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Nagoya Math. J.
Vol. 52 (1973), 85-95

HIRONAKA'S ADDITIVE GROUP SCHEMES

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In [1] and [2], Hironaka referred to the importance of an additive group scheme $B_{p_n,\mathfrak{p}}$, which is associated with a point $\mathfrak{p}$ in $\boldsymbol{P}_n$, in connection with the resolution of singularities in characteristic $p > 0$. Also he showed that if the dimension of $B_{p_n,\mathfrak{p}}$ is not greater than p, then it is a vector group.

By Oda [3], these schemes can be characterized in terms of vector spaces and differential operators of the coefficient field, as we recall in section 1. Moreover Oda classified these schemes in dimension $\leqq 5$ completely and conjectured that;

(1) If $\dim B_{p_n,\mathfrak{p}} < 2p - 1$, then it is a vector group,

(2) If $\dim B_{p_n,\mathfrak{p}} = 2p - 1$ and $B_{p_n,\mathfrak{p}}$ is not a vector group, then its type is unique.

In this paper we see that this conjecture is true, using some tools in Oda [3].

The author wishes to thank Professor T. Oda who taught this concept to him and gave him many suggestions.

Section 1.

Let $S = k[X_0, \cdots, X_n] = \sum_{m \geqq 0} S_m, \boldsymbol{P}_n = \operatorname{Proj}(S)$, and $\mathfrak{p} \in \boldsymbol{P}_n$. A graded subalgebra $U(\mathfrak{p}) = \sum_{m \geqq 0} U_m(\mathfrak{p})$ of S is defined as follows:

$$U_m(\mathfrak{p}) = \{f \mid f \in S_m, \operatorname{mult}_{\mathfrak{p}}(\operatorname{Proj}(S/fS)) \geqq m\} .$$

Then $U(\mathfrak{p})$ is generated as a k-algebra by purely inseparable forms in S, i.e. elements of the form $a_0 X_0^{p^e} + \cdots + a_n X_n^{p^e}$ with $a_i \in k, p = \operatorname{ch}(k)$. (See [2], Th. 1, Cor.)

DEFINITION 1.1. *A Hironaka scheme $B_{p_n,\mathfrak{p}}$ associated with $\mathfrak{p}$ in $\boldsymbol{P}_n$* is a homogeneous additive subgroup scheme of the vector group Spec(S) defined by

Received June 18, 1973.

$$B_{p_n,\mathfrak{p}} = \mathrm{Spec}\,(S/U_+(\mathfrak{p})\cdot S)\ , \qquad \text{where } U_+(\mathfrak{p}) = \sum_{m>0} U_m(\mathfrak{p})\ .$$

For simplicity, we call $B_{p_n,\mathfrak{p}}$ the H-scheme associated with $\mathfrak{p}$.

In order to mention the following theorem, which is the main theorem of Oda's characterization in [3], we recall some terminologies.

(a) $L = \sum_{i \geqq 0} L_i$ is a graded k-subspace of S, where L_i is the subset of S_{p^i} consisting of all the purely inseparable forms of degree p^i. Then L is a graded left $k[F]$-module, with F acting as the p-th power map.

(b) $\mathrm{Diff}(k)$ and $\mathrm{Diff}_m(k)$ are the left k-vector spaces of differential operators over $\boldsymbol{Z}$ of k into itself, and those of order $\leqq m$, respectively. When V is a subset of L_e, the following vector subspaces of L_e are defined for $i \leqq e$:

$$\mathscr{D}_i(V) = \mathrm{Diff}_{p^i-1}(k)V$$
$$\mathscr{N}_i(V) = \{f \mid f \in L_i, \mathscr{D}_i(f) \subset k\cdot V\}\ .$$

(c) When $Q = \sum_{i \geqq 0} Q_i$ is a graded left $k[F]$-submodule of L, we can find an integer e such that $Q_{i+1} = k\cdot FQ_i$ ($i \geqq e$) and $Q_e \underset{\neq}{\supset} k\cdot FQ_{e-1}$. We call such e the exponent of Q and write $e(Q)$. We define the exponent of $B_{p_n,\mathfrak{p}}$ to be $e(U(\mathfrak{p}) \cap L)$.

(d) We call $\mathfrak{p}$ in $\boldsymbol{P}_n$ the most generic point associated with an H-scheme B in $\mathrm{Spec}\,(S)$ when $B_{p_n,\mathfrak{p}} = B$ and an arbitrary $\mathfrak{p}' \in \boldsymbol{P}_n$, which satisfies $B_{p_n,\mathfrak{p}'} = B$, contains $\mathfrak{p}$.

Remark 1.2. B is a vector group if and only if the exponent of B equals 0.

THEOREM 1.3. (Oda [3], Th. 2.5). *Let N be a graded left $k[F]$-submodule of L. Then* $\mathrm{Spec}\,(S/N\cdot S)$ *is an H-scheme of exponent e if and only if* $e(N) = e, N_e \subsetneqq L_e, \mathscr{N}_e\mathscr{D}_e(N_e) = N_e$ *and* $N = \mathrm{rad}_L\,(k[F]N_e)$, *where we define* $\mathrm{rad}_L\,(Q) = \{f \in L \mid$ *there exists a non-negative integer j such that* $F^j f \in Q\}$. *Moreover* $\mathrm{rad}_S\,(\mathscr{D}_e(N_e)\cdot S)$ *is the most generic point associated with* $\mathrm{Spec}\,(S/N\cdot S)$ *and* $\dim\,(\mathrm{Spec}\,(S/N\cdot S)) = \dim_k\,(L_e/N_e)$.

By this theorem H-schemes can be written in terms of vector spaces and differential operators as follows:

(*) Let W be a finite dimensional k^q-vector space and let V be a k-subspace of $k \underset{k^q}{\otimes} W$, with $q = p^e$. Then an H-scheme of exponent e is in one to one correspondence with a pair (V, W) satisfying the following conditions:

(i) $\mathscr{N}_e\mathscr{D}_e(V) = V$,

(ii) $V \subsetneqq k \otimes_{k^q} W$,

(iii) $V \supsetneqq k(V \cap (k^p \otimes_{k^q} W))$ if $e \geqq 1$.

Here $\dim(H\text{-scheme}) = \dim_k (k \otimes_{k^q} W/V)$. Since $\mathrm{Diff}_{q-1}(k)$ acts trivially on k^q, it is considered to act on $k \otimes_{k^q} W$ through the first factor. In this paper $H(V, W)$ means an H-scheme which is determined by a pair (V, W) satisfying (i) (ii) (iii). Also, when $e \geqq 1$, we sometimes assume the condition (iv) below for the sake of convenience,

(iv) $V \cap W = 0$ and W is minimal (i.e. $k \otimes_{k^q} W' \not\supset V$, for any proper k^q-subspace W' of W).

The former condition of (iv) means that we are dealing with the smallest ambient vector group containing the H-scheme, and the latter means that we neglect the part of the vector group when we represent the H-scheme as (vector group) $\times$ (not vector group).

Remark 1.4. When $e \geqq 1$, it is evident that if (V, W) satisfies (iii) then (V, W) automatically satisfies (ii).

(V, W) and (V', W') are said to be of the same type when there exist a field automorphism σ of k and a k^q-semi-linear isomorphism $\psi: W \to W'$ such that the induced map $\sigma \otimes \psi: k \otimes_{k^q} W \to k \otimes_{k^q} W'$ sends V onto V'.

Section 2.

EXAMPLE 2.1. (See Oda [3].) Let W be a k^p-vector space of $\dim W = 2p$ with basis X_i, Z_i $(i = 0, \cdots, p-1)$. Let c_1 and c_2 be elements of k, p-independent over k^p. If $V = k\cdot f$ with $f = \sum_{i=0}^{p-1} c_1^i(X_i + c_2 Z_i)$, then $H = H(V, W)$ is an H-scheme of exponent $e(H) = 1$ and $\dim H = 2p - 1$. Furthermore $\mathscr{D}_1(V) = \sum_{i=0}^{p-1} k\cdot(X_i + c_1^{p-1-i}c_2 Z_{p-1}) \oplus \sum_{i=0}^{p-2} k\cdot(Z_i - c_1^{p-1-i}Z_{p-1})$. The H-scheme corresponding to this pair is

$$\mathrm{Spec}\left(k[x_i, z_i]\Big/\sum_{i=0}^{p-1} c_1^i(x_i^p + c_2 z_i^p)\right) ,$$

with x_i, z_i $(i = 0, \cdots, p-1)$ indeterminates. This is the most typical example of those H-schemes which are not vector groups and associated with a closed point in $\boldsymbol{P}_{2p-1}$.

Now let W^* be the dual space of a k^q-vector space W with $q = p^e$. Since $\mathrm{Diff}_{q-1}(k)$ acts on $k \otimes_{k^q} W^*$, we can define $\mathscr{D}_i^*$ and $\mathscr{N}_i^*$ in the same way as $\mathscr{D}_i$ and $\mathscr{N}_i$ for $i \leqq e$.

DEFINITION 2.2. For a pair (V, W) we define (V^*, W^*) to be a pair where W^* is the dual k^q-vector space of W and $V^* = \mathscr{D}_e(V)^\perp$. We define conditions (i*) (ii*) (iii*) (iv*) in the same way as in (*) of § 1.

LEMMA 2.3. (Oda [3], Lemma 2.8.). *For a k-subspace U of $k \otimes_{k^q} W$, we have*

$$\mathscr{N}_i(U)^\perp = \mathscr{D}_i^*(U^\perp) \quad and \quad \mathscr{D}_i(U)^\perp = \mathscr{N}_i^*(U^\perp) .$$

LEMMA 2.4. *When $q = p^e$ and $q' = p^{e'}$ with $e' \leqq e$, we have $\mathscr{D}_e(V) = \mathrm{Diff}_{q-q'}(k)\mathscr{D}_{e'}(V)$.*

Proof. Since $k \cdot V$ is a finite dimensional k-vector space, we can choose a base f_β $(\beta = 1, \cdots, s)$. There exists a finite set $c_1, \cdots, c_m$ of elements of k, p-independent over k^p so that $K = k^q(c_1, \cdots, c_m)$ contains the coefficients of f_β $(\beta = 1, \cdots, s)$. Since $\mathrm{Diff}_{q-1}(k)V = k \cdot \mathrm{Diff}_{q-1}(K/k^q)V$, it is enough to show

$$\mathrm{Diff}_{q-1}(K/k^q) = \mathrm{Diff}_{q-q'}(K/k^q)\,\mathrm{Diff}_{q'-1}(K/k^q) .$$

Let D_{ij} $(1 \leqq i \leqq m, 0 \leqq j \leqq e-1)$ be the k^q-linear map of K into itself defined by

$$D_{ij}\Big(\prod_{1 \leqq \alpha \leqq m} c_\alpha^{t_\alpha}\Big) = \begin{cases} 0 & (t_i < p^j) \\ \dbinom{t_i}{p^j} c_i^{t_i - p^j} \prod\limits_{\substack{1 \leqq \alpha \leqq m \\ \alpha \neq i}} c_\alpha^{t_\alpha} & (t_i \geqq p^j) . \end{cases}$$

Then D_{ij} is a differential operator of K over k^q of order p^j. Moreover D_{ij}'s commute with each other. When t_{ij} $(1 \leqq i \leqq m, 0 \leqq j \leqq e-1)$ vary among integers satisfying

$$0 \leqq t_{ij} \leqq p - 1$$

and

$$\sum_{\substack{1 \leqq i \leqq m \\ 0 \leqq j < e}} t_{ij} p^j \leqq p^e - 1 ,$$

the operators $D = \prod D_{ij}{}^{t_{ij}}$ $(1 \leqq i \leqq m, 0 \leqq j < e)$ form a K-basis of $\mathrm{Diff}_{q-1}(K/k^q)$. Then we see easily that D can be written as $D'D''$ with

D' in $\mathrm{Diff}_{q-q'}(K/k^q)$ and D'' in $\mathrm{Diff}_{q'-1}(K/k^q)$. Thus the lemma is proved.

PROPOSITION 2.5. *(V, W) satisfies* (i) *of* (*) *in* §1 *if and only if (V^*, W^*) satisfies* (i*). *Under this condition, when $e \geqq 1$, (V, W) satisfies* (iii) (*resp.* (iv)) *if and only if (V^*, W^*) satisfies* (iii*) (*resp.* (iv*)).

Proof. If $\mathcal{N}_e\mathcal{D}_e(V) = V$, we have by Lemma 2.3 $\mathcal{D}_e^*(V^*)^\perp = \mathcal{D}_e^*(\mathcal{D}_e(V)^\perp)^\perp = \mathcal{N}_e\mathcal{D}_e(V) = V$, and $\mathcal{N}_e^*\mathcal{D}_e^*(V^*) = \mathcal{N}_e^*(V^\perp) = \mathcal{D}_e(V)^\perp = V^*$. Thus the equivalence of (i) and (i*) is proved. To prove the equivalence of (iii) (resp. (iv)) with (iii*) (resp. (iv*)) it is enough to show the only if parts. If $V^* = k\cdot(V^* \cap (k^p \otimes W^*))$, then we have $\mathcal{D}_e^*(V^*) = k\cdot(\mathcal{D}_e^*(V^*) \cap (k^p \otimes W^*))$ by the fact in the proof of Lemma 2.4. Thus $V^\perp = k\cdot(V^\perp \cap (k^p \otimes W^*))$ and $V = k\cdot(V \cap (k^p \otimes W))$, and hence (iii) and (iii*) are equivalent. If (V, W) satisfies (i), we have $V \cap W = \mathcal{D}_e(V) \cap W$ and similarly in the dual space $V^* \cap W^* = V^\perp \cap W^*$ by the remark below Proposition 3.1 in Oda [3]. Thus W is not minimal if and only if there exists $0 \neq f \in W^*$ such that $\langle V, f\rangle = 0$, i.e. if and only if $\{0\} \neq V^\perp \cap W^* = V^* \cap W^*$. Thus W is minimal if and only if $V^* \cap W^* = \{0\}$. By the duality the equivalence of (iv) and (iv*) is proved.

Thus, when $e \geqq 1$, we can associate the dual H-scheme $H(V^*, W^*)$, which we denote also by H^*, with $H = H(V, W)$. Evidently we have $e(H) = e(H^*)$ and $H^{**} = H$.

As was seen in Oda [3], $V \cap W = \mathcal{D}_e(V) \cap W$ is one of the handiest necessary conditions for a pair (V, W) to correspond to an H-scheme.

LEMMA 2.6. (Oda [3], Proposition 3.1.) *Let $H = H(V, W)$ be an H-scheme with $\dim H = d$, $e(H) = e$ and $\dim_k(V) = v$. Then there exists a k^{p^e}-basis $\{X_i, Y_j\}_{(i=1,\dots,d,j=1,\dots,v)}$ of W and a k-basis $\{f_j\}_{j=1,\dots,v}$ of V such that*

$$f_j = Y_j + c_{1j}X_1 + \cdots + c_{dj}X_d (c_{ij} \in k) \quad \text{and} \quad \mathcal{N}_e\mathcal{D}_e(f_j) = k\cdot f_j.$$

Moreover we can choose f_1 so that $H_1 = H(k\cdot f_1, k^{p^e}\cdot Y_1 \oplus \sum_{i=1}^d k^{p^e}\cdot X_i)$ is an H-scheme with $\dim H_1 = d$ and $e(H_1) = e$.

LEMMA 2.7. *Let $H = H(V, W)$ be an H-scheme with $e(H) \geqq 1$. When $0 \leqq e' \leqq e$, $H' = H(V, W')$ is an H-scheme with $e(H') = e'$ and $\dim H' = \dim H$, where $W' = k^{p^{e'}} \otimes_{k^{p^e}} W$.*

Proof. The conditions (ii) (iii) of (*) being trivially verified, it is enough to show that $\mathcal{N}_{e'}\mathcal{D}_{e'}(V) = V$ if $\mathcal{N}_e\mathcal{D}_e(V) = V$. By Lemma 2.4 above and Lemma 2.9 in Oda [3], we have $\mathcal{D}_e\mathcal{N}_{e'}\mathcal{D}_{e'}(V) =$

$\mathrm{Diff}_{p^e-p^{e'}}(k)\mathscr{D}_{e'}\mathscr{N}_{e'}\mathscr{D}_{e'}(V) = \mathrm{Diff}_{p^e-p^{e'}}(k)\mathscr{D}_{e'}(V) = \mathscr{D}_e(V)$. Thus $\mathscr{N}_{e'}\mathscr{D}_{e'}(V) \subset \mathscr{N}_e\mathscr{D}_e(V) = V$. The inverse inclusion is trivial. The exponent and the dimension are easy to calculate.

This lemma means that the image H' of H by the Frobenius morphism $F^{e-e'}$ of the ambient vector group defined by $(x_0, \cdots, x_n) \to (x_0^{p^{e-e'}}, \cdots, x_n^{p^{e-e'}})$ is again H-scheme of exponent e'.

THEOREM 2.8. *If $H = H(V, W)$ is not a vector group* (i.e. $e(H) \geqq 1$), *then* $\dim H \geqq 2p - 1$. *Moreover if* $\dim H = 2p - 1$ *and H is not a vector group with $V \cap W = \{0\}$, then H is of the same type as Example* 2.1.

Proof. Let m be the smallest dimension of H-schemes with positive exponents. Then by Lemma 2.6 and Lemma 2.7 there exists $H_f = H(k\cdot f, W)$ such that $\dim H_f = m$ and $e(H_f) = 1$. Moreover it is an immediate consequence of the minimality of m that H_f satisfies (iv) of (*), hence in particular $\mathscr{D}_1(f) \cap W = \{0\}$. Now let us observe dimensions over k of the sequence

$$k\cdot f \subset \mathrm{Diff}_1(k)f \subset \mathrm{Diff}_2(k)f \subset \cdots \subset \mathrm{Diff}_{p-1}(k)f = \mathscr{D}_1(f) .$$

We claim $\dim_k \mathrm{Diff}_{i+1}(k)f \geqq \dim_k \mathrm{Diff}_i(k)f + 2$ $(i = 0, \cdots, p-2)$. If $\dim_k \mathrm{Diff}_i(k)f = t$, then we may assume that $\mathrm{Diff}_i(k)f$ is generated by $X_j + h_j$ $(j = 0, \cdots, t-1)$ over k, where h_j is a k-linear combination of $X_t, \cdots, X_m$ and $\{X_j\}_{j=0,\cdots,m}$ is a k^p-basis of W. We define $c(g)$ to be the k^p-vector subspace of k spanned by the coefficients of $g \in k \otimes_{k^p} W$. There are the following three possibilities:

(1) There exists j $(0 \leqq j < t)$ such that there is no intermediate subfield of the form $k^p(a)$ containing $c(X_j + h_j)$. In this case, we may assume that there exist D_1, D_2 in $\mathrm{Der}(k/k^p)$ with $D_1(X_j + h_j) = X_t + h'$ and $D_2(X_j + h_j) = X_{t+1} + h''$, where h' and h'' are linear combinations of $X_{t+2}, \cdots, X_m$. The above statement is obvious in this case.

(2) For each j there exists an intermediate subfield $k^p(a_j)$ containing $c(X_j + h_j)$.

(i) If there exist $j \neq j'$ such that $k^p(a_j) \neq k^p(a_{j'})$, then we can choose $D_j, D_{j'}$ in $\mathrm{Der}(k/k^p)$ satisfying $D_j(a_j) = 1$ and $D_{j'}(a_{j'}) = 1$. It is enough to show that $D_j(h_j)$ and $D_{j'}(h_{j'})$ are linearly independent over k. If $D_j(h_j) = u\cdot D_{j'}(h_{j'})$ with $u \in k$, then

$$c(D_j(h_j)) = u\cdot c(D_{j'}(h_{j'})) \subset k^p(a_j) \cap u\cdot k^p(a_{j'}) .$$

But it is easy to show that

$$\dim_{k^p}(k^p(a_j) \cap u\cdot k^p(a_{j'})) \leqq 1 .$$

Hence we readily get a contradiction in view of the property $\mathscr{D}_1(f) \cap W = \{0\}$.

(ii) For all $j, k^p(a_j) = k^p(a)$ with $a \in k$.

Then $\mathscr{D}_1(f) = k\cdot(\mathscr{D}_1(f) \cap (k^p(a) \otimes W))$ and thus we have $(k\cdot f)^* = k\cdot((k\cdot f)^* \cap (k^p(a) \otimes W))$, since $(k\cdot f)^* = \mathscr{D}_1(f)^{\perp}$. Hence $(k\cdot f)^{\perp} = \mathscr{D}_1^*((k\cdot f)^*) = k\cdot(\mathscr{D}_1^*((k\cdot f)^*) \cap (k^p(a) \otimes W))$. Thus we may assume $c(f) \subset k^p(a)$. If D is a derivation with $D(a) = 1$, there exists an integer $s \leqq p - 1$ such that $D^s(f) \neq 0$ and $D^{s+1}(f) = 0$. So $0 \neq D^s(f) \in \mathscr{D}_1(f) \cap W$, a contradiction. Hence (ii) does not happen.

Thus we conclude that $\dim \mathscr{D}_1(f) \geqq 2p - 1$ and $\dim W \geqq 2p$. Hence $\dim H_f = \dim W - \dim k\cdot f \geqq 2p - 1$ and $m \geqq 2p - 1$. But the dimension of the H-scheme in Example 2.1 is $2p - 1$, hence $m = 2p - 1$. The first part of the theorem is thus proved. Now let us prove the second part of Theorem 2.5. When $p = 2$, Hironaka already proved this theorem (Hironaka [2], Th. 3.). From now on we assume $p \neq 2$.

Step (I): The case where the H-scheme is of the form $H = H(k\cdot f, W)$ with $\dim H = 2p - 1$ and $e(H) = 1$. (Then H automatically satisfies (iv) of (*).) In this case the codimension of $\mathscr{D}_1(f)$ in $k \otimes_{k^p} W$ equals 1, i.e. the most generic point associated with H is a closed point, since $\dim_{k^p} W = \dim W^* = 2p$ and $(k\cdot f)^* \neq 0$, thus $2p - 1 \leqq \dim H^* < 2p$, hence $\dim H^* = 2p - 1$ and $\mathrm{codim}_k \mathscr{D}_1(f) = \dim_k (k\cdot f)^* = 1$. By the proof of the first part, the sequence of the dimensions of $k\cdot f \subset \mathrm{Diff}_1(k)f \subset \cdots \subset \mathrm{Diff}_{p-1}(k)f$ is necessarily $1, 3, 5, \cdots, 2p - 1$. In particular

$$\dim \mathrm{Diff}_1(k)f = 3 \quad \text{and} \quad \dim \mathrm{Diff}_2(k)f = 5 .$$

We put $K = k^p(c(f))$. Then $[K : k^p] = p^2$, since $\dim \mathrm{Diff}_1(k)f = r + 1$ if $[K : k^p] = p^r$. Since $\mathrm{Diff}_i(k)f = k\cdot \mathrm{Diff}_i(K/k^p)f$ with arbitrary $i \geqq 0$, we have

$$\dim_k \mathrm{Diff}_2(k)f = \dim_K \mathrm{Diff}_2(K/k^p)f = 5 .$$

But $\dim_K \mathrm{Diff}_2(K/k^p) = 6$, thus there exists D in $\mathrm{Diff}_2(K/k^p)$ such that $D \neq 0$ and $D(f) = 0$. Since W is minimal, we have

$$\dim_{k^p} c(f) = \dim_{k^p} W = 2p .$$

We may assume $c(f) \ni 1$. Hence by Lemma 2.9 below there exists D_0 in $\mathrm{Der}(K/k^p)$ such that $D = u \cdot D_0^2$ with $u \in K$ and $D_0(c_1) = 0$, $D_0(c_2) = 1$ where $K = k^p(c_1, c_2)$. Thus

$$c(f) = k^p(c_1) \oplus c_2 \cdot k^p(c_1) \ ,$$

and H is of the same type as Example 2.1.

Step (II): The general case $H = H(V, W)$ with $\dim H = 2p - 1$, $e(H) = 1$, and $V \cap W = \{0\}$. Then H-schemes $H_j = H(k \cdot f_j, k^p Y_j \oplus \sum_{i=1}^{2p-1} k^p \cdot X_i)$ of dimension $2p - 1$ in Lemma 2.6 ($j = 1, \cdots, v$) have exponent $e(H_j) = 1$, since $V \cap W = \{0\}$. Thus the codimension of $\mathscr{D}_1(V)$ in $k \otimes W$ is 1, since by the proof of step (I) $\mathscr{D}_1(f_j)$ are of codimension one in $k^p \cdot Y_j \oplus \sum_{i=1}^{2p-1} k^p X_j$ and have the property $\mathscr{D}_1(f_j) \cap W = \{0\}$ for all j. Hence $V^* = k \cdot f^*$ and $\dim \mathscr{D}_1^*(f^*) = \dim V^{\perp} = \dim H = 2p - 1$. By applying the proof of the first part to $H(k \cdot f^*, W^*)$, we have

$$\dim \mathrm{Diff}_1(k) f^* = 3 \quad \text{and} \quad \dim \mathrm{Diff}_2(k) f^* = 5 \ .$$

Thus by Lemma 2.9 below $\dim c(f^*) \leqq 2p$. Since $V \cap W = \{0\}$ if and only if W^* is minimal, we have $2p \geqq \dim c(f^*) = \dim W^* = \dim W$, hence $\dim V = v = 1$. (II) is thus reduced to (I).

Step (III): The case $H = H(V, W)$ where $\dim H = 2p - 1$ and $e(H) = e > 1$. If there exists such $H(V, W)$, then by Lemma 2.6 there exists $H' = H(k \cdot f, W')$ with $\dim H' = 2p - 1$ and $e(H') = e$ satisfying (iv). Then by Lemma 2.7 and the minimality of $2p - 1$, $H'' = H(k \cdot f, W'')$ satisfies $\dim H'' = 2p - 1$, $e(H'') = 1$ and (iv), where $W'' = k^p \otimes_{k^{p^e}} W'$. Thus by (I) H'' is of the same type as Example 2.1. But it is easy to calculate that

$$\mathscr{D}_e(f) \supset \mathrm{Diff}_p(k) f = k \otimes_{k^p} W'' = k \otimes_{k^{p^e}} W' \ .$$

Thus we have a contradiction to the property $\mathscr{D}_e(f) \cap W' = \{0\}$.

It remains to prove the following lemma to conclude the proof of Theorem 2.8.

LEMMA 2.9. *Let $k \supset K \supset k_p$ with $[K : k_p] = p^2$ and $p \neq 2$, and let D be an element of* $\mathrm{Diff}_2(K/k_p)$ *with $D \neq 0$ and $D(1) = 0$. Then D satisfies the followings:*

(1) $\dim_{k^p} \ker(D) \leqq 2p$ *when D is considered to be a k^p-linear map from K to itself,*

(2) *the equality holds if and only if there exists* $D_0 \in \operatorname{Der}(K/k^p)$ *with the property* $D_0(c_1) = 0$ *and* $D_0(c_2) = 1$ *where* $k^p(c_1, c_2) = K$, *such that* $D = u \cdot D_0^2$ *with* $u \in K$.

Proof. We put $T = \ker(D) \subset K$. Then T contains 1. If T is contained in a proper subfield of K, then (1) is obvious. Otherwise we may choose elements t_1 and t_2 of T with $k^p(t_1, t_2) = K$. Let D_1, D_2 be elements of $\operatorname{Der}(K/k^p)$ defined by $D_i(t_j) = \delta_{i,j}$ $(i, j = 1, 2)$. Then $D = a'D_1^2 + b'D_1D_2 + c'D_2^2$. If $a' = c' = 0$, then $\dim T = 2p - 1$. We may thus assume

$$D = D_1^2 + aD_1D_2 + bD_2^2 \quad \text{with} \quad a, b \quad \text{in} \quad K\,.$$

To an element $\Delta = \sum_{i,j=1}^{p} a_{i,j} D_2^{i-1} D_1^{j-1}$ of $\operatorname{Diff}(K/k^p)$ we associate a (p, p)-matrix $\rho(\Delta) = (a_{i,j})$. Then ρ is an isomorphism from $\operatorname{Diff}(K/k^p)$ to the set $\mathfrak{M}(K; p, p)$ of (p, p)-matrices with coefficients in K as vector spaces over K. Then

$$\rho(D) = \begin{pmatrix} 0 & 0 & 1 & \\ 0 & a & & \\ b & & & \\ & & & 0 \end{pmatrix}.$$

Let I be the left ideal $\operatorname{Diff}(K/k^p) \cdot D$ of the ring $\operatorname{Diff}(K/k^p)$. Then $\rho(\Delta_{i,j})$ is of the form

$$\begin{array}{c} \qquad\qquad\qquad\qquad j \\ \qquad\qquad\qquad\qquad \vdots \end{array}$$
$$\left[\begin{array}{cccccc} 0 & 0 & \cdots\cdots & 0 & 0 & \\ 0 & * & \cdots\cdots & * & 0 & \\ * & * & \cdots\cdots & * & 0 & \\ \vdots & \vdots & & \vdots & & \\ & & & * & 0 & 0 \\ * & * & & * & 1 & \cdots \\ * & * & \cdots * & a & 0 & \\ * & * & \cdots * \;\; b & 0 & 0 & \\ & & 0 & & & 0 \end{array}\right] \cdots i$$

where $\Delta_{i,j} = D_2^{i-1} D_1^{j-3} D$ is an element of I $(1 \leqq i \leqq p, 3 \leqq j \leqq p)$. Since $\rho(\Delta_{i,j})$ $(i = 1, \cdots, p, j = 3, \cdots, p)$ are linearly independent over K, we have $\dim_K I \geqq p(p-2)$.

By a theorem of Jacobson, we can identify the ring $\operatorname{Diff}(K/k^p)$ with $\operatorname{Hom}_{k^p}(K, K)$. Let $\pi: K \to K/T$ be the natural projection and let $n =$

$\dim_{k^p} T$. From the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{k^p}(K/T, K) \xrightarrow{\pi^*} \mathrm{Hom}_{k^p}(K, K) \longrightarrow \mathrm{Hom}_{k^p}(T, K) \longrightarrow 0$$

we get $\pi^*(\mathrm{Hom}_{k^p}(K/T, K)) \supset I$ and $\dim_K \mathrm{Hom}_{k^p}(K/T, K) = \dim_K \mathrm{Hom}_{k^p}(K, K) - \dim_K \mathrm{Hom}_{k^p}(T, K) = p^2 - n$. Thus $p^2 - n = \dim_K \mathrm{Hom}(K/T, K) \geqq \dim_K I \geqq p(p-2)$. Hence $n \leqq 2p$ and we get (1). Moreover $n = 2p$ if and only if $\dim_K I = p(p-2)$, hence I is generated by $\Delta_{i,j}$ $(i = 1 \cdots p, j = 3 \cdots p)$ as a K-vector space. To show (2) it is sufficient to show the existence of $D_0 \in \mathrm{Der}(K/k^p)$ with $D = u \cdot D_0^2$, since $2p = \dim_{k^p} \ker(D) = \dim_{k^p} \ker(D_0^2) \leqq 2 \dim_{k^p} \ker(D_0) \leqq 2p$. Hence $\dim \ker(D_0) = p$ and $\mathrm{Im}(D_0) \supset \ker(D_0) \ni 1$. Thus we can find such c_1, c_2 that $D_0(c_1) = 0, D_0(c_2) = 1$, and $k^p(c_1, c_2) = K$. In order to seek such D_0, we use a primitive method depending on complicated calculations, of which we indicate only an outline below.

Since $\rho(D_1^{p-2}D) = \begin{pmatrix} 0 & 0 \cdots\cdots\cdots & 0 \\ 0 & * \cdots\cdots * & a \\ * \cdots\cdots * & b & 0 \\ & 0 & \end{pmatrix}$ is in $\rho(I)$, we can write

$$\rho(D_1^{p-2}D) = \sum_{\substack{1 \leqq i \leqq p \\ 3 \leqq j \leqq p}} x(i, j) \cdot \rho(\Delta_{i,j}) \quad \text{with} \quad x(i, j) \in K .$$

Comparing the $(i, p - i + 2)$-components $(i = 2, \cdots, p)$ of both sides, we get

$(a^2 - 4b)^{1/2(p-1)} = 0$, hence $b = (\frac{1}{2}a)^2$. Thus

$$D = (D_1 + \tfrac{1}{2}aD_2)^2 - \tfrac{1}{2}(D_1(a) + \tfrac{1}{2}aD_2(a))D_2 .$$

$$\text{Similarly } \rho(I) \ni \rho(D_1^{p-1}D) = \begin{pmatrix} 0 & 0 \cdots\cdots\cdots\cdots\cdots\cdots & 0 \\ 0 & * \cdots\cdots * & ,(p-1)D_1(a) \\ * \cdots * & (p-1)\frac{1}{2}aD_1(a), (\frac{1}{2}a)^2 & \\ & 0 & \end{pmatrix}$$

is of the form $\sum_{\substack{1 \leqq i \leqq p \\ 3 \leqq j \leqq p}} y(i, j) \cdot \rho(\Delta_{i,j})$ with $y(i, j) \in K$. From the comparison of the $(i, p - i + 3)$-components $(i = 3, \cdots, p)$ and $(i, p - i + 2)$-components $(i = 2, \cdots, p)$, we get

$$(\tfrac{1}{2}a)^{p-2}(D_1(a) + (\tfrac{1}{2}a)D_2(a)) = 0 .$$

If $a = 0$, then $D = D_1^2$, and if $D_1(a) + (\frac{1}{2}a)D_2(a) = 0$, then $D = (D_1 + \frac{1}{2}aD_2)^2$. Thus Lemma 2.9 is proved.

Remark 2.10. In general let $m(e)$ be the smallest dimension of H-schemes whose exponents are not less than e. By Lemma 2.7 we have $m(1) \leqq m(2) \leqq \cdots \leqq m(e) \leqq \cdots$. It is quite likely that $m(e) = 2p^e - 1$. This is in fact the case for $e = 1$ as we saw in Theorem 2.8, as well as for $e = 0$ (for obvious reasons). Now let $H = H(k \cdot f, \sum_\alpha k^p \cdot X_\alpha)$ be an H-scheme with $e(H) = 1$ and $f = \sum_\alpha a_\alpha X_\alpha$, which is associated with a closed point. Suppose there exists a p-basis Λ of k over k^p such that a_α's are in $k^{p^2}(\Lambda')$ with $\Lambda' \subsetneqq \Lambda$. Let c be an element of Λ not in Λ', and define

$$F = \sum_{\beta=0}^{p-1} (c^p)^\beta f_\beta \quad \text{with} \quad f_\beta = \sum_\alpha a_\alpha Y_{\alpha,\beta} \,.$$

Then $H_2 = H(k \cdot F, \sum_{\alpha,\beta} k^{p^2} \cdot Y_{\alpha,\beta})$ is an H-scheme with $e(H_2) = 2$ and is associated with a closed point. If we take the H-scheme in Example 2.1 as H, then H_2 is an H-scheme with $e(H_2) = 2$ and $\dim H_2 = 2p^2 - 1$. Thus inductively we can construct examples $H_2, H_3, \cdots, H_e$ such that

$$e(H_e) = e \quad \text{and} \quad \dim H_e = 2p^e - 1 \,.$$

Obviously we no longer have the uniqueness of type when $e > 1$.

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