N. Mochizuki
Nagoya Math. J.
Vol. 52 (1973), 53–59

HOLOMORPHIC EXTENSION OF CONTINUOUS, WEAKLY HOLOMORPHIC FUNCTIONS ON CERTAIN ANALYTIC VARIETIES

NOZOMU MOCHIZUKI

§ 1. Introduction

Let M, N be connected complex submanifolds of a neighborhood of the origin $0 \in \boldsymbol{C}^d$, the space of d complex variables, such that $0 \in M \cap N$. We shall suppose throughout that $M \not\subset N$ and $N \not\subset M$ in any neighborhood of 0. Let $X = M \cup N$. X is an analytic subvariety with the irreducible branches M and N. Let Δ be a neighborhood of 0 in $\boldsymbol{C}^d$. We consider the following proposition:

($*$) Let f be any complex-valued function defined on $X \cap \Delta$ such that the restrictions $f | M \cap \Delta$ and $f | N \cap \Delta$ are holomorphic. Then f extends to a holomorphic function on Δ.

If X is quasi-normal at 0, then ($*$) holds for a suitable polydisk Δ (for the definition of quasi-normality, see § 2).

It is the object of the present paper to deal with a property of varieties introduced above which implies the quasi-normality. This problem has been discussed in [2] in a restricted case. Observing examples such as Theorem 6 in [1] or Corollary 6 in [2], we are led to infer that a sense of orthogonality of M to N or maximality of the embedding of X into $\boldsymbol{C}^d$ at 0 has some connection with the quasi-normality of X and that such a situation will be well described by tangent spaces of M and N; so we shall give a sufficient condition by use of them.

§ 2. A lemma

We denote by ${}_n\mathcal{O}_0$ the ring of germs of holomorphic functions at 0 in $\boldsymbol{C}^n$ and by $\mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_m)$ the germ of the variety defined by the ideal $(\mathbf{f}_1, \cdots, \mathbf{f}_m)$. $\mathrm{id}\,\mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_m)$ denotes the ideal of ${}_n\mathcal{O}_0$ consisting of germs

Received May 7, 1973.

which vanish on the variety $\mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_m)$. Let Δ be a polydisk with center 0 in $\boldsymbol{C}^d$; i.e., $\Delta = \{(z_1, \cdots, z_d) \in \boldsymbol{C}^d \,||z_i| < r,\ i = 1, \cdots, d\}$ for some $r > 0$. Then Δ_k will stand for the polydisk of the same radius in $\boldsymbol{C}^k$, $1 \leqq k < d$. The ring of germs of holomorphic functions of an analytic space X at a point $p \in X$ will be denoted by ${}_X\mathcal{O}_p$, and the ring of germs of continuous, weakly holomorphic functions will be denoted by ${}_X\mathcal{O}'_p$. X is said to be quasi-normal at p if ${}_X\mathcal{O}'_p = {}_X\mathcal{O}_p$ ([1]).

The following is a generalization of Theorem 2 in [2].

LEMMA. *Let M, N be complex submanifolds of $\boldsymbol{C}^d$ of dimension m, n, respectively, and let $0 \in M \cap N$. Suppose that there exist a polydisk Δ and a nonsingular holomorphic map $\alpha = (f_1, \cdots, f_d) \colon \Delta_n \to \boldsymbol{C}^d$, $\alpha(0) = 0$, such that*

$$M = \Delta_m \times \{0\} \subset \boldsymbol{C}^d\ , \qquad N \cap U = \alpha(\Delta_n)$$

for a neighborhood U of 0 in $\boldsymbol{C}^d$. Let $X = M \cup N$. Then, X is quasi-normal at 0 if and only if

$$\mathrm{id}\, \mathbf{V}(\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d) = (\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d)\ .$$

Proof. Since α is nonsingular, there exists a nonsingular holomorphic map $\tilde{\alpha} \colon \Delta \to \boldsymbol{C}^d$ such that $\tilde{\alpha} | \Delta_n \times \{0\} = \alpha$. We may suppose that $\tilde{\alpha}$ is a biholomorphic map of Δ onto U.

Suppose that X is quasi-normal at 0 and let $\mathbf{g} \in \mathrm{id}\, \mathbf{V}(\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d)$. There exists a polydisk $\Delta' \subset \Delta$ such that the following hold:

(1) $\tilde{\alpha}(\Delta') \subset U \cap \Delta$.

(2) There is a holomorphic function g on Δ'_n inducing the given germ $\mathbf{g}$; and $g = 0$ on the subvariety $V = \{z' \in \Delta'_n \,|\, f_j(z') = 0,\ j = m + 1, \cdots, d\}$. Let $U' = \tilde{\alpha}(\Delta')$. Then,

$$\alpha^{-1}(N \cap U') = \Delta'_n\ , \qquad \alpha^{-1}(M \cap N \cap U') = V\ .$$

We define a function G on $X \cap U'$ by

$$G(z) = \begin{cases} 0\ , & \text{for } z \in M \cap U' \\ g \circ \alpha^{-1}(z)\ , & \text{for } z \in N \cap U'\ . \end{cases}$$

G is continuous and weakly holomorphic; hence there exist a polydisk $\Delta'' \subset U' \cap \Delta'$ and a holomorphic function $\tilde{G}$ on Δ'' such that $\tilde{G} = G$ on $X \cap \Delta''$. The power series expansion of $\tilde{G}$ is expressed in the form

$$\tilde{G}(z) = g_0(z_1, \cdots, z_m) + \sum_{j=1}^{r} g_j(z_1, \cdots, z_m, z_{m+j}, \cdots, z_d)z_{m+j} ,$$

$z \in \Delta''$, where $r = d - m$. Let $\alpha(\Delta_n''') \subset \Delta''$ for a suitable polydisk $\Delta''' \subset \Delta''$. In the above, we see that $g_0 = 0$ on Δ_m'''. Putting $z = \alpha(z')$, $z' \in \Delta_n'''$, we obtain $\tilde{G}(z) = g(z')$; hence we have

$$\begin{aligned} g(z') &= \sum_{j=1}^{r} g_j(f_1(z'), \cdots, f_m(z'), f_{m+j}(z'), \cdots, f_d(z'))f_{m+j}(z') \\ &= \sum_{j=1}^{r} a_j(z')f_{m+j}(z') , \end{aligned}$$

where $a_j, j = 1, \cdots, r$, are holomorphic functions on Δ_n'''. It follows that $\mathbf{g} \in (\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d)$.

To prove the converse, suppose that $\mathrm{id}\,\mathbf{V}(\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d) = (\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d)$ and let $\mathbf{G} \in {}_X\mathcal{O}_0'$. There exists a polydisk $\Delta' \subset U \cap \Delta$ such that the germ $\mathbf{G}$ is induced from a continuous function G defined and weakly holomorphic on $X \cap \Delta'$. We choose a polydisk $\Delta'' \subset \Delta'$ such that $\tilde{\alpha}(\Delta'') \subset \Delta'$. Let π be the projection: $\boldsymbol{C}^d \to \boldsymbol{C}^m \times \{0\} \subset \boldsymbol{C}^d$ and define a holomorphic function $G \circ \alpha - G \circ \pi \circ \alpha$ on Δ_n''. Since $\alpha = \pi \circ \alpha$ on the subvariety

$$V = \{z' \in \Delta_n'' \mid f_j(z') = 0,\ j = m + 1, \cdots, d\} ,$$

it follows from the assumption that

$$\mathbf{G} \circ \alpha - \mathbf{G} \circ \pi \circ \alpha \in (\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d) .$$

There exist a polydisk $\Delta''' \subset \Delta'' \cap \tilde{\alpha}(\Delta'')$ and holomorphic functions a_j, $j = 1, \cdots, r$, on Δ_n''' such that

$$G \circ \alpha(z') = G \circ \pi \circ \alpha(z') + \sum_{j=1}^{r} a_j(z')f_{m+j}(z') , \qquad z' \in \Delta_n''' .$$

Let π' be the projection: $\boldsymbol{C}^d \to \boldsymbol{C}^n$. We define a holomorphic function $\tilde{G}$ on a polydisk Δ'''', $\Delta'''' \subset \tilde{\alpha}(\Delta''')$, by

$$\tilde{G}(z) = G \circ \pi(z) + \sum_{j=1}^{r} (a_j \circ \pi' \circ \tilde{\alpha}^{-1})(z)z_{m+j}, \qquad z \in \Delta'''' .$$

$\tilde{G}$ is an extension of $G \mid X \cap \Delta''''$. In fact, for $z \in \Delta_m'''' \times \{0\}$, we have $\tilde{G}(z) = G(z)$; on the other hand, if $z \in N \cap \Delta''''$, then $z = \alpha(z')$ for some $z' \in \Delta_n'''$, hence we have

$$\tilde{G}(z) = G \circ \pi \circ \alpha(z') + \sum_{j=1}^{r} a_j(z')f_{m+j}(z') = G(z) .$$

This completes the proof.

§ 3. Theorems

For a complex submanifold M of C^d with $0 \in M$, the tangent space $T_0(M)$ to M at 0 is defined to be the collection of all derivations of the ring ${}_M\mathcal{O}_0$. We shall regard $T_0(M)$ as a vector subspace of C^d. Let $\dim_0 M = m$, $\dim_0 N = n$. Let $T_0(M, N)$ denote $T_0(M) + T_0(N)$, the subspace spanned by $T_0(M)$ and $T_0(N)$. Then, $\dim T_0(M, N) = m + n$ if and only if $T_0(M) \cap T_0(N) = (0)$; and $m + n \leqq d$ in this case. Also we have $\dim T_0(M, N) = d$ if and only if $T_0(M) + T_0(N) = C^d$; and $d \leqq m + n$ in this case. This means that the embedding map $\alpha: \Delta_n \to C^d$ is transversal to M at 0.

THEOREM 1. *Let M, N be complex submanifolds of C^d such that $0 \in M \cap N$ and $\dim_0 M = m$, $\dim_0 N = n$. Let $X = M \cup N$. If $\dim T_0(M, N) = \min(m + n, d)$, then X is quasi-normal at* 0.

Proof. M and N are locally represented as follows in general. There exist neighborhoods U, U' of 0 in C^d, a polydisk Δ, nonsingular holomorphic maps $\alpha = (f_1, \cdots, f_d): \Delta_m \to C^d$, $\alpha(0) = 0$, and $\beta = (g_1, \cdots, g_d): \Delta_n \to C^d$, $\beta(0) = 0$, such that

$$M \cap U = \alpha(\Delta_m), \qquad N \cap U' = \beta(\Delta_n).$$

Let $J_\alpha(0)$ and $J_\beta(0)$ denote the Jacobian matrices at 0 of α and β, respectively. The column vectors of the matrix $J_\alpha(0)$ constitute a basis of the tangent space $T_0(M)$; the situation is the same for $J_\beta(0)$ and $T_0(N)$. Hence, $\dim T_0(M, N)$ is equal to the rank of the matrix consisting of the columns of both $J_\alpha(0)$ and $J_\beta(0)$. From this follows that $\dim T_0(M, N)$ is invariant under any nonsingular change of local coordinates at 0 in C^d.

Now, we shall see that there exist complex submanifolds M', N' of C^d, a neighborhood U'' of 0 in C^d, a polydisk Δ'' and a nonsingular holomorphic map $\gamma: \Delta''_n \to C^d$ with $\gamma(0) = 0$ such that

$$M' = \Delta''_m \times \{0\} \subset C^d, \qquad N' \cap U'' = \gamma(\Delta''_n),$$

and that we can find neighborhoods W, W' of 0 and biholomorphic map λ of W onto W' for which we have $\lambda(X \cap W) = X' \cap W'$ where $X' = M' \cup N'$. In fact, let $\tilde{\alpha}, \tilde{\beta}$ be nonsingular holomorphic maps: $\Delta \to C^d$ such that

$$\tilde{\alpha} | \Delta_m \times \{0\} = \alpha, \qquad \tilde{\beta} | \Delta_n \times \{0\} = \beta.$$

We may assume that $\tilde{\alpha}, \tilde{\beta}$ are biholomorphic maps of Δ onto U, U', respectively. Let Δ' be a polydisk such that $\tilde{\alpha}(\Delta') \subset U \cap U'$; let $U_0 = \tilde{\alpha}(\Delta')$. Let $N' = \tilde{\alpha}^{-1}(N \cap U_0)$. We define $U'' = \tilde{\alpha}^{-1}\tilde{\beta}(\Delta'')$ for a polydisk $\Delta'' \subset \tilde{\beta}^{-1}(U_0)$. We have then $N' \cap U'' = \tilde{\alpha}^{-1}\beta(\Delta''_n)$. Let $M' = \Delta''_m \times \{0\} \subset C^d$, $X' = M' \cup N'$. Then, we have $\tilde{\alpha}^{-1}(X \cap U_0) = X'$. Therefore, it suffices to put $W = U_0$, $W' = \Delta'$ and $\lambda = \tilde{\alpha}^{-1}$ on U_0.

Consequently, we have only to prove that X' is quasi-normal at 0 under the assumption that $\dim T_0(M', N') = \min(m + n, d)$. Let $\gamma = (f_1, \cdots, f_d)$. We define matrices J and J' by

$$J = \left(\frac{\partial f_i}{\partial z_j}(0)\right), \qquad i = 1, \cdots, m;\ j = 1, \cdots, n,$$

$$J' = \left(\frac{\partial f_{m+i}}{\partial z_j}(0)\right), \qquad i = 1, \cdots, r;\ j = 1, \cdots, n,$$

where $r = d - m$. Let A be the following $d \times (m + n)$ matrix where I denotes the $m \times m$ identity matrix:

$$A = \begin{pmatrix} I & J \\ 0 & J' \end{pmatrix}.$$

The condition that $\dim T_0(M', N') = \min(m + n, d)$ is equivalent to the condition that A has the maximal rank, which is easily seen to be equivalent to maximality of the rank of J'; this means that the map $(f_{m+1}, \cdots, f_d)$ is nonsingular. Thus, as in the proof of Corollary 4 in [2], we have $\mathrm{id}\, \mathbf{V}(\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d) = (\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d)$. This completes the proof.

Since the points where a space is quasi-normal constitute an open subset, we have the following

COROLLARY. *Let M, N be submanifolds satisfying the condition of Theorem* 1. *Then there exists a polydisk Δ such that the proposition* $(*)$ *holds.*

THEOREM 2. *Let* $\dim_0(M \cap N) = 0$. *Then X is quasi-normal at* 0 *if and only if* $T_0(M) \cap T_0(N) = (0)$.

Proof. Let X be quasi-normal at 0; let $X' = M' \cup N'$ and $N' \cap U'' = \gamma(\Delta''_n)$ where $\gamma = (f_1, \cdots, f_d)$ as in the proof of Theorem 1. The assumption implies that $\dim_0 \mathbf{V}(\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d) = 0$, so $(\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d)$ is the maximal ideal of ${}_n\mathcal{O}_0$ and $n \leqq d - m$. We have

$$\mathbf{z}_i \in (\mathbf{f}_{m+1}, \cdots, \mathbf{f}_d)\ , \qquad i = 1, \cdots, n\ ,$$

so that rank $J' = n$ as in the proof of Proposition 5 in [2]. The result follows from $\dim T_0(M, N) = m + n$. This completes the proof.

The converse of Theorem 1 does not hold in general as is seen from the example after Corollary 4 in [2].

The variety X considered in Theorem 1 has the property that $M \cap N$ is a submanifold of a neighborhood of 0. But, we cannot expect anything significant concerning the relation between the variety $M \cap N$ and quasi-normality of $M \cup N$.

THEOREM 3. *Let V be an analytic subvariety of a neighborhood of* 0 *in C^n, and let $0 \in V$. Then there exist submanifolds M, N, N' of C^d, $n < d$, such that $0 \in M \cap N \cap N'$, $M \cap N = M \cap N' = V \times \{0\} \subset C^d$; and $M \cup N$ is quasi-normal, yet $M \cup N'$ is not quasi-normal, at* 0.

Proof. Take a neighborhood U of 0 and holomorphic functions $f_1, \cdots, f_r$ on U such that $V \cap U = \{z' \in U \mid f_j(z') = 0,\ j = 1, \cdots, r\}$. First, let $\mathrm{id}\,\mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_r) \neq (\mathbf{f}_1, \cdots, \mathbf{f}_r)$. Let $\mathrm{id}\,\mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_r) = (\mathbf{g}_1, \cdots, \mathbf{g}_s)$ for $\mathbf{g}_i \in {}_n\mathcal{O}_0$, $i = 1, \cdots, s$. Then we have

$$\mathbf{V}(\mathbf{g}_1, \cdots, \mathbf{g}_s) = \mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_r)\ , \qquad \mathrm{id}\,\mathbf{V}(\mathbf{g}_1, \cdots, \mathbf{g}_s) = (\mathbf{g}_1, \cdots, \mathbf{g}_s)\ .$$

Let $\Delta_n \subset U$ be a suitable polydisk such that $V \cap \Delta_n = \{z' \in \Delta_n \mid g_j(z') = 0,\ j = 1, \cdots, s\}$ where g_j are holomorphic functions on Δ_n which are representatives of germs $\mathbf{g}_j$. We define submanifolds M, N, N' as follows:

$$\begin{aligned} M &= \Delta_n \times \{0\} \subset C^d, \text{ where } d = n + r + s\ , \\ N &= \{(z', g_1(z'), \cdots, g_s(z'), 0, \cdots, 0) \in C^d \mid z' \in \Delta_n\}\ , \\ N' &= \{(z', f_1(z'), \cdots, f_r(z'), 0, \cdots, 0) \in C^d \mid z' \in \Delta_n\}\ . \end{aligned}$$

We have $M \cap N = M \cap N' = (V \cap \Delta_n) \times \{0\} \subset C^d$; $M \cup N$ is quasi-normal but $M \cup N'$ is not quasi-normal, at 0.

Next, let $\mathrm{id}\,\mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_r) = (\mathbf{f}_1, \cdots, \mathbf{f}_r)$. It suffices to find germs $\mathbf{f}'_1, \cdots, \mathbf{f}'_t \in {}_n\mathcal{O}_0$ such that $\mathbf{V}(\mathbf{f}'_1, \cdots, \mathbf{f}'_t) = \mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_r)$, $\mathrm{id}\,\mathbf{V}(\mathbf{f}'_1, \cdots, \mathbf{f}'_t) \neq (\mathbf{f}'_1, \cdots, \mathbf{f}'_t)$. If $(\mathbf{f}_1, \cdots, \mathbf{f}_r) = (\mathbf{f})$, $\mathbf{f} \neq \mathbf{0}$, we have only to take $\mathbf{f}^2$. If $(\mathbf{f}_1, \cdots, \mathbf{f}_r)$ is not a principal ideal, we may assume that there is an integer t, $2 \leqq t \leqq r$, such that

$$\mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_t) = \mathbf{V}(\mathbf{f}_1, \cdots, \mathbf{f}_r)\ , \qquad \mathbf{f}_1 \notin (\mathbf{f}_2, \cdots, \mathbf{f}_t)\ .$$

It follows that

$$\mathbf{V}(\mathbf{f}_1^2, \mathbf{f}_2, \cdots, \mathbf{f}_t) = \mathbf{V}(\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_r) , \qquad \mathrm{id}\,\mathbf{V}(\mathbf{f}_1^2, \mathbf{f}_2, \cdots, \mathbf{f}_t) \neq (\mathbf{f}_1^2, \mathbf{f}_2, \cdots, \mathbf{f}_t) ,$$

since $\mathbf{f}_1 \notin (\mathbf{f}_1^2, \mathbf{f}_2, \cdots, \mathbf{f}_t)$. The proof is completed.

REFERENCES

[1] N. Mochizuki, Quasi-normal analytic spaces, Proc. Japan Acad., **48** (1972), 181–185.
[2] ——, Quasi-normal analytic spaces. II, Kōdai Math. Sem. Rep., **25** (1973), 362–366.

College of General Education
Tôhoku University