cdot|)_\varepsilon$

(ii) $|G_w(v) - G_w(v')| \leqq 1/2\, |v - v'|$
for any $w \in (X_\ell(M), |\cdot|_\ell)^\circ_\delta$ and $v,\ v' \in (X^0(M), |\cdot|)_\varepsilon$

And so by the contraction principle

(iii) for any $w \in (X_\ell(M), |\cdot|_\ell)_\delta$ there exists a unique $v \in X^0(M)$ such that $|v| < \varepsilon$ and $G_w(v) = v$ i.e.

$$\exp w \circ f \circ \exp v \circ f^{-1} = \exp v\ .$$

Note that $\exp v$ is onto since $\exp v$ is homotopic to the identity. Hence the proof of theorem is complete except for proving the injectivity of $u = \exp v$, remarking several facts that for any $g \in L(M)$ and $u \in C^0(M)$ $g \circ u = u \circ f$ if and only if $(g \circ f^{-1}) \circ f \circ u \circ f^{-1} = u$, that if $d_\ell(g, f)$ is sufficiently small there exists a unique $w \in X_\ell(M)$ with $|w|_\ell$ sufficiently small such that $g \circ f^{-1} = \exp w$ (see Prop. 2–2), that if $d_0(u, 1_M)$ is sufficiently small there exists a unique $v \in X^0(M)$ with $|v|$ sufficiently small such that $u = \exp v$ and that $|\cdot|$ and $\|\cdot\|$ are equivalent. To prove the injectivity let g be a lipeomorphism of M and u be in $C^0(M)$ with $d_0(u, 1_M) < \lambda_0/2$ and assume $g \circ u = u \circ f$. Choose x, $y \in M$ with $u(x) = u(y)$. If $x \neq y$ there exists an integer n_0 such that $d(f^{n_0}(x), f^{n_0}(y)) \geqq \lambda_0$ by Lemma 3–4. As $g^{n_0} \circ u = u \circ f^{n_0}$ we have $u \circ f^{n_0}(x) = g^{n_0} \circ u(x) = g^{n_0} \circ u(y) = u \circ f^{n_0}(y)$. On the other hand as $d_0(u, 1_M) < \lambda_0/2$ and $d(f^{n_0}(x), f^{n_0}(y)) \geqq \lambda_0$ we have $u \circ f^{n_0}(x) \neq u \circ f^{n_0}(y)$. This is a contradiction. Hence $x = y$.

q.e.d.

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