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ON A θ-WEYL SUM

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0°. We treat the sum $\theta(\alpha^{-1},\gamma;N,X) \underset{\text{def.}}{=} \sum_{X\leqq n\leqq X+N} e((2\alpha)^{-1}(n+\gamma)^2)$, where α and γ are real with α positive.*) This sum was treated first by Hardy and Littlewood [4], and after them, by Behnke [1] and [2], Mordell [9], Wilton [11] and others. The reader will find its history in [7] and in the comments of the Collected Papers [4]. Here we show that the sum can be expressed explicitly, together with an error term $O(N^{1/2})$, using the regular continued fraction expansion of α. As the statements have complications we will divide them into two theorems. In the followings all letters except $\vartheta, i, \sigma, \zeta, \chi$ and those in 3° are real, N is a positive real, and always k, n, a, A, B, C, D and E denote integers. The author expresses his thanks to Professor Tikao Tatuzawa and Professor Tomio Kubota for their encouragements.

1°. LEMMA 1. *Let α, α', γ and γ' be reals such that*

$$\alpha^{-1} \equiv \alpha'^{-1} \qquad \text{mod. } 1$$

and

$$(2\alpha)^{-1}(1+2\gamma) \equiv (2\alpha')^{-1}(1+2\gamma') \qquad \text{mod. } 1\,,$$

then we have

$$(2\alpha)^{-1}(n+\gamma)^2 \equiv (2\alpha')^{-1}(n+\gamma')^2 + (2\alpha)^{-1}\gamma^2 - (2\alpha')^{-1}\gamma'^2 \qquad \text{mod. } 1$$

for any integer n.

Proof. It is easy.

LEMMA 2 (Hardy-Littlewood, Mordell and Wilton). *If* $0 < \omega \leq 2$,

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*) In this note $e(\alpha)$ means $e^{2\pi i\alpha}$ for real α. $\boldsymbol{N}$ is the set of positive integers. $\boldsymbol{Z}$ is the set of all integers. The implied positive numerical constants in the symbol "$\ll$" in the statements and proofs of (Case 2) of Theorem 1 can be given arbitrarily. Other implied constants in the symbols "$\ll$", "$O(\ )$" and "$\underset{\cap}{\cup}$" are absolute or can be explicitly calculated.

$-\frac{1}{2} \leq x \leq \frac{1}{2}$, $N' - \frac{1}{2} \leq \omega N + x < N' + \frac{1}{2}$ with integral N and N', then we have

$$\sum_{n=0}^{N}{}' e(\tfrac{1}{2}\omega n^2 + xn) = e(\tfrac{1}{8})\cdot\omega^{-1/2} \sum_{n=0}^{N'}{}' e(-\tfrac{1}{2}\omega^{-1}(n-x)^2) + \vartheta(3 + 2\omega^{-1/2}),$$

where $|\vartheta| \leq 1$. Here $\sum'$ means that the first and last terms of the sum are to be halved.

Proof. This is the Theorem in [11].

LEMMA 3. *Let α_0, N_0 and X_0 be reals with $\alpha_0 \geqq \frac{1}{2}$, $N_0 \geqq 0$ and $N_0 \geqq 2\alpha_0$. Expand α_0 as $\alpha_0 = a_0 + \alpha_1^{-1}$ with an integer a_0. Here we suppose α_0 not to be an integer. Let γ_0 and γ_1 be reals with $\frac{1}{2}a_0 - \gamma_0 \equiv \alpha_1^{-1}\gamma_1 \bmod 1$. Put $X_1 = \alpha_0^{-1}(X_0 + \gamma_0)$ and $N_1 = \alpha_0^{-1}N_0$. Then, for $\varepsilon = \pm 1$, we have*

$$\theta(\varepsilon\alpha_0^{-1}, \gamma_0; N_0, X_0) = e(\varepsilon(\tfrac{1}{8} + (2\alpha_1)^{-1}\gamma_1^2))\cdot\alpha_0^{1/2}\cdot\theta(-\varepsilon\alpha_1^{-1}, \gamma_1; N_1, X_1) + O(1 + \alpha_0^{1/2}).$$

Proof. This can be obtained from Lemmas 1 and 2.

LEMMA 4 (van der Corput). *Let $f(x)$ be a real valued function on the interval $[X, Y]$, whose first derivative $f'(x)$ is monotonic, not decreasing and such that $0 \leq f'(x) \leq \frac{1}{2}$ on the interval. Then we have*

$$\sum_{X \leq n \leq Y} e(f(n)) = \int_X^Y e(f(u))\cdot du + \vartheta\left(\frac{1}{2} + \frac{1}{\pi} + \left(\frac{1}{4} + \frac{1}{\pi^2}\right)^{1/2}\right),$$

where $|\vartheta| \leq 1$.

Proof. This is "Satz 1" in [5]. A little less precise statements can be found in [10], Chap. 4.

LEMMA 5. *Let α_0, N_0 and X_0 be reals with $\alpha_0 > 0$, $N_0 \geq 0$ and $\frac{1}{2}\alpha_0 \geqq N_0$. Let γ_0 be given. Choose $\tilde{\gamma}_0$ so that $\tilde{\gamma}_0 \equiv \gamma_0 \bmod \alpha_0$ and that the interval $[\alpha_0^{-1}(X_0 + \tilde{\gamma}_0), \alpha_0^{-1}(X_0 + \tilde{\gamma}_0 + N_0)]$ is contained in the interval $[-\frac{3}{4}, \frac{3}{4}]$. Then, for $\varepsilon = \pm 1$, we have*

$$\theta(\varepsilon\alpha_0^{-1}, \gamma_0; N_0, X_0) = e(\varepsilon(2\alpha_0)^{-1}(\gamma_0^2 - \tilde{\gamma}_0^2)) \int_{X_0+\tilde{\gamma}_0}^{X_0+\tilde{\gamma}_0+N_0} e(\varepsilon(2\alpha_0)^{-1}u^2)du + O(1).$$

Proof. This is obtained from Lemmas 1 and 4.

We regard $\theta(\varepsilon\alpha_0^{-1}, \gamma_0; N_0, X_0)$ to be $\sum_{X_0 \leq n \leq X_0+N_0} 1$ for $\alpha_0 = +\infty$. Then Lemma 5 holds also for $\alpha_0 = +\infty$.

LEMMA 6. *Let α_0, γ_0, N_0 and X_0 be reals with $\alpha_0 > 0$, $N_0 > 0$ and $2\alpha_0 \geqq N_0 \geqq \frac{1}{2}\alpha_0$. Then, for $\varepsilon = \pm 1$, we have*

$$\theta(\varepsilon\alpha_0^{-1}, \gamma_0; N_0, X_0) = O(1 + \alpha_0^{1/2}) .$$

Proof. If $1 \gg \alpha_0 > 0$, the result is obvious. Suppose we have $\alpha_0 \geqq 4$. We express the interval $[X_0, X_0 + N_0]$ as a union of at most $O(1)$ subintervals, each of length $\leqq \frac{1}{2}\alpha_0$ and $\gg \alpha_0$. In each subinterval we can apply Lemma 5. The contribution of the terms containing integrals are $O(\sqrt{\alpha_0})$ by the convergence of the integral $\int_{-\infty}^{\infty} e(u^2)du$, and so we have the result.

2°. We define several numbers concerning continued fraction expansion of α. Let α be positive. Choose α_0 uniquely so that $\alpha_0^{-1} \equiv \alpha^{-1} \bmod. 1$ and $+\infty \geqq \alpha_0 > 1$. Expand α_0 as $\alpha_k = a_k + (\alpha_{k+1})^{-1}$ with $a_k \in N$ and $+\infty \geqq \alpha_{k+1} > 1$, beginning with $k = 0$. If $\alpha_{k+1} = +\infty$ for some k, we stop the expansion at this k. Define integers A_k, B_k and $C_j^{(k+1)}$ as follows: $A_{-1} = 1$, $A_0 = a_0$ and $A_k = a_k A_{k-1} + A_{k-2}$ for $k \geqq 1$; $B_{-1} = 0$, $B_0 = 1$ and $B_k = a_k B_{k-1} + B_{k-2}$ for $k \geqq 1$; $C_{k+1}^{(k+1)} = 1$, $C_k^{(k+1)} = a_k$ and $C_j^{(k+1)} = a_j C_{j+1}^{(k+1)} + C_{j+2}^{(k+1)}$ for $k - 1 \geqq j \geqq 0$. Define a matrix ζ_k to be

$$\begin{pmatrix} A_k & -B_k \\ (-1)^k A_{k-1} & (-1)^{k+1} B_{k-1} \end{pmatrix} .$$

This belongs to $SL(2, Z)$, as can be seen from (2) of Lemma 7. Define Ξ_k and H_k as follows: $\Xi_k = 0$ or 1 with $\Xi_k \equiv A_k B_k \bmod. 2$ for $k \geqq -1$ and $H_k = (-1)^k \Xi_{k-1}$ for $k \geqq 0$. We have the following lemmas.

LEMMA 7. (1) *A_k and B_k increase monotonically as k increases.*

(2) $A_k B_{k-1} - A_{k-1} B_k = (-1)^{k+1}$ *and* $(A_k, B_k) = 1$ *for* $k \geqq 0$.

(3) $C_1^{(k+1)} = B_k$ *and* $C_0^{(k+1)} = A_k$ *for* $k \geqq -1$.

(4) $A_k + \alpha_{k+1}^{-1} A_{k-1} = \alpha_k \cdots \alpha_0$,
$B_k + \alpha_{k+1}^{-1} B_{k-1} = \alpha_k \cdots \alpha_1$, *for* $k \geqq 0$, *and*
$B_k - \alpha_0^{-1} A_k = (-1)^k (\alpha_{k+1} \cdots \alpha_0)^{-1}$, *for* $k \geqq -1$.

(5) $\alpha_k \cdot \alpha_{k+1} > 2$ *for* $k \geqq 0$.

(6) $\alpha_{k+1} \cdots \alpha_0 \underset{\cap}{\cup} A_{k+1}$ *for* $k \geqq -1$.

LEMMA 8 (best approximation). *Let α_0 be > 1, and make A_k and B_k from α_0 as above. Let also a rational number $B^{-1}A$ be given, where B and A are its irreducible denominator and numerator respectively, so that, for any rationals $B'^{-1}A'$ with $0 < B' \leq B$ and $B'^{-1}A' \neq B^{-1}A$, we* *have* $|B\alpha_0 - A| \leq |B'\alpha_0 - A'|$. *Then the pair* (A, B) *is equal to* (A_k, B_k) *for some* k.

Proof. All statements of Lemmas 7 and 8 are well-known or can be easily shown. See, for instance, [6]. Lemma 8 is included here to suggest the nature of A_k and B_k.

LEMMA 9. *We have*

$$\sum_{h;k\geqq h\geqq j-1} (\alpha_{k+1} \cdots \alpha_{h+2})^{-1}\{(-\alpha_h) \cdots (-\alpha_j)\} = (-1)^{k+1-j}C_j^{(k+1)}$$

for $0 \leq j \leq k+1$, *where* $\alpha_{k+1} \cdots \alpha_{h+2} = 1$ *for* $h = k$ *and* $(-\alpha_h) \cdots (-\alpha_j) = 1$ *for* $h = j-1$.

Proof. If we put $\delta_j^{(k+1)} = \sum_{h;k\geqq h\geqq j-1} (\alpha_{k+1} \cdots \alpha_{h+2})^{-1}\{(-\alpha_h) \cdots (-\alpha_j)\}$, we have $\delta_j^{(k+1)} = -a_j\delta_{j+1}^{(k+1)} + \delta_{j+2}^{(k+1)}$ for $k-2 \geqq j \geqq 0$. Also $\delta_{k+1}^{(k+1)} = 1$ and $\delta_k^{(k+1)} = -a_k$. Thus $(-1)^{k+1-j}\delta_j^{(k+1)}$ has the same properties as $C_j^{(k+1)}$. Hence they are identical.

Let a real γ be given. Using α_k, a_k etc., we define γ_k as follows: γ_0 is any real number satisfying

$$(2\alpha_0)^{-1}(1 - 2\gamma_0) \equiv (2\alpha)^{-1}(1 - 2\gamma) \bmod. 1 ,$$

and

$$\gamma_{k+1} = (-1)^{k+1}\alpha_{k+1}(B_k\gamma_0 - \tfrac{1}{2}\Xi_k) + (-1)^{k+1}B_{k-1}\gamma_0 + \tfrac{1}{2}H_k$$

for $k \geqq 0$. Given a real X, we define X_k inductively by $X_0 = X$ and $X_{k+1} = \alpha_k^{-1}(X_k + \gamma_k)$ for $k \geqq 0$.

LEMMA 10. *We have the following equalities*:

(1) $$\alpha_{k+1}^{-1}\gamma_{k+1} = -\gamma_k + \tfrac{1}{2}a_k + (-1)^k D_k$$

for $k \geqq 0$, *where* D_k *is an integer defined by*

$$D_k = \tfrac{1}{2}(\Xi_k - a_k\Xi_{k-1} + (-1)^k H_{k-1} + (-1)^{k+1}a_k) .$$

(2) $$X_{k+2} = (\alpha_0 \cdots \alpha_{k+1})^{-1}X_0 + (-1)^k\alpha_0^{-1}\gamma_0 A_k + (-1)^{k+1}\tfrac{1}{2}(A_k + B_k) + (-)^k E_k$$

for $k \geqq 0$, *where* E_k *is an integer defined by*

$$E_k = \sum_{j=1}^{k+1} C_j^{(k+1)} D_{j-1} .$$

Proof. The fact that $\Xi_k - a_k\Xi_{k-1} + (-1)^k H_{k-1} + (-1)^{k-1}a_k$ is an even integer follows from the definitions and (2) of Lemma 7. Therefore D_k and E_k are integers. The number $\alpha_{k+1}^{-1}\gamma_{k+1}$ is equal to

$$\begin{aligned}&(-1)^{k+1}(B_k\gamma_0 - \tfrac{1}{2}\Xi_k) + \alpha_{k+1}^{-1}((-1)^{k+1}B_{k-1}\gamma_0 + \tfrac{1}{2}H_k)\\ &= (-1)^{k+1}(B_k\gamma_0 - \tfrac{1}{2}\Xi_k) + (\alpha_k - a_k)((-1)^{k+1}B_{k-1}\gamma_0 + \tfrac{1}{2}H_k)\\ &= (-1)^{k+1}\alpha_k(B_{k-1}\gamma_0 + \tfrac{1}{2}(-1)^{k+1}H_k) + (-1)^{k+1}B_{k-2}\gamma_0\\ &\quad - \tfrac{1}{2}(-1)^{k+1}\Xi_k - \tfrac{1}{2}a_kH_k\,.\end{aligned}$$

The last sum is equal to $-\gamma_k + \frac{1}{2}H_{k-1} - \frac{1}{2}(-1)^{k+1}\Xi_k - \frac{1}{2}a_kH_k$, by $H_k = (-1)^k\Xi_{k-1}$. Thus the right hand side of (1) is easily obtained. As for (2), we see, by direct calculations, that X_{k+2} is equal to $(\alpha_0 \cdots \alpha_{k+2})^{-1}X_0 + \beta_{k+2}$, where β_{k+2} is $\alpha_{k+1}^{-1}\gamma_{k+1} + (\alpha_{k+1}\alpha_k)^{-1}\gamma_k + \cdots + (\alpha_{k+1}\cdots\alpha_0)^{-1}\gamma_0$. Then β_{k+2} is equal to

$$\begin{aligned}&\sum_{h;k\geqq h\geqq 0}(\alpha_{k+1}\cdots\alpha_{h+2})^{-1}\{(-\alpha_h)\cdots(-\alpha_1)\}(-\gamma_0 + \tfrac{1}{2}\alpha_0 - \tfrac{1}{2})\\ &\quad + (\alpha_{k+1}\cdots\alpha_0)^{-1}(\gamma_0 - \tfrac{1}{2}\alpha_0)\\ &\quad + \left(\begin{aligned}&(-1)^kD_k + \cdots + (-1)^{j-1}D_{j-1}\sum_{h;k\geqq h\geqq j-1}(\alpha_{k+1}\cdots\alpha_{h+2})^{-1}\\ &\times\{(-\alpha_h)\cdots(-\alpha_j)\} + \cdots + D_0\sum_{h;k\geqq h\geqq 0}(\alpha_{k+1}\cdots\alpha_{h+2})^{-1}\\ &\times\{(-\alpha_h)\cdots(-\alpha_1)\}\end{aligned}\right).\end{aligned}$$

By Lemma 9, this sum is equal to

$$\begin{aligned}&(-1)^kB_k(-\gamma_0 + \tfrac{1}{2}\alpha_0 - \tfrac{1}{2}) + (\alpha_{k+1}\cdots\alpha_0)^{-1}(\gamma_0 - \tfrac{1}{2}\alpha_0)\\ &\quad + [(-1)^kD_k + \cdots + (-1)^{j-1}D_{j-1}(-1)^{k+1-j}C_j^{(k+1)} + \cdots\\ &\quad + D_0(-1)^{(k+1)-1}C_1^{(k+1)}]\,,\end{aligned}$$

for $k \geqq 0$. Substituting the third formula of (4) of Lemma 7 with $(\alpha_{k+1}\cdots\alpha_0)^{-1}$ in the second term of the above sum, we have the result (2).

The formula (2) of Lemma 7 and the fact that E_k is an integer are fundamental.

3°. Let τ be a complex variable whose imaginary part is positive. Let x and y be any complex numbers, and $\sigma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ be any matrix in $SL(2, \boldsymbol{Z})$. Define $\sigma\langle\tau\rangle$ to be $(a\tau + b)(c\tau + d)^{-1}$. Then we see that

$$\theta(\tau\,; x, y) \underset{\text{def.}}{=} \sum_{m\in\boldsymbol{Z}} e^{\pi i\tau(m-y)^2 + 2\pi imx - \pi ixy}$$

is equal to

$$\chi(\sigma)\cdot e^{(\pi i/2)\{\eta(ax+by)-\xi(cx+dy)\}}\cdot(c\tau+d)^{-1/2}\theta(\sigma\langle\tau\rangle;\,ax+by-\tfrac{1}{2}\xi,\,cx+dy-\tfrac{1}{2}\eta)\,,$$

where $\xi\equiv ab$ mod. 2 and $\eta\equiv cd$ mod. 2. Also $\chi(\sigma)$ is a certain eighth root of the unity which does not depend on x, y and τ. This formula is well-known. See, for instance, [3], pp. 47–66.

We restrict ξ,η and the branch of $(c\tau+d)^{1/2}$ as follows: $\xi=0$ or 1, $\eta=0$ or ±1 where the signature in ±1 is given in advance for each σ, and, as for $(c\tau+d)^{1/2}$,

$$\begin{aligned}&(c\tau+d)^{1/2}=1 && \text{if } c=0 \text{ and } d=1\,,\\ &(c\tau+d)^{1/2}=e^{(1/2)\pi i} && \text{if } c=0 \text{ and } d=-1\,,\\ &0<\arg(c\tau+d)^{1/2}<\pi/2 && \text{if } c>0\,,\end{aligned}$$

and

$$0>\arg(c\tau+d)^{1/2}>-\pi/2 \qquad \text{if } c<0\,.$$

Then we can write $\chi(\sigma)$ explicitly in terms of a, b, c and d, if we use the Jacobi symbol. The reader will find some of them, that is, those for $\sigma\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}$ or $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ mod. 2, in [8], for instance.

Rewriting the θ-formula, we have

$$\sum_{m\in\boldsymbol{Z}} e^{\pi i\tau(m+\gamma)^2}=\chi(\sigma)(c\tau+d)^{-1/2}\cdot e^{\pi i(b\gamma+(1/2)\xi)\gamma_\sigma}e^{(\pi i/2)(d\xi-b\eta)\gamma}\sum_{m\in\boldsymbol{Z}} e^{\pi i\sigma\langle\tau\rangle(m+\gamma_\sigma)^2}\,,$$

where γ_σ is $(d\gamma+\frac{1}{2}\eta)-(\sigma\langle\tau\rangle)^{-1}(b\gamma+\frac{1}{2}\xi)$.

LEMMA 11. *Let σ be ζ_k, that is, $\begin{pmatrix}A_k & -B_k\\ (-1)^kA_{k-1} & (-1)^{k+1}B_{k-1}\end{pmatrix}$. Choose α,γ,ξ and η to be $\alpha_0^{-1}+i\cdot0+$, γ_0, Ξ_k and H_k respectively, with the notations defined in 2°. Then we have*

$$\sigma\langle\alpha_0^{-1}+i\cdot0+\rangle=(-1)^{k+1}\alpha_{k+1}^{-1}+i\cdot0+$$

and

$$\gamma_\sigma=\gamma_{k+1}+i\cdot0\pm\,.$$

Here $0+$ or $0\pm$ stands for a sufficiently small positive or a real number respectively.

Proof. It is easy to check the assertion about $\sigma\langle\alpha_0^{-1}+i\cdot0+\rangle$ by the third formula of (4) of Lemma 7. Then the other part clearly holds.

4°. Now we proceed to the sum $\theta(\alpha^{-1},\gamma;N,X)$. We suppose that

N is not smaller than 1. We use those notations in 2° relating to α, γ and X. Also we put $N_{k+1} = (\alpha_k \cdots \alpha_0)^{-1}N$ with $N_0 = N$. If $N_1 < \frac{1}{2}$, we define k_0 to be -1. But, if $N_1 \geqq \frac{1}{2}$, there is, by (5) of Lemma 7, some k_0 with $0 \leqq k_0 \ll \log N$ so that $N_{k_0+1} \geqq \frac{1}{2}$ but $0 \leq N_{k_0+2} < \frac{1}{2}$. We divide the statements into two theorems. We suppose $\alpha > 0$ and that $\alpha_0 \neq +\infty$.

THEOREM 1. *(Case 1) If $k_0 = -1$, we have*

$$\theta(\alpha^{-1}, \gamma; N, X) = e((2\alpha)^{-1}\gamma^2 - (2\alpha_0)^{-1}\tilde{\gamma}_0^2) \int_{X+\tilde{\gamma}_0}^{X+\tilde{\gamma}_0+N} e((2\alpha_0)^{-1}u^2)du + O(1) ,$$

where $\tilde{\gamma}_0$ is so chosen that $\tilde{\gamma}_0 \equiv \gamma_0 \operatorname{mod.} \alpha_0$ and that the interval

$$[\alpha_0^{-1}(X + \tilde{\gamma}_0), \alpha_0^{-1}(X + \tilde{\gamma}_0 + N)]$$

is contained in the interval $[-\frac{3}{4}, \frac{3}{4}]$.

(Case 2) If $k_0 \geqq -1$ and if $N_{k_0+1} \ll 1$ or $N_{k_0+2} \gg 1$, then we have

$$\theta(\alpha^{-1}, \gamma; N, X) = O(N^{1/2}) .$$

Proof. If $k_0 = -1$ the result is obtained from Lemma 5 and Lemma 1. If $k_0 \geqq 0$ but if $N_{k_0+1} \ll 1$, we can apply Lemma 3 repeatedly $(k_0 + 1)$ times, as is ensured by (1) of Lemma 10, and can use the fact that

$$\theta((-1)^{k_0+1}\alpha_{k_0+1}^{-1}, \gamma_{k_0+1}; N_{k_0+1}, X_{k_0+1}) = O(1) .$$

We have an estimate $O(1 + \sum_{h=0}^{k_0} (\alpha_0 \cdots \alpha_h)^{1/2})$, which is $O(1 + (\alpha_0 \cdots \alpha_{k_0})^{1/2})$ by (5) of Lemma 7. But $\alpha_0 \cdots \alpha_{k_0} \underset{\cap}{\cup} N$, so we have done in this case. If $k_0 \geqq 0$ and if $N_{k_0+2} \gg 1$, we again apply Lemma 3 repeatedly $(k_0 + 1)$ times and Lemma 6 after that. We have $O(1 + (\alpha_0 \cdots \alpha_{k_0+1})^{1/2})$ as an estimate in this case, which is $O(N^{1/2})$ again.

THEOREM 2. *(Case 3) If $k_0 \geqq 0$, $N_{k_0+1} > 2$ and $0 < N_{k_0+2} < \frac{1}{4}$, we have*

$$\begin{aligned}\theta(\alpha^{-1}, \gamma; N, X) = {} & \chi(\zeta_{k_0}) \cdot e^{\pi i \Delta_{k_0}} \cdot e((2\alpha)^{-1}\gamma^2 - (2\alpha_0)^{-1}\gamma_0^2) \\ & \times (\alpha_0 \cdots \alpha_{k_0})^{1/2} \int_{X_{k_0+1}+\tilde{\gamma}_{k_0+1}}^{X_{k_0+1}+\tilde{\gamma}_{k_0+1}+N_{k_0+1}} e((-1)^{k_0+1}(2\alpha_{k_0+1})^{-1}u^2) \cdot du \\ & + O(1 + A_{k_0}^{1/2}) ,\end{aligned}$$

where Δ_{k_0} is

$$\begin{aligned}&\gamma_{k_0+1}(-B_{k_0}\gamma_0 + \tfrac{1}{2}\Xi_{k_0}) + (-1)^{k_0+1}\tfrac{1}{2} \cdot (B_{k_0-1}\Xi_{k_0} + A_{k_0-1}H_{k_0})\gamma_0 \\ &\quad + (-1)^{k_0+1}(2\alpha_{k_0+1})^{-1}(\gamma_{k_0+1}^2 - \tilde{\gamma}_{k_0+1}^2) .\end{aligned}$$

Also ζ_{k_0} *is* $\begin{pmatrix} A_{k_0} & -B_{k_0} \\ (-1)^{k_0}A_{k_0-1} & (-1)^{k_0+1}B_{k_0-1} \end{pmatrix}$, *and the value of* $\chi(\zeta_{k_0})$ *is that in* $3°$ *corresponding to* $\xi = \Xi_{k_0}$, $\eta = H_{k_0}$ *and the branch of* $(c\tau + d)^{1/2}$ *is restricted as is stated there. The value* $\tilde{\gamma}_{k_0+1}$ *is so chosen that* $\tilde{\gamma}_{k_0+1} \equiv \gamma_{k_0+1} \bmod . \alpha_{k_0+1}$ *and that the interval* $[\alpha_{k_0+1}^{-1}(X_{k_0+1} + \tilde{\gamma}_{k_0+1}), \alpha_{k_0+1}^{-1}(X_{k_0+1} + \tilde{\gamma}_{k_0+1} + N_{k_0+1})]$ *is contained in the interval* $[-\frac{3}{4}, \frac{3}{4}]$.

(*Case* 4) *If* $k_0 \geqq 0$ *but* $N_{k_0+2} = 0$, *then, with the same* $\chi(\zeta_{k_0})$ *as above, we have*

$$\theta(\alpha^{-1}, \gamma; N, X) = \chi(\zeta_{k_0})e^{\pi i \Delta'_{k_0}} \cdot e((2\alpha)^{-1}\gamma^2 - (2\alpha_0)^{-1}\gamma_0^2) \cdot A_{k_0}^{1/2} \times \sum_{X_{k_0+1} \le n \le X_{k_0+1} + N_{k_0+1}} e((B_{k_0}\gamma_0 - \tfrac{1}{2}\Xi_{k_0})n) + O(1 + A_{k_0}^{1/2}) ,$$

where Δ'_{k_0} *is*

$$(B_{k_0}\gamma_0 - \tfrac{1}{2}\Xi_{k_0})((-1)^{k_0+1}B_{k_0-1}\gamma_0 + \tfrac{1}{2}H_{k_0}) + (-1)^{k_0+1}\tfrac{1}{2} \cdot (B_{k_0-1}\Xi_{k_0} + A_{k_0-1}H_{k_0})\gamma_0 .$$

In this case α_0 *is* $B_{k_0}^{-1}A_{k_0}$ *with* $A_{k_0} \le 2N$.

Proof. (Case 3) Suppose $N_{k_0+2} \neq 0$. We use Lemma 3 repeatedly $(k_0 + 1)$ times and Lemma 5 after that. As $\gamma_\sigma, \sigma\langle\tau\rangle$ and $(c\tau + d)^{1/2}$ for $\sigma = \zeta_{k_0}$ and $\tau = \alpha_0^{-1} + i \cdot 0+$ are equal to $\gamma_{k_0+1} + i \cdot 0\pm$, $(-1)^{k_0+1}\alpha_{k_0+1}^{-1} + i \cdot 0+$ and $(\alpha_0 \cdots \alpha_{k_0} + i \cdot 0\pm)^{1/2}$ respectively, we have, from θ-formula in $3°$, the main term in the result. We have $O(1 + (\alpha_0 \cdots \alpha_{k_0})^{1/2})$ as its errors, which is $O(1 + A_{k_0}^{1/2})$ by (6) of Lemma 7. (Case 4) Now we suppose $N_{k_0+2} = 0$, i.e., $\alpha_{k_0+1} = +\infty$. We have $\zeta_{k_0}\langle\alpha_0^{-1} + i \cdot 0 + \rangle = i \cdot 0+$. We rewrite the θ-formula in $3°$ as follows:

$$\sum_{m \in Z} e^{\pi i \tau (m+\gamma)^2} = \chi(\sigma)(c\tau + d)^{-1/2} e^{(\pi i/2)(d\xi - b\eta)\gamma - \pi i (b\gamma + (1/2)\xi)(d\gamma + (1/2)\eta)} \times \sum_{m \in Z} e^{\pi i \sigma\langle\tau\rangle (m + d\gamma + (1/2)\eta)^2 - 2\pi i (b\gamma + (1/2)\xi) m}$$

Then we obtain the result in this case also by the similar considerations.

In the integrals in Cases 1 and 3, $\alpha_{k_0+1}^{-1}(X_{k_0+1} + \tilde{\gamma}_{k_0+1})$ is to be determined mod. 1. But it is equal to $X_{k_0+2} + \alpha_{k_0+1}^{-1}(\tilde{\gamma}_{k_0+1} - \gamma_{k_0+1})$; then X_{k_0+2} and the integer $\alpha_{k_0+1}^{-1}(\tilde{\gamma}_{k_0+1} - \gamma_{k_0+1})$ can be determined by (2) of Lemma 10.

5°. We fix an irrational number α_0 arbitrarily which is larger than 1. Make those numbers defined in $2°$ from $\alpha = \alpha_0$. Let $\psi(k)$ be a real valued function on $k = -1, 0, 1, 2, \cdots$, whose value is larger than 2. If we suppose that N_{k_0+2} is larger than or equal to $(2\psi(k_0))^{-1}$, then we have $A_{k_0+1} \ll N\psi(k_0)$, as $N_{k_0+2} \underset{\cap}{\cup} NA_{k_0+1}^{-1}$. Thus, by the convergence of $\int_{-\infty}^{\infty} e(u^2) \cdot du$, we have

$$(1)\qquad (\alpha_0 \cdots \alpha_{k_0})^{1/2} \int e((2\alpha_{k_0+1})^{-1}u^2)du \ll (\alpha_0 \cdots \alpha_{k_0})^{1/2}(\alpha_{k_0+1})^{1/2} \ll A_{k_0+1}^{1/2} \ll (N\psi(k_0))^{1/2} .$$

Let us, on the contrary, suppose that N_{k_0+2} is smaller than $(2\psi(k_0))^{-1}$. Suppose also that we have a real β_0 which satisfies the following conditions, where $\{x\}$ denotes the fractional part of x:

$$(2)\qquad \begin{array}{ll} |\{\beta_0 A_k\} - \frac{1}{2}| \geqq \psi(k)^{-1} & \text{if } A_k + B_k \text{ is odd with } k \geqq 0, \\ \min(\{\beta_0 A_k\}, 1 - \{\beta_0 A_k\}) \geqq \psi(k)^{-1} & \text{if } k = -1 \text{ or if } A_k + B_k \text{ is even with } k \geqq 0 . \end{array}$$

Then, if we substitute $X_0 = 0$ and $\gamma_0 = \alpha_0\beta_0$ in (2) of Lemma 10, the interval $[\{X_{k_0+2}\}, \{X_{k_0+2}\} + N_{k_0+2}]$ is contained in the interval $[(2\psi(k_0))^{-1}, 1 - (2\psi(k_0))^{-1}]$ for $k_0 \geqq -1$. By the mean-value theorem on integrals, we have

$$(3)\qquad (\alpha_0 \cdots \alpha_{k_0})^{1/2} \int_{X_{k_0+1}+\tilde{\gamma}_{k_0+1}}^{X_{k_0+1}+\tilde{\gamma}_{k_0+1}+N_{k_0+1}} e((2\alpha_{k_0+1})^{-1}u^2)\cdot du \ll (\alpha_0 \cdots \alpha_{k_0})^{1/2} \times (\alpha_{k_0+1})^{1/2}(\alpha_{k_0+1}\psi(k_0)^{-2})^{-1/2} \ll (\alpha_0 \cdots \alpha_{k_0})^{1/2}\psi(k_0) \ll N^{1/2}\psi(k_0) .$$

Therefore, if we suppose the existence of a β_0 satisfying the condition (2), it follows, from (1) and (3) applied to Cases 1 or 3 and also from Case 2 of Theorems 1 and 2, that

$$(4)\qquad \theta(\alpha_0^{-1}, \alpha_0\beta_0\,; 0, N) \ll N^{1/2}\psi(k_0)$$

for any $N \geqq 1$.

The measure of the set of β_0 in the interval $[0, 1)$ which do not satisfy (2) for some $k \geqq -1$ is obviously not larger than $\sum_{k=-1}^{\infty} 2\psi(k)^{-1}$. Therefore, if we suppose that

$$(5)\qquad \sum_{k=-1}^{\infty} 2\psi(k)^{-1} < 1 ,$$

the measure of the set of β_0 in $[0, 1)$ which satisfy the condition (2) for every $k \geqq -1$ is not smaller than $1 - \sum_{k=-1}^{\infty} 2\psi(k)^{-1} > 0$. If we give $\psi(k)$ the values $ck(\log k)^2$ for $k \geqq 3$ with a large positive constant c, and some appropriate values for $2 \geqq k \geqq -1$, then the inequality (5) is satisfied. But $k_0 = O(\log N)$. Therefore we have the following

THEOREM 3. *If we are given any real irrational α_0 which is larger than* 1, *then there exists a set I_{α_0} of reals in the interval* $[0, 1)$ *whose* *measure is larger than* $\frac{1}{2}$, *so that we have*

$$\theta(\alpha_0^{-1}, \alpha_0\beta_0; 0, N) \ll N^{1/2} (\log 10N) (\log\log 10N)^2 ,$$

for all β_0 *in* I_{α_0}, *where the implied constant is absolute.*

This result is an improvement on the existence of an irrational $\alpha_0^{-1}\gamma_0$ such that we have $\theta(\alpha_0^{-1}, \gamma_0; 0, N) \ll N^{3/4}$, shown in [1], p. 294, Satz XV.

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