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THE SCHEME OF LIE SUB-ALGEBRAS OF A LIE ALGEBRA AND THE EQUIVARIANT COTANGENT MAP*

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Introduction. The main object of this paper is to develop techniques for investigating the local properties of actions of an algebraic group on an algebraic variety. Our main tools are certain schemes which may be associated to Lie algebras. Let X be a scheme and let $\mathscr{L}$ be a locally free sheaf of Lie algebras over $\mathcal{O}_X$ of rank n. Let $\underline{\mathrm{Lie}}^r_{\mathscr{L}/X}$ denote the functor which assigns to each X-scheme Y with structure morphism, $p: Y \to X$, the set of all coherent sheaves of Lie subalgebras of $p^*\mathscr{L}$ which are sub-bundles of $p^*\mathscr{L}$ of rank r. Sections 1 and 2 are devoted to demonstrating that $\underline{\mathrm{Lie}}^r_{\mathscr{L}/X}$ is a representable functor and that representing scheme, $\mathrm{Lie}^r_{\mathscr{L}/X}$, is projective over X. Section 3 is devoted to a description of the tangent space to $\mathrm{Lie}^r_{\mathscr{L}/k}$ at a closed point corresponding to a Lie subalgebra of $\mathscr{L}$ of dimension r in the case where X is the spectrum of a field.

Section 4 is devoted to the construction of a certain canonical map associated to actions of algebraic groups with equidimensional orbits. The map may be interpreted in the following way. Let V be a variety over k. Let G be a linear algebraic group and let $\alpha: G \times_k V \to V$ be an action with equidimensional orbits. The map is the well known map which associates to each point of V, v, the Lie algebra of the stabilizer of v.

Section 5 is an attempt to calculate the dimension of the connected component of $\mathrm{Lie}^r_{\mathscr{L}/k}$ in which a Lie subalgebra, H, lies. Some conditions are given which imply that the component in which H lies is the closure of the orbit of H under the adjoint action in the case where $\mathscr{L}$ is the

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Lie algebra of an algebraic group. These conditions may be interpreted, under suitable circumstances, as principal orbit theorems.

Also, it is hoped that section 2 in which the results of section 1 are utilized to construct $\mathrm{Lie}^r_{\mathscr{X}/X}$ may be of some independent interest as a way of making certain relative constructions.

The results in this paper are really special to characteristic zero. However, the corresponding results in positive characteristic may be proven by entirely analogous methods. Over a field of positive characteristic, p, it is necessary to consider deformations of closed local sub-groups whose rings of functions are of the form $k[x_1, \cdots, x_r]/(x_1^{p^\nu}, \cdots, x_r^{p^\nu})$ with the x_i indeterminates, rather than the scheme of Lie sub-algebras of dimension r in a Lie algebra of dimension n. The interested reader is referred to a forthcoming paper for the details (Haboush [5]).

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§ 1. Lie Algebras.

Let R be a commutative ring with unit and let L be a projective Lie algebra of rank n over R. Then the bracket product on L is a map from the second exterior power of L, $\Lambda^2_R L$, to L. This map dualizes to a map, $D^1_L: L^* \to (\Lambda^2_R L)^*$, where the asterisk denotes the linear dual. By alternation, $(\Lambda^2_R L)^*$ may be identified with $\Lambda^2_R L^*$. The map D^1_L extends uniquely to an anti-derivation, D_L, of degree $+1$ of $\Lambda_R L^*$, the full exterior algebra of L^* over R, in itself. (An antiderivation of degree $+1$ is a graded S_0-map, D, of a graded algebra, $S = \bigoplus_{\nu \in Z} S_\nu$, into itself satisfying $D(u_p \cdot u_q) = D(u_p)\cdot u_q + (-1)^p u_p \cdot D(u_q)$ whenever $u_p \in S_p$ and $u_q \in S_q$.)

1.1. DEFINITION. Let R, L, $\Lambda_R L^*$, D_L be as above. Then D_L is called *the Lie anti-derivation associated to L.*

Let H be another projective, finite, Lie algebra over R and let $u: H \to L$ be a Lie algebra homomorphism. Then u dualizes to a map $u^*: L^* \to H^*$ and hence gives an algebra homomorphism, $\Lambda u^*: \Lambda_R L^* \to \Lambda_R H^*$. The fact that u is a Lie algebra homomorphism is expressed by the commutativity of the following diagram:

$$\begin{array}{ccc} H & \xrightarrow{u} & L \\ \uparrow & & \uparrow \\ \Lambda_R^2 H & \xrightarrow{\Lambda_R^2 u} & \Lambda_R^2 L \end{array} .$$

The vertical arrows are given by the brackets in H and L respectively. Dualizing gives the relation, $D_H^1 \circ u^* = \Lambda_R^2 u^* \circ D_L^1$. Thus, $D_H \cdot (\Lambda u^*) = (\Lambda u^*) D_L$. It follows that if $\mathfrak{A} = \ker \Lambda u^*$, $D_L(\mathfrak{A}) \subset \mathfrak{A}$. Now assume that H is a projective R-submodule of L of rank r which is a direct summand of L. Let $j: H \to L$ be the natural injection. Then $j^*: L^* \to H^*$ is surjective. Now j^* extends to $\Lambda(j^*): \Lambda_R L^* \to \Lambda_R H^*$ which is also surjective. Let $\mathfrak{A} = \ker \Lambda(j^*)$. Suppose that $D_L(\mathfrak{A}) \subset \mathfrak{A}$. Then D_L induces a derivation, D, on $\Lambda_R H^*$. In degree 1, this yields a commutative diagram,

$$\begin{array}{ccc} H^* & \xleftarrow{j^*} & L^* \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D_L} \\ \Lambda_R^2 H^* & \xleftarrow{\Lambda^2 j^*} & \Lambda_R^2 L^* \end{array} .$$

As j and $\Lambda^2 j$ are the natural injections and D_L^* is the bracket on L we have proven the following:

1.2. LEMMA. *Let R be a commutative ring with unit, let L be a projective Lie algebra over R and let $H \subset L$ be a direct summand of L. Let $\mathfrak{A} = \ker(\Lambda j^*)$ where j is the injection of H in L. Then H is a Lie subalgebra of L if and only if $D_L(\mathfrak{A}) \subset \mathfrak{A}$.*

Now let $N = \ker j^*$. As H is a direct summand of L, $\mathfrak{A} = N \cdot \Lambda_R L^*$. A moment's consideration will reveal that $D_L(\mathfrak{A}) \subset \mathfrak{A}$ if and only if $D_L(N) \subset N \wedge L^*$.

For the next portion of the discussion assume that L, H and N are free. Then the rank of N is $s = n - r$ and we may assume that $\Lambda^s N$ is generated by one element $\omega \in \Lambda^s L^*$. Moreover $N = \{m \in L^* : \omega \wedge m = 0\}$. Assume that $n_1, \cdots, n_s$ is a basis for N and that $\omega = n_1 \wedge \cdots \wedge n_s$. Then $D_L(\omega) = \sum_{j=1}^s (-1)^{j-1} n_1 \wedge \cdots \wedge D(n_j) \wedge \cdots \wedge n_s$, and so it is clear that if H is a subalgebra of $L, D_L(\omega) \in \omega \wedge L^*$. Now assume that $D_L(\omega) \in \omega \wedge L^*$. Then if $m \in N$, $m \wedge \omega = 0$. Thus $D_L(m \wedge \omega) = D_L(m) \wedge \omega - m \wedge D_L(\omega) = 0$. As $D_L(\omega) \in \omega \wedge L^*$, $m \wedge D_L(\omega) = 0$ and so $D_L(m) \wedge \omega = 0$. But $\{x \in \Lambda^2 L^* : x \wedge \omega = 0\} = N \wedge L^*$. Consequently $D_L(m) \in N \wedge L^*$.

It follows that $D_L(N) \subset N \wedge L^*$ if and only if $D_L(\omega) \in \omega \wedge L^*$. Moreover this is equivalent, by the preceding, to the condition that H be a subalgebra of L. Though we have assumed that H, N and L are free, by localizing sufficiently we have proven the following.

1.3. PROPOSITION. *Let R be commutative with unit and let L be a projective Lie algebra of rank n. Let H be a direct summand of L of rank r. Let $N = (L/H)^* \subset L^*$ and let $s = n - r$. Then H is a subalgebra of L if and only if $D_L(\Lambda^s N) \subset (\Lambda^s N) \cdot L^*$.*

§ 2. The Representibiliy Theorem.

2.1. DEFINITION. Let X be a scheme and let $\mathscr{L}$ be a locally free sheaf of Lie algebras over $\mathcal{O}_X$ of rank n. Then $\underline{\mathrm{Lie}}^r_{\mathscr{L}/X}$ is the functor which assigns to each X-scheme, Y with structure morphism $p: Y \to X$, the set of all coherent sheaves of Lie subalgebras of $p^*\mathscr{L}$ which are sub-bundles of $p^*\mathscr{L}$ of rank r.

2.2. THEOREM. *Let X be a scheme and let $\mathscr{L}$ be a locally free sheaf of Lie algebras over $\mathcal{O}_X$ of rank n. Then*

i) $\underline{\mathrm{Lie}}^r_{\mathscr{L}/X}$ *is a representable functor.*

ii) *If* $\mathrm{Lie}^r_{\mathscr{L}/X}$ *is the scheme representing* $\underline{\mathrm{Lie}}^r_{\mathscr{L}/X}$, $\mathrm{Lie}^r_{\mathscr{L}/X}$ *is a projective scheme over X.*

iii) *If $q: \mathrm{Lie}^r_{\mathscr{L}/X} \to X$ is the structure morphism, there is a subbundle of Lie subalgebras of $q^*(\mathscr{L})$ of rank r, denoted $\mathfrak{H}_{\mathscr{L}/X}$, so that the isomorphism of functors between $\underline{\mathrm{Lie}}^r_{\mathscr{L}/X}$ and $\mathrm{Hom}_X(-, \mathrm{Lie}^r_{\mathscr{L}/X})$ is given by $f \to f^*\mathfrak{H}_{\mathscr{L}/X}$ for $f \in \mathrm{Hom}_X(Y, \mathrm{Lie}^r_{\mathscr{L}/X})$.*

Proof. Our proof is an explicit construction. First we review our notation, which though standard is not without its eccentricities. Let X be a scheme and let $\mathscr{E}$ be a locally free sheaf of $\mathcal{O}_X$ modules. Then $P(\mathscr{E})$ denotes $\mathrm{Proj}\,(S_{\mathcal{O}_X}(\mathscr{E}))$. If $q: P(\mathscr{E}) \to X$ is the structure morphism and $\mathscr{F}$ is a coherent sheaf on $P(\mathscr{E})$, $\tilde{\Gamma}_*(\mathscr{F})$ will denote $\coprod_{n \in Z^+} q_*(\mathscr{F}(n))$ and $\tilde{\Gamma}_n(\mathscr{F})$ will denote $q_*(\mathscr{F}(n))$. If $f: Y \to X$ is an X-scheme, $\mathscr{E}_Y$ will occasionally be used in place of $f^*\mathscr{E}$.

First we observe that the constructions of § 1 apply to our situation. Namely, the bracket on $\mathscr{L}$ induces an anti-derivation, $D_{\mathscr{L}}$, of $\Lambda_{\mathcal{O}_X}\mathscr{L}^*$ in itself.

Consider $P(\Lambda^s_{\mathcal{O}_X}\mathscr{L})$ where $s = n - r$, and let $q: P(\Lambda^s_{\mathcal{O}_X}\mathscr{L}) \to X$. Then

$\tilde{\Gamma}_1(q^*\Lambda^s_{\mathcal{O}_X}\mathcal{L}^*) \simeq \Lambda^s_{\mathcal{O}_X}\mathcal{L}^* \otimes_{\mathcal{O}_X} S^1_{\mathcal{O}_X}(\Lambda^s_{\mathcal{O}_X}\mathcal{L}) \simeq \Lambda^s_{\mathcal{O}_X}\mathcal{L}^* \otimes_{\mathcal{O}_X}\Lambda^s_{\mathcal{O}_X}\mathcal{L}$ which, by alternation and a standard functorial isomorphism, is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\Lambda^s_{\mathcal{O}_X}\mathcal{L}, \Lambda^s_{\mathcal{O}_X}\mathcal{L})$. There is a section, denoted $\omega_{\mathcal{L}}$, in $\tilde{\Gamma}_1(q^*\Lambda^s\mathcal{L}^*)$, which corresponds to the identity map on $\Lambda^s_{\mathcal{O}_X}\mathcal{L}$.

Now observe that $\tilde{\Gamma}_*(q^*\Lambda_{\mathcal{O}_X}\mathcal{L}^*) \simeq \Lambda_{\mathcal{O}_X}\mathcal{L}^* \otimes_{\mathcal{O}_X} S_{\mathcal{O}_X}(\Lambda^s\mathcal{L})$. This can be identified with $\Lambda_{S_{\mathcal{O}_X}(\Lambda^s\mathcal{L})}(\tilde{\Gamma}_* q^*\mathcal{L}^*)$. Consequently $\omega_{\mathcal{L}}$ may be used to define a morphism, $\psi: q^*\mathcal{L}^* \to q^*\Lambda^{s+1}_{\mathcal{O}_X}\mathcal{L}^*$, by setting $\psi(u) = \omega_{\mathcal{L}} \wedge u$. Moreover $q^*(D_{\mathcal{L}})$ is an anti-derivation of $q^*\Lambda_{\mathcal{O}_X}\mathcal{L}^*$ in itself and $q^*(D_{\mathcal{L}})(\omega_{\mathcal{L}}) \in q^*(\Lambda^{s+1}_{\mathcal{O}_X}\mathcal{L}^*)$. Then $q^*(D_{\mathcal{L}})(\omega_{\mathcal{L}})$ and the image of $q^*\mathcal{L}^*$ under ψ generate a coherent subsheaf of $q^*\Lambda^{s+1}_{\mathcal{O}_X}\mathcal{L}^*$. Denote it $\mathcal{A}$. Set $\boldsymbol{P} = P(\Lambda^s\mathcal{L})$, let $\mathcal{G} = \Lambda^{r+1}_{\mathcal{O}_P}\mathcal{A}$ and let $\mathcal{F} = \Lambda^{r+1}_{\mathcal{O}_P}(q^*\Lambda^{s+1}_{\mathcal{O}_X}\mathcal{L}^*)$. As there is a map $q: \mathcal{G} \to \mathcal{F}$, there is a pairing $\mathcal{F}^* \otimes_{\mathcal{O}_P} \mathcal{G} \to \mathcal{O}_{\boldsymbol{P}}$. Let $\mathcal{I}$ be the image of $\mathcal{F}^* \otimes_{\mathcal{O}_P} \mathcal{G}$ in $\mathcal{O}_{\boldsymbol{P}}$. Thus $\mathcal{I}$ is a sheaf of ideals in $\mathcal{O}_{\boldsymbol{P}}$ which consequently defines a closed subscheme of $\boldsymbol{P}$, $\boldsymbol{L}$. We observe that our entire construction commutes with base extension. That is, if $f: Y \to X$ is a morphism, and we were to proceed with our construction using Y and $f^*\mathcal{L}$ rather than X and $\mathcal{L}$, we would obtain $Y \times_X \boldsymbol{L}$.

We assert that $\boldsymbol{L}$ represents $\underline{\mathrm{Lie}}^r_{\mathcal{L}/X}$. The fact that our construction commutes with base extension makes two simplifications possible. The first is that it suffices to show that the sections of X in $\boldsymbol{L}$ correspond to the subalgebras of $\mathcal{L}$ which are sub-bundles of $\mathcal{L}$ of rank r. The fact that $\boldsymbol{L}$ represents $\underline{\mathrm{Lie}}^r_{\mathcal{L}/X}$ then follows by functoriality. The second simplification consists in the observation that if for an open cover of X, $\{U_i\}_{i\in I}$, the sections of U_i in $\boldsymbol{L}\,|\,U_i$ correspond to Lie subalgebras of $\mathcal{L}\,|\,U_i$ which are rank r sub-bundles of $\mathcal{L}$, then the same thing is true for X and $\mathcal{L}$. Thus we may assume that X is affine, and that $\mathcal{L}$ is free.

We shall hence give an explicit description of $\boldsymbol{L}$ when $X = \mathrm{Spec}\, R$ and $\mathcal{L}$ is the sheaf associated to L a free Lie algebra over R with basis $x_1, \cdots, x_n$. Let J be the set of strictly increasing sequences of integers between one and n of length s, and let $e_1, \cdots, e_n$ be the basis for L^* dual to $x_1, \cdots, x_n$. If $T = (i_1, \cdots, i_s) \in J$, then let $x_T = x_{i_1} \wedge, \cdots, \wedge\, x_{i_s}$ and let $e_T = e_{i_1} \wedge, \cdots, \wedge\, e_{i_s}$. Then $\boldsymbol{P} = \mathrm{Proj}\, R[X_T]_{T\in J}$ and $\tilde{\Gamma}_*(q^*\Lambda^s\mathcal{L}^*) = R[X_T]_{T\in J} \otimes_R \coprod_{T\in J} Re_T$. Moreover $\omega_{\mathcal{L}} = \sum_{T\in J} x_T \otimes e_T$. Let J' be the set of all strictly increasing $(s+1)$-tuples of integers between one and n and choose a suitable ordering for J'. Let $e_{T'}$ denote the basis element of $\Lambda^{s+1}L^*$ corresponding to T' (as above). Then we may write:

$$\begin{aligned}
\omega \wedge e_1 &= \sum_{T' \in J'}{}' \ell_{1,T'}(X_T)_{T \in J} \otimes e_{T'} \\
&\vdots \\
\omega \wedge e_n &= \sum_{T' \in J'}{}' \ell_{n,T'}(X_T)_{T \in J} \otimes e_{T'} \\
(1 \otimes D_L)(\omega) &= \sum_{T' \in J'}{}' \ell_{n+1,T'}(X_T)_{T \in J} \otimes e_{T'}
\end{aligned}$$

where the prime on the sum indicates that the sum is appropriately ordered, and the $\ell_{j,T'}$ are linear forms in the X_T. Then the ideal $\mathscr{I}$ described above correspond to the graded ideal, $\tilde{\mathscr{I}}$, generated in $R[X_T]_{T \in J}$ by the $r+1$ by $r+1$ minors of the matrix, $(\ell_{j,T'})$ where the T' are suitably ordered, and $L = \operatorname{Proj} R[X_T]_{T \in J}/\tilde{\mathscr{I}}$. Now let a section, σ, of $\operatorname{Spec} R$ in L be given. Then, σ determines a locally free rank one sheaf of R-modules, P and a surjective morphism $\Lambda^s_R L \xrightarrow{\tilde{\sigma}} P \to 0$. Consequently P^* is a direct summand of $\Lambda^s_R L^*$. Let $N = \{x \in L^* : x \wedge P^* = 0\}$. We assert that N is a direct summand of L^* of rank s and that $D_L(P^*) \subset P^* \wedge L^*$. Clearly it suffices to prove this statement locally and so we may assume that P is free with generator e. In this case, set $\tilde{\sigma}(X_T) = a_T$. Then since σ corresponds to a section of L, the $r+1$ by $r+1$ minors of the matrix $(\ell_{j,T'}(X_T)_{T \in J})$ vanish when the a_T are substituted for the X_T. Moreover the image of P^* in $\Lambda^s L^*$ is $\sum_{T \in J} a_T e_T = \omega_\sigma$. Now clearly

$$\omega_\sigma \wedge e_i = \sum_{T' \in J'}{}' \ell_{i,T'}(a_T)_{T \in J} e_{T'}$$

and

$$D_L(\omega_\sigma) = \sum_{T' \in J'}{}' \ell_{n+1,T'}(a_T)_{T \in J} e_{T'} \ .$$

The fact that the $r+1$ by $r+1$ minors of the matrix of coefficients of these expressions vanishes implies that $\omega_\sigma \wedge L^*$ is a projective rank r submodule of $\Lambda^{s+1} L^*$, by the general theory of Grassman varieties, and further implies that $D_L(\omega_\sigma) \in \omega_\sigma \wedge L^*$. This proves the assertion concerning N. It follows by Proposition 1.3 that $(L^*/N)^*$ is a Lie subalgebra of L which is a direct summand of L. Thus statements i) and ii) of the theorem are proven. Statement iii) may be proven quite simply by taking $\mathfrak{H}_{\mathscr{L}/X}$ to be the sub-bundle of Lie algebras of $q^*\mathscr{L}$ associated to the identity map of $\mathrm{Lie}^r_{\mathscr{L}/X}$ into itself. More explicitly we may return to our construction and observe that on $\mathrm{Lie}^r_{\mathscr{L}/X}$ the sheaf $\mathscr{A}$

is equal to $\omega_{\mathscr{L}} \wedge q^*\mathscr{L}^*$. Thus, restricting to $\mathrm{Lie}^r_{\mathscr{L}/X}$, ψ becomes surjective. Dualizing one obtains $0 \to (\mathscr{A} \mid \mathrm{Lie}^r_{\mathscr{L}/X})^* \to ((q^*\mathscr{L}^*) \mid \mathrm{Lie}^r_{\mathscr{L}/X})^*$. It is easily verified that $(\mathscr{A} \mid \mathrm{Lie}^r_{\mathscr{L}/X})^*$ is precisely $\mathfrak{H}_{\mathscr{L}/X}$. We conclude with a corollary to the proof of Theorem 2.2 which needs no further explanation.

2.3. COROLLARY. *Let $f: Y \to X$ be a morphism of schemes and let $\mathscr{L}$ be a locally free rank n sheaf of Lie algebras over $\mathcal{O}_X$. Then* $\mathrm{Lie}^r_{f^*(\mathscr{L})/Y} \simeq Y \times_X \mathrm{Lie}^r_{\mathscr{L}/X}$.

3. The Tangent Space

In this section, we shall consider the case where X is the prime spectrum of a field, k, and $\mathscr{L} = L$ is just a Lie algebra over k. Then $\mathrm{Lie}^r_{L/X}$ is just a projective scheme over k whose closed points correspond to the Lie subalgebras of L of dimension r.

Let $k[\varepsilon]$ denote the ring of dual numbers; that is $\varepsilon^2 = 0$. Then if X is a k-scheme and $x \in X$ is a point rational over k, the tangent space to X at x consists of those elements of $\mathrm{Hom}_k(\mathrm{Spec}\, k[\varepsilon], X)$ which project to x under the natural map induced by the map $p_1: k[\varepsilon] \to k$. Suppose now that $X = \mathrm{Lie}^r_{L/k}$ and x is the point determined by the Lie subalgebra H. Then, $\mathrm{Hom}_k(\mathrm{Spec}\, k[\varepsilon], \mathrm{Lie}^r_{L/k})$ is, by the definition of $\mathrm{Lie}^r_{L/k}$, the set of $k[\varepsilon]$ Lie subalgebras of $k[\varepsilon] \otimes_k L$ which are free $k[\varepsilon]$-modules (Note that $k[\varepsilon]$ is artinian local) of rank r and $k[\varepsilon]$ direct summands of $k[\varepsilon] \otimes_k L$. The tangent space at x corresponds to the set of subalgebras $H' \subset k[\varepsilon] \otimes_k L$ such that $(p_1 \otimes 1)(H') = H$, where p_1 is the natural map from $k[\varepsilon]$ to k.

We observe that if H is a Lie subalgebra of L, then L/H admits a natural H-representation, given by $h \circ \bar{u} = \overline{[h, u]}$ where $\bar{x}$ denotes the class of x in L/H and $h \in H$. We use $Z^1(H, L/H)$ and $B^1(H, L/H)$ to denote the Whitehead one cocycles and co-boundaries of H in L/H in the standard complex. That is $Z^1(H, L/H) = \{\delta \in \mathrm{Hom}_k(H, L/H) : \delta([u, v]) = u \circ \delta(v) - v \circ \delta(u)\}$ and $B^1(H, L/H)$ is the set of maps of the form $\delta_u(h) = h \circ u$ for $u \in L/H$. There is a natural map, γ, from L to $B^1(H, L/H)$ given by $\gamma(m) = \delta_m$ where $\delta_m(h) = h \circ \bar{m}$ and $\bar{m}$ is the class of m in L/H. Now, $\gamma(m) = 0$ only if $\delta_m = 0$. But $\delta_m = 0$ if and only if $[H, m] \subset H$. Thus $B^1(H, L/H)$ is isomorphic to $L/N_L(H)$ where $N_L(H)$ denotes the normalizer of H in L.

3.1. THEOREM. *Let L be a Lie algebra of dimension n over k, and* *let $x = x_H$ denote the k-valued point of $\mathrm{Lie}^r_{L/k}$ determined by the r dimensional subalgebra H.*

Let $T_H = T_x$ denote the tangent space to $\mathrm{Lie}^r_{L/k}$ at x. Then T_H is isomorphic to $Z^1(H, L/H)$.

Proof. Let $k[\varepsilon]$ be the dual numbers. Then $k[\varepsilon] = k \oplus k \cdot \varepsilon$ and there are two projections $p_1, p_2 \colon k[\varepsilon] \to k$.

We must construct an element of $Z^1(H, L/H)$ for each element of T_H. As has been previously remarked the elements correspond to free rank r $k[\varepsilon]$-subalgebras of $k[\varepsilon] \otimes L = L'$, H' which are $k[\varepsilon]$-direct summands of L' and which project to H under $p_1 \otimes 1$. We shall construct the correspondence. Let $p_1' = p_1 \otimes 1$ and let $p_2' = p_2 \otimes 1$. Given H', p_1' is a surjective morphism from H' to H. As H and H' are k-vector spaces, there is a map $u \colon H \to H'$ such that $p_1' \circ u = \mathrm{id}$. Thus $p_2' \circ u$ maps H into $p_2'(H')$. As $\varepsilon \cdot H' = \varepsilon \otimes H$, $H \subset p_2'(H')$. The class of $p_2' \circ u \bmod H$ is a map δ from H into L/H. We assert that δ is the required element of $Z^1(H, L/H)$.

First we shall show that δ is unique. Thus we must show that for any two sections $u, u' \colon H \to H'$ such that $p_1' \circ u = p_1' \circ u' = \mathrm{id}_H$, we obtain the same map δ. Now, $u - u'$ maps H into $\ker p_1' | H'$, and since δ is obtained by composing with p_2' and taking the residue class $\bmod H$, it suffices to show that $P_2'(\ker (p_1 | H')) \subset H$.

Thus we may assume that $h_1, \cdots, h_r$ is a base of H, and that $1 \otimes h_1 + \varepsilon \otimes g_1, \cdots, 1 \otimes h_r + \varepsilon \otimes g_r$ is a base of H' over $k[\varepsilon]$. Then what we must show is that $\varepsilon \otimes L \cap H' \subset \varepsilon \otimes H$. Suppose that $\sum_{i=1}^r (\alpha_i + \beta_i \varepsilon)(1 \otimes h_i + \varepsilon \otimes g_i) = \varepsilon \otimes m$. Then $\sum_{i=1}^r (1 \otimes \alpha_i h_i) + \varepsilon \otimes (\beta_i h_i + \alpha_i g_i) = \varepsilon \otimes m$. As the h_i are linearly independent the α_i are zero and so $m = \sum \beta_i h_i \in H$. This establishes the uniqueness of δ.

To show that $\delta \in Z^1(H, L/H)$ note that $\delta(h)$ is obtained explicitly by choosing $1 \otimes h + \varepsilon \otimes g \in H'$ and setting $\delta(h)$ equal to the class of g in L/H. Thus suppose $1 \otimes h + \varepsilon \otimes g$ and $1 \otimes h' + \varepsilon \otimes g'$ are in H'. Then $[1 \otimes h + \varepsilon \otimes g, 1 \otimes h' + \varepsilon \otimes g'] = 1 \otimes [h, h'] + \varepsilon \otimes ([h, g'] - [h', g])$. It immediately follows that $\delta([h, h']) = h\delta(h') - h'\delta(h)$.

To construct an H' given a δ, just choose $\sigma \colon L/H \to L$ which splits the natural projection. Let $u = \sigma \circ \delta$. Then take $H' = \{1 \otimes h_1 + \varepsilon \otimes (u(h_1) + h_2) \colon h_1, h_2 \in H\}$. It is readily verified that H' is a subalgebra of L' of the requisite type, that it is independent of the choice of σ and that its associated cocycle is δ. The theorem is proven.

3.2. COROLLARY. *Let L be a Lie algebra over k and let H be a subalgebra of L of dimension r. Let x be the k-point of* $\mathrm{Lie}^r_{L/k}$ *corresponding to H. Let $N_L(H)$ be the normalizer of H in L. Then if $H^1(H, L/H) = 0$, T_x, the tangent space to* $\mathrm{Lie}^r_{L/k}$ *at x, is isomorphic to $L/N_L(H)$.*

Proof. If $H^1(H, L/H) = 0$, $B^1(H, L/H) = Z^1(H, L/H)$. By theorem 3.1, T_x is isomorphic to $Z^1(H, L/H)$ and as we have observed $B^1(H, L/H)$ is isomorphic to $L/N_L(H)$.

3.3. COROLLARY. *Let L be a Lie algebra of dimension n over k and let H be an ideal of dimension r in L. Let $x \in \mathrm{Lie}^r_{L/k}$ be the k-point corresponding to H. Let $q = \dim_k H[H, H]$. Then $\dim T_x = q\cdot(n - r)$. In particular if $H = [H, H]$, x is an isolated point of* $\mathrm{Lie}^r_{L/k}$.

Proof. If H is an ideal in L, the natural representation of H on L/H is the zero representation. Consequently if $\delta \in Z^1(H, L/H)$, $\delta([h, h'])$ $= h\delta(h') - h'\delta(h) = 0$. It follows that $Z^1(H, L/H) = \mathrm{Hom}_k (H/[H, H], L/H)$ where "Hom_k" denotes vector homomorphisms. The result follows at once.

§ 4. The Equivariant Cotangent Map

Let X be a reduced and irreducible scheme of finite type over a field, k, and let G be a linear algebraic group over k. Let $\alpha: G \times_k X \to X$ be an action of G on X. Then the set of points lying in orbits of highest dimension is an open subset of X, X_0. We shall always be concerned with the properties of α in X_0, and so, replacing X by X_0 if necessary, we shall assume that X has equidimensional orbits. Let $p_2: G \times_k X \to X$ be projection on the second factor. Let $\Phi: G \times_k X \to X \times_k X$ be defined by $\Phi = (\alpha, p_2)$. Let $\Delta: X \to X \times X$ be the diagonal morphism. Set $G_\Delta =$ $\Delta^{-1}(G \times_k X) = X \times_{X \times X} (G \times X)$ where Φ is the structure morphism for $G \times X$. Then, in a natural way G_Δ is a closed sub-group scheme of $G \times X$ where the latter is regarded as an X-group with structure morphism p_2. Let $e_X: X \to G \times X$ be the identity section of X in $G \times X$ and let $e_\Delta: X \to G_\Delta$ be the identity section of X in G_Δ. Let I_X be the sheaf of ideals defining $e_X(X)$ and let I_Δ be the sheaf of ideals defining $e_\Delta(X)$. Let $\omega_\Delta = e^*_\Delta(I_\Delta)$ and let $\omega_X = e^*_X(I_X)$. Let L be the Lie algebra of G over k; let $\xi: X \to k$ be the structure morphism for X over k and let

$j_\Delta: G_\Delta \to G \times X$ be the inclusion. Then observe that $e_X = j_\Delta \circ e_\Delta$, and that $\omega_X = \xi^*(L^*)$. As G_Δ is closed in $G \times X$ there is a natural surjection of sheaves of $\mathcal{O}_X$-modules, $\pi_\alpha: \omega_X \to \omega_\Delta$. As orbits in X are equidimensional, stabilizers are likewise and the fibre of the dual of ω_Δ at x is the Lie algebra of the stabilizer of x. Consequently the fibres of ω_Δ are equidimensional. As X is reduced and irreducible, it follows by Mumford [3], Lemma 1, page 51 that ω_Δ is a locally free sheaf of $\mathcal{O}_X$-modules. Furthermore dualizing π_α, one obtains a morphism of sheaves:

$$0 \longrightarrow \omega_\Delta^* \xrightarrow{\pi_\alpha^*} \omega_X^* \,.$$

However ω_X^* is the sheaf of Lie algebras associated to $G \times X$ and π_α^* is the inclusion of the Lie algebra of G_Δ in the Lie algebra of $G \times X$. Let the dimension of G be n, and let the dimension of the orbits in X be s. Let $r = n - s$. What we have shown is that $\omega_X^* = \xi^*L$ is a locally free sheaf of Lie algebras and that ω_Δ^* is a sub-bundle of Lie subalgebras of dimension r. Consequently, by Theorem 2.2, there is a map $\ell_\alpha: X \to \mathrm{Lie}^r_{\omega_X^*/X}$. Noting that $\omega_X^* = \xi^*L$, and applying Corollary 2.3 we conclude that $Lie^r_{\omega_X^*/X} = X \times_k \mathrm{Lie}^r_{L/k}$. If $q_2: \mathrm{Lie}^r_{\omega_X^*/X} \to \mathrm{Lie}^r_{L/k}$ denotes projection on the second factor $q_2 \circ \ell_\alpha$ maps X to $\mathrm{Lie}^r_{L/k}$.

4.1. DEFINITION. Let G be a linear algebraic group over k, let X be a reduced and irreducible algebraic variety over k, and let $\alpha: G \times_k X \to X$ be an action of G on X with equidimensional orbits. Then the equivariant cotangent map associated to α is the map $q_2 \circ \ell_\alpha$ defined above. It will be written λ_α.

Observe that if G_x is the stabilizer of x, then $\sigma G_x \sigma^{-1}$ is the stabilizer of σx. If the Lie algebra of G is L, inner automorphism by G induces the adjoint representation on L and G operates by Lie automorphisms. Consequently, the adjoint action carries Lie sub-algebras to Lie subalgebras and so induces an action of G on $\mathrm{Lie}^r_{L/k}$. By the remark at the beginning of this paragraph, λ_α is a G-equivariant map.

For the remainder of this paper, if G is a linear algebraic group, with Lie algebra L, we shall write $\mathrm{Lie}^r_{G/k}$ in lieu of $\mathrm{Lie}^r_{L/k}$. The following facts are obvious consequences of standard facts and Theorem 2.2.

4.2. Let G be a linear algebraic group of dimension n over k, let H be a subalgebra of the Lie algebra of G and let $x = x(H)$ be the k point of $\mathrm{Lie}^r_{G/k}$ corresponding to H. Then G_x, the stabilizer of x, is just

$N_G(H)$, the normalizer of H in G. Thus $G(x)$, the orbit of x, is just $G/N_G(H)$.

4.3. If H is the Lie algebra of K, a closed subgroup of G, and x is as above, then G_x is equal to $N_G(K^0)$ where K^0 is the connected component of K and $N_G(K^0)$ is the G-normalizer of K^0.

§ 5. Applications

We shall use the definition in section 4 to translate the results of sections 3 and 4 into the language of algebraic groups.

5.1. PROPOSITION. *Let G be a linear algebraic group over k and let K be a normal semi-simple k-subgroup of G of dimension r. Let H be the Lie algebra of K. Let x be the point of* $\mathrm{Lie}^r_{G/k}$ *corresponding to H. Then x is an isolated point in* $\mathrm{Lie}^r_{G/k}$.

Proof. The conditions of Corollary 3.3 are satisfied.

5.2. COROLLARY. *Let G be a linear algebraic group over k, of characteristic* 0, *and let K be a connected normal semi-simple k-subgroup of G. Let B be a connected linear algebraic group of automorphisms of G. Then B leaves K stable.*

Proof. As B operates on G, it operates on $\mathrm{Lie}^r_{G/k}$. Let x be the point of $\mathrm{Lie}^r_{G/k}$ corresponding to the Lie algebra of K. Then, as B is connected, so is the orbit of x. Consequently, $B(x) = x$. As the characteristic of K is zero, B must leave K stable. Q.E.D.
(We remark that this implies that if K is a connected semi-simple subgroup of G and N is the normalizer of K in G, then N is self normalizing)

5.3. THEOREM. *Let G be a connected linear algebraic group of dimension n over k, let K be a closed k-subgroup of G of dimension r and let H be the Lie algebra of K. Let $x = x(H)$ be the point of* $\mathrm{Lie}^r_{G/k}$ *corresponding to H. Suppose that $H^1(H, L/H) = 0$. Then the closure of the orbit $G(x)$ in* $\mathrm{Lie}^r_{G/k}$ *is an irreducible component of* $\mathrm{Lie}^r_{G/k}$.

Proof. By 3.2, T_x the tangent space at x, is isomorphic to $L/N_L(H)$ where $N_L(H)$ is the Lie normalizer of H in L. If N is the Lie algebra of $N_G(H)$, then $N \subset N_L(H)$. Let $q = \dim_k N_L(H)$ and let $q_1 = \dim_k N$.

Then $n - q \leq n - q_1$. On the other hand, $G/N_G(H)$ is isomorphic to the orbit of x and as it is of dimension $n - q_1$, its dimension is greater than or equal to $\dim_k T_x$. Thus $q = q_1, N_L(H) = N$ and the theorem is an immediate consequence.

5.4. THEOREM. *Let G be a connected linear algebraic group and let X be a reduced and irreducible variety, both over k, of characteristic 0. Let $\alpha: G \times_k X \to X$ be an action of G on X with equidimensional orbits. Suppose that for some point $t \in X$, $H^1(H, L/H) = 0$ where H is the Lie algebra of G_t and L is the Lie algebra of G. Then there is a dense open subset of X, U, such that $t \in U$, and for any $u \in U$, $(G_u)^0$ is conjugate to $(G_t)^0$ where $(G_u)^0$ is the connected component of the identity in G_u.*

Proof. Let $\lambda_\alpha: X \to \mathrm{Lie}^r_{G/k}$ denote the equivariant cotangent map associated to α. Let x be the point in $\mathrm{Lie}^r_{G/k}$ corresponding to H. Then as $\lambda_\alpha(t) = x$, we may apply 5.3 and 5.4 follows at once.

In conclusion, it should be remarked that theorems 5.3 and 5.4 may be interpreted as principal orbit theorems for actions of reductive groups on affine varieties in characteristic zero. Results along these lines have been recently proven by Richardson in Richardson [4].

In the terminology of that paper we have shown that if $H^1(H, L/H) = 0$ then H is a "principal" isotropy subalgebra.

We further observe that our results trivially imply that if G is a connected algebraic group over a field of characteristic zero, then G contains only a finite number of conjugacy classes of reductive subgroups of any given dimension.

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