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# COHOMOLOGICAL DIMENSION OF GROUP SCHEMES

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In Umemura [9], we calculated the invariants algcd$(G)$, $p(G)$, $q(G)$ for a commutative algebraic group $G$. We remark that all the results hold for a group scheme which is not necessarily commutative.

To determine $p(G)$, I cannot succeed in dropping the hypothesis "quasi-projective" but this assumption is satisfied in the characteristic 0 case.

## 1. Notation and definition

(1.1) All schemes are connected and of finite type over a fixed field $k$ which we assume to be algebraically closed. Let $X$ be a scheme. The algebraic cohomological dimension of $X$ denoted by algcd$(X)$ is, by definition, $\min\{n \in N \mid H^j(X, F) = 0$ for all $j > n$ and all coherent sheaves $F$ on $X\}$. We need two more invariants $p(X)$ and $q(X)$ defined by the following equations:

$$p(X) = \max\{n \in N \cup \{\infty\} \mid H^i(X, F) \text{ is a finite dimensional } k\text{-vector space for all } i < n \text{ and all locally free sheaves } F \text{ on } X\}\,.$$

$$q(X) = \min\{n \in N \cup \{-1\} \mid H^i(X, F) \text{ is a finite dimensional } k\text{-vector space for all } i > n \text{ and for all coherent sheaves } F \text{ on } X\}\,.$$

Let $Y$ be a complex analytic space then the analytic cohomological dimension of $Y$ denoted by ancd$(Y)$ is by definition $\min\{n \in N \mid H^i(Y, F) = 0$ for all $i > n$ and all coherent sheaves $F$ on $Y\}$.

(1.2) *Remark* 1. Since a quasi-coherent sheaf is a direct limit of coherent sheaves and the functor $H^i(X,\ )$ commutes with direct limits, algcd$(X) = \min\{n \in N \mid H^i(X, F) = 0$ for all $i > n$ and all quasi-coherent sheaves $F$ on $X\}$.

*Remark* 2. Let $F$ be a coherent sheaf on $X$, then $F$ has a filtration

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such that each of the quotients is a coherent sheaf on $X_{\mathrm{red}}$. Conversely a coherent sheaf on $X_{\mathrm{red}}$ is naturally a coherent sheaf on $X$. Hence $\operatorname{algcd}(X) = \operatorname{algcd}(X_{\mathrm{red}})$ and $q(X) = q(X_{\mathrm{red}})$.

## 2. Algebraic cohomological dimension

(2.1) THEOREM 1. *Let $G$ be a group scheme. Then we have:*

$$\operatorname{algcd}(G) = \max\{\dim A \mid A \text{ is an abelian variety such that there exists a surjective homomorphism of group schemes } G_{\mathrm{red}} \to A\}$$

$$p(G) = \begin{cases} 0 & \text{if } G \text{ is quasi-projective and not complete} \\ \infty & \text{if } G \text{ is complete} \end{cases}$$

$$q(G) = \begin{cases} \operatorname{algcd}(G) & \text{if } G \text{ is not complete} \\ -1 & \text{if } G \text{ is complete}\,. \end{cases}$$

*Proof.* We proved this theorem for commutative algebraic groups in Umemura [9]. In view of Remark 2, to prove the assertions concerning $\operatorname{algcd}(G)$ and $q(G)$, we may assume that $G$ is reduced. If $G$ is complete, $H^i(G, F)$ is finite dimensional for all $i$ and all coherent sheaves and by Lichtenbaum's theorem (Hartshorne [4]) we have $\operatorname{algcd}(G) = \dim G$ and $q(G) = -1$. We may also assume $G$ is not complete. First we prove the assertions on $\operatorname{algcd}(G)$ and $q(G)$ under the hypothesis that $G$ is reduced and not complete. Then by Chevalley's theorem we have an exact sequence

$$\text{(a)} \qquad 1 \longrightarrow B \longrightarrow G \xrightarrow{\pi} A \longrightarrow 1\,.$$

where $B$ is an affine group scheme and $A$ is an abelian variety. Since the morphism $\pi$ is affine, we have $H^i(G, F) = H^i(A, \pi_* F)$ for a coherent sheaf $F$ on $G$. Since $\pi_* F$ is quasi-coherent, we have $\operatorname{algcd}(G) \leq \dim A$. In general we have $q(G) \leq \operatorname{algcd}(G)$ from the definition. It is sufficient to show that $q(G) = \operatorname{algcd}(G) = \dim A$. Let $n$ be the dimension of $A$. We have to prove that there exists a coherent sheaf $F$ on $G$ such that $H^n(G, F)$ is an infinite dimensional $k$-vector space. We need

THEOREM (Rosenlicht [8] p. 432). *Let $C$ be the center of $G$. Then $G/C$ is a linear algebraic group.*

COROLLARY. *The restriction of $\pi$ to $C$ is surjective.*

*Proof of the corollary.* By the above Theorem $G/C$ is linear. $A/\pi(C)$ is an abelian variety. Hence the surjective homomorphism $G/C \to A/\pi(C)$ is trivial and we have $A = \pi(C)$.

If $C$ is not complete, by Umemura [9] 2.7 Corollaire 1, there exists a coherent sheaf $F$ on $C$ and an integer $m \geq n$ such that $H^m(C, F)$ is infinite dimensional. $F$ can be regarded as a coherent sheaf on $G$ and we have $H^m(G, F) = H^m(C, F)$. As we have seen above algcd $(G) \leq n$. We conclude that $m = n$. Hence the coherent sheaf $F$ on $G$ has the required property.

If $C$ is complete, then by Rosenlicht's theorem above, $G/C$ is a linear algebraic group of positive dimension since we assume $G$ is not complete.

(b) $$1 \longrightarrow C \longrightarrow G \xrightarrow{\varphi} G/C \longrightarrow 1.$$

Since $\varphi$ is flat, by base change theorem, $R^q\varphi_* O_G$ is a locally free sheaf on $G/C$ of rank $\binom{\dim C}{q}$ (see Mumford [6] p. 50 Corollary 2 and p. 129 Corollary 2). Since $G/C$ is affine, we have $H^0(G/C, R^q\varphi_* O_G) \simeq H^q(G, O_G)$ by E. G. A. III (1.4.11). Let $m$ be the dimension of $C$. Then $R^m\varphi_* O_G$ is locally free sheaf of rank 1 and $H^0(G/C, R^m\varphi_* O_G)$ is infinite dimensional since $G/C$ is affine and of positive dimension. Hence $H^m(G, O_G)$ is an infinite dimensional $k$-vector space. It is sufficient to show that $m = \dim C = \dim A$. In fact the restriction of $\pi$ to $C$ is an isogeny of abelian varieties $C$ and $A$. The restriction of $\pi$ to $C$ is surjective by the Corollary above and its kernel $C \cap B$ is finite.

Now we calculate $p(G)$. If $G$ is complete, the assertion is well known. So we may assume $G$ is not complete but quasi-projective. Since $G_{\text{red}}$ is not complete, $G_{\text{red}}$ contains an affine closed subgroup $H$ of positive dimension by Chevalley's theorem. Let $L$ be an ample line bundle on $G$. We denote by $I$ the ideal sheaf of $H$ in $G$. So we have an exact sequence:

(c) $$0 \longrightarrow I \longrightarrow O_G \longrightarrow O_H \longrightarrow 0\ .$$

We have $H^1(G, I \otimes L^{\otimes \ell}) = 0$ for a sufficiently large integer $\ell$ since $L$ is ample. We fix such an integer $\ell$. Tensoring $L^{\otimes \ell}$ with (c), we have

$$0 \longrightarrow I \otimes L^{\otimes \ell} \longrightarrow L^{\otimes \ell} \longrightarrow O_H \otimes L^{\otimes \ell} \longrightarrow 0\ .$$

The exact sequence of cohomology is

(d) $$H^0(L^{\otimes \ell}) \longrightarrow H^0(O_H \otimes L^{\otimes \ell}) \longrightarrow H^1(I \otimes L^{\ell}) = 0\ .$$

Since $H$ is affine and of positive dimension and since $O_H \otimes L^{\otimes \ell}$ is a line bundle, $H^0(O_H \otimes L^{\otimes \ell})$ is infinite dimensional. By the exact sequence (d), $H^0(G, L^{\otimes \ell})$ is infinite dimensional. Hence $p(G) = 0$. This completes the proof of the Theorem.

(2.2) *Remark.* I don't know if every group scheme over an algebraically closed field $k$ is quasi-projective. If $G$ is reduced, then $G$ is quasi-projective( Chow [2]). If the characteristic of $k$ is 0, a group scheme is reduced (Oort [7]). Hence a group scheme is quasi-projective in characteristic 0.

### 3. Analytic cohomological dimension

(3.1) We need Matsushima's results (Matsushima [5]).

THEOREM A. *Let $G$ be a complex Lie group and $N$ a normal subgroup of $G$. We suppose the quotient group $G/N$ is a complex torus $T$. Let $\varphi: N \to GL(m, \boldsymbol{C})$ be a linear representation of $N$. Then the principal $GL(m, \boldsymbol{C})$-bundle on $T$ associated to this representation has a holomorphic connection.*

THEOREM B. *An indecomposable principal $GL(m, \boldsymbol{C})$-bundle $P$ over a complex torus with a holomorphic connection can be written in the form;*

$$P = P_1 \otimes P_2$$

*where the transition matrices of $P_1$ are upper triangular matrices whose diagonal components are 1 and $P_2$ is a principal $\boldsymbol{C}^*$-bundle with trivial Chern class.*

COROLLARY. *A principal $GL(m, \boldsymbol{C})$-bundle over a complex torus $T$ with a holomorphic connection is $C^\infty$-trivial.*

*Proof of Corollary.* We may assume that $P$ is indecomposable. Then $P$ is isomorphic to $P_1 \otimes P_2$ by Theorem B. It is easy to see that $P_1$ and $P_2$ are $C^\infty$-trivial.

(3.2) THEOREM 2. *Let $G$ be a group scheme defined over $\boldsymbol{C}$. Then* $\text{algcd}\,(G) \geq \text{ancd}\,(G^{an})$.

*Proof.* By (2.2) $G$ is reduced. By Chevalley's theorem, we have an exact sequence (a). $B$ is a closed sub-group scheme of $GL(m, \boldsymbol{C})$ for a certain number $m$. Hence we can associate to this representation the principal $GL(m,\boldsymbol{C})$-bundle $P_G$ over $A$. By Theorem A, $P_G$ has a holomorphic connection. Hence by the Corollary to Theorem B. $P_G$ is $C^\infty$-trivial. On $A \times GL(m,\boldsymbol{C})$, we put

$$f(z, x_{11}, \cdots, x_{ij}, \cdots, x_{mm}) = \sum_{1\le i,j \leqslant n} |x_{ij}|^2 + \left|\det\begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mm}\end{bmatrix}\right|^2$$

where

$$\left(z, \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mm}\end{bmatrix}\right) \in A \times GL(m,\boldsymbol{C}) .$$

Let $g$ be a $\boldsymbol{C}^\infty$-isomorphism from the principal $GL(m,\boldsymbol{C})$-bundle $P_G$ to $A \times GL(m,\boldsymbol{C})$. Let $F$ be the composition $f \circ g$. Then it is easy to see that the closed analytic sub-set $G^{an}$ of $P_G$ is $\dim A + 1$-complete by considering the restriction of $f \circ g$ to $G^{an}$ (cf. Umemura [9]). Hence by a theorem of Andreotti and Grauert [1] p. 250, we have $\operatorname{ancd}(G^{an}) \le \dim A$. On the other hand $\operatorname{algcd} G = \dim A$ by Theorem 1. q.e.d.

(3.3) APPLICATION. Hartshorne's conjecture is true for group schemes. (cf. Hartshorne [4], p. 230 and Umemura [9]).

COROLLARY TO THEOREM 1 AND THEOREM 2 (Hartshorne's conjecture). *Let $G$ be a group scheme over $\boldsymbol{C}$. Consider the natural maps*

$$\alpha_i : H^i(G, F) \longrightarrow H^i(G^{an}, F^{an})$$

*for any coherent sheaf $F$ on $G$.*

(1) *$\alpha_i$ is an isomorphism for all $i < p(G)$.*

(2) *$\alpha_i$ is an isomorphism for all $i > q(G)$.*

(3) *$F \mapsto F^{an}$ is an equivalence of the category of coherent algebraic sheaves on $G$ and the category of coherent analytic sheaves on $G^{an}$ if $p(G) \ge 1$.*

*Proof.* If $G$ is complete, we have nothing to prove. If $G$ is not complete, $p(G) = 0$ by Theorem 1. Hence (1) and (3) are trivial. $q(G) = \operatorname{algcd}(G)$ by Theorem 1, and $\operatorname{algcd}(G) \ge \operatorname{ancd}(G^{an})$ by Theorem 2. Hence (2) follows.

(3.4) *Remark.* In [9], we show that, for any integer $n \ge 0$, there exists an algebraic variety (indeed, a commutative algebraic group) $G$

defined over $C$ such that algcd $(G) = n$ and ancd $(G^{an}) = 0$. By considering the product with a complete variety, for any pair of integers $n \geq m \geq 0$, there exists an algebraic variety $G$ such that algcd $(G) = n$ and ancd $(G^{an}) = m$.

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