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A HETEROGENEOUS INTERPOLANT

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In this note we exhibit an interpolant for a certain valid implication $\models \varphi \rightarrow \psi$, where φ and ψ come from the infinitary language $L_{\omega_1\omega_1}$. The existence of this interpolant follows from Takeuti's heterogeneous interpolation theorem [5], but unfortunately the proof in [5] is not explicit enough to allow one to find the interpolant explicitly. Takeuti's theorem asserts the existence of an interpolant in the class $\bar{L}_{\omega_1\omega_1}$ of *heterogeneous formulas*, which admits the rules of formation of $L_{\omega_1\omega_1}$ plus the following additional rule: if $\varphi \in \bar{L}_{\omega_1\omega_1}$ and $\langle Q_\alpha \rangle_{\alpha<\beta}$ is a sequence of quantifiers (i.e. $Q_\alpha = \exists$ or $Q_\alpha = \forall$) then $Q_0 x_{i_0} \cdots Q_\alpha x_{i_\alpha} \cdots {}_{(\alpha<\beta)}\varphi \in \bar{L}_{\omega_1\omega_1}$. (The semantic interpretation is the obvious one; consult [2, § C] or [4].)

The present interpolation example (whose investigation was suggested by J. Malitz) will be presented *as a definability theorem*. Namely we give a formula explicitly defining an isomorphism between two isomorphic well-founded extensional relations.

We take unary predicates A_1, A_2 and binary predicates E_1, E_2, F (written medially). Let σ be the conjunction of the following sentences.

$$\forall x[(A_1x \vee A_2x) \wedge \neg(A_1x \wedge A_2x)]$$
$$\forall x\forall y[xE_iy \rightarrow A_ix \wedge A_iy] \qquad (i = 1, 2)$$
$$\forall x\forall y[A_ix \wedge A_iy \rightarrow [x \simeq y \leftrightarrow \forall z(zE_ix \leftrightarrow zE_iy)]] \qquad (i = 1, 2)$$
$$\forall x_0\forall x_1 \cdots [\neg \bigwedge_{j\in\omega} x_{j+1}E_ix_j] \qquad (i = 1, 2)$$
$$\forall x\forall y[xFy \rightarrow A_1x \wedge A_2y]$$
$$\forall x[A_1x \rightarrow \exists!\, y(xFy)]$$
$$\forall y[A_2y \rightarrow \exists!\, x(xFy)]$$
$$\forall u\forall v\forall x\forall y[xFy \wedge uFv \rightarrow (uE_1x \leftrightarrow vE_2y)]\ .$$

One may easily check that if

$$\langle U\,; A_1, A_2, E_1, E_2, F\rangle \models \sigma$$

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and

$$\langle U ; A_1, A_2, E_1, E_2, F' \rangle \models \sigma ,$$

then $F = F'$. Thus F is implicitly defined by σ, and hence by Takeuti's theorem together with the usual argument for Beth's theorem, there is a (heterogeneous) formula $\Phi(x, y)$ such that $\sigma \models xFy \leftrightarrow \Phi(x, y)$, with F not appearing in Φ. (Such Φ cannot be in any $L_{\kappa\lambda}$, as follows from the proof of Malitz [3, Theorem 4.2].) The aim of this note is to explicitly exhibit Φ.

Let C be the set of finite sequences of 0's and 1's (including the empty sequence $\Box$). For $\Box \neq \sigma = a_0 a_1 \cdots a_{n-1} a_n \in C$, we let $\hat{\sigma} = a_0 \cdots a_{n-1}$. For all $\sigma \in C$ we take variables x_σ and y_σ. For all $\sigma \in C$, let Q_σ stand for[1]

$$(\forall x_{\sigma 0} \in A_1)(\exists y_{\sigma 0} \in A_2)(\forall y_{\sigma 1} \in A_2)(\exists x_{\sigma 1} \in A_1) .$$

Now let $\Phi(x, y)$ be

$$(A_1 x) \wedge (A_2 y) \wedge Q_\Box \cdots \underset{\substack{\sigma, \tau \in C \\ |\sigma| \leq |\tau|}}{Q_\sigma} \cdots Q_\tau \cdots \bigwedge_{\substack{\sigma \in C \\ \sigma \neq \Box}} (x_\sigma E_1 x_{\hat{\sigma}} \leftrightarrow y_\sigma E_2 y_{\hat{\sigma}}) ,$$

where $|\sigma|$ denotes the length of σ. We claim that $\sigma \models (\Phi(x, y) \leftrightarrow xFy)$. To see this, let $\mathfrak{A} = \langle U ; A_1, A_2, E_1, E_2, F \rangle$ be any model of σ. We then need to see that for $a, b \in U, \mathfrak{A} \models \Phi \; [a, b]$ if and only if $(a, b) \in F$. Certainly if $(a, b) \in F$, then we may obviously use the isomorphism F to continue to establish the satisfaction of Φ in $\mathfrak{A}$. Conversely suppose that for some a and b, $a \in A_1$ and $b \in A_2$,

$$(*) \qquad \mathfrak{A} \models \Phi[a, b] \qquad \text{but } (a, b) \notin F .$$

We let a be E_1-minimal among those a for which such b exists, and let b be E_2-minimal such that (*) holds for b and this value of a. Notice that $\Phi(x, y)$ is logically equivalent to

$$\forall x_0 \exists y_0 [(x_0 E_1 x \leftrightarrow y_0 E_2 y) \wedge \Phi(x_0, y_0)] \wedge$$
$$\forall y_1 \exists x_1 [(x_1 E_1 x \leftrightarrow y_1 E_2 y) \wedge \Phi(x_1, y_1)] .$$

Thus we know that

$$\mathfrak{A} \models \begin{bmatrix} \forall x_0 \exists y_0 [(x_0 E_1 a \leftrightarrow y_0 E_2 b) \wedge \Phi(x_0, y_0)] \wedge \\ \forall y_1 \exists x_1 [(x_1 E_1 a \leftrightarrow y_1 E_2 b) \wedge \Phi(x_1, y_1)] \end{bmatrix}$$

[1] We express the quantifiers Q_σ with "$\in$" for cognitive purposes. Clearly it is possible to write Φ in strict accordance with the formation rule mentioned above.

By the minimality of a and b, we know that therefore

$$\mathfrak{A} \models \begin{bmatrix} \forall x_0 \exists y_0 [(x_0 E_1 a \leftrightarrow y_0 E_2 b) \wedge x_0 F y_0] \wedge \\ \forall y_1 \exists x_1 [(x_1 E_1 a \leftrightarrow y_1 E_2 b) \wedge x_1 F y_1] \end{bmatrix}.$$

But since F is an isomorphism and since E_1 and E_2 are each extensional, we see that $(a, b) \in F$.

References

[1] Barwise, J., ed., The syntax and semantics of infinitary languages, Lecture notes in mathematics **#72**, Springer-Verlag, Berlin, 1968.

[2] Henkin, L., Some remarks on infinitely long formulas, pp. 167–183 in: Infinitistic methods (Proceedings of the symposium on foundations of mathematics, Warsaw, September, 1959), Pergamon, Państwowe Wydawnictwo Naukowe, Warsaw, 1961.

[3] Malitz, J., Infinitary analogs of theorems from first order model theory, J. Symbolic Logic **36** (1971), 216–228.

[4] Mycielski, J., On the axiom of determinateness, Fund. Math., **53** (1964), 205–224.

[5] Takeuti, G., A determinate logic, Nagoya Math. J. **38** (1970), 113–138. (Essentially the same article appears as pp. 237–264 in [1].)

Added July 18, 1973 The following articles give further information on Takeuti's (and other) interpolation theorems:

[6] Kueker, D. W., Löwenheim-Skolem and interpolation theorems in infinitary languages, Bull. Amer. Math. Soc. **78** (1972), 211–215.

[7] Nebres, B. F., Herbrand uniformity theorems for infinitary languages, J. Math. Soc. Japan **24** (1972), 1–19.

[8] Swett, A. K., Interpolation theorems for languages with game quantifiers, ms., Toronto, 1973.

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