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HOLOMORPHIC MAPPINGS INTO PROJECTIVE SPACE WITH LACUNARY HYPERPLANES

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1. Introduction

In this note, we shall examine some results of Bloch [2] and Cartan [3] concerning complex projective space minus hyperplanes in general position. The purpose is to restate their results in a more general setting by using the intrinsic pseudo-distance defined on a complex space [16] and the concept of tautness introduced by Wu in [18]. In the process we shall generalize some results of Dufresnoy [4] and Fujimoto [7].

Before investigating projective space minus hyperplanes in general position, we consider in § 2 a more general situation.

2. Hyperbolically or tautly imbedded complex spaces

Throughout this section, let Y be a complex space, M a relatively compact open subset of Y and Δ a closed complex subspace of Y. (The example we have in mind is the one where Y is $P_n(C)$, M is the complement of $n+2$ hyperplanes in general position and Δ is the union of a certain set of hyperplanes to be defined in § 3).

We denote the open unit disk in C by D, the polydisk $D \times \cdots \times D$ in C^k by D^k, the disk of radius r by D_r and the intrinsic pseude-distance of M by d_M (see [16] for its definition and basic properties). The space of holomorphic maps from N to M with the compact-open topology will be denoted by $\mathrm{Hol}(N, M)$.

When Δ is empty, most of the following definitions reduce to familiar ones which have been studied extensively in [1], [5], [12], [13], [14], [16] and [18]. The motivation for these modified definitions will become apparent in § 3 when we consider the theorems of Bloch and Cartan.

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DEFINITION 1. M is *hyperbolic modulo* Δ if for every pair of distinct points p, q of M not both contained in Δ, we have $d_M(p, q) > 0$.

DEFINITION 2. M is *hyperbolically imbedded modulo* Δ *in* Y if, for every pair of distinct points p, q of Y not both contained in Δ, there exist neighborhoods V_p and V_q of p and q in Y such that $d_M(V_p \cap M, V_q \cap M) > 0$.

DEFINITION 3. M is *complete hyperbolic modulo* Δ if it is hyperbolic modulo Δ and if for each sequence $\{p_n\}$ in M which is Cauchy with respect to the pseudo-distance d_M, we have one of the following:

(a) $\{p_n\}$ converges to a point $p \in M$;

(b) for every open neighborhood U of Δ in Y, there exists an integer n_0 such that $p_n \in U$ for $n \geq n_0$.

DEFINITION 4. M is *taut modulo* Δ (resp. *tautly imbedded modulo* Δ *in* Y) if for each positive integer k and each sequence $\{f_n\}$ in $\mathrm{Hol}(D^k, M)$, we have one of the following:

(a) $\{f_n\}$ has a subsequence which converges in $\mathrm{Hol}(D^k, M)$ (resp. in $\mathrm{Hol}(D^k, Y)$);

(b) for each compact set $K \subset D^k$ and each compact set $L \subset M - \Delta$ (resp. $L \subset Y - \Delta$), there exists an integer n_0 such that $f_n(K) \cap L = \phi$ for $n \geq n_0$.

DEFINITION 5. M is *s-taut modulo* Δ (respectively *s-tautly imbedded modulo* Δ *in* Y) if for each compact set $K \subset D$ and each compact set $L \subset M - \Delta$ (resp. $L \subset Y - \Delta$), there exist compact subsets $L_1, \cdots, L_m$ of L and taut open subset $U_1, \cdots, U_m$ of M (resp. Y) such that:

(a) $L = \bigcup_{j=1}^m L_j$ and $L_j \subset U_j$

(b) if $f: D \to M$ is holomorphic and $f(0) \in L_j$, then $f(K) \subset U_j$.

(The s stands for strongly).

DEFINITION 6. M is *locally taut* (resp. *locally complete*) if for each point $p \in Y$ there exists a neighborhood V_p of p such that $V_p \cap M$ is taut (resp. complete hyperbolic). (Note that the condition is trivially satisfied for all $p \notin \partial M$.)

DEFINITION 7. Let Z be a closed analytic subvariety of Y. If Z is defined locally as the zeros of a single nonconstant holomorphic function, then we call Z a *principal hypersurface* of Y. In particular, if Y is nonsingular, then every hypersurface is principal.

We shall first relate definitions 6 and 7 to each other.

PROPOSITION 1. *If M is locally complete, then M is locally taut. Let $Z = Y - M$. If Z is a principal hypersurface, then M is locally complete. In particular, if $Z = Y - M$ has pure codimension 1 and Y is a compact complex manifold, then M is locally complete.*

Proof. The first statement follows immediately from the fact that a complete hyperbolic space is always taut. This was proved in [12] for complex manifolds and the same proof is valid for complex spaces. However, it does depend on Barth's result in [1] that the intrinsic distance d_X on a hyperbolic complex space X induces the natural topology on X.

To prove the second statement, it suffices to consider an arbitrary point $p \in Z$. Let V_p be a neighborhood of p in Y which can be written as a closed analytic subspace of a ball B in some C^n. Since V_p is a closed analytic subspace of the complete hyperbolic space B, it is complete hyperbolic. Taking V_p small, we may assume that $Z \cap V_p$ is defined as the zeros of a bounded holomorphic function $f: V_p \to D$. This implies that $V_p \cap M$ is complete hyperbolic by Theorem 4.10, page 60 of [16].

The last statement follows from the fact that a subvariety of pure codimension 1 in a complex manifold is always a principal hypersurface.

Q.E.D.

THEOREM 1. *Let Y, M and Δ be as above. The concepts defined above are related by the following table:*

$$\begin{tikzcd}
s\text{-taut imbd mod } \Delta \arrow[r] \arrow[d, dashed] & \text{taut imbd mod } \Delta \arrow[r] \arrow[d, dashed] & \text{hyp imbd mod } \Delta \arrow[d] \\
s\text{-taut mod } \Delta \arrow[r] & \text{taut mod } \Delta \arrow[r] & \text{hyp mod } \Delta
\end{tikzcd}$$

where a dotted arrow means "... implies ... if M is locally taut in Y."

Proof. (1) s-taut imbd mod $\Delta \to$ taut imbd mod Δ. Let $\{f_n\}$ be a sequence in $\mathrm{Hol}(D^k, M)$. Assume that there exist compact sets $K \subset D^k$ and $L \subset Y - \Delta$ such that $f_n(K) \cap L \neq \phi$ for arbitrarily large n. By taking a subsequence if necessary, we may assume that $f_n(K) \cap L \neq \phi$ for all n. Since D^k is homogeneous, there exists an automorphism g_n of D^k for each n such that $g_n(0) \in K$ and $f_n \circ g_n(0) \in L$, where 0 is the origin in D^k. By taking a subsequence if necessary, we may assume that the sequence

$\{g_n\}$ converges to a mapping $g \in \mathrm{Hol}(D^k, D^k)$. If we prove that a subsequence of $\{f_n \circ g_n\}$ converges to $h \in \mathrm{Hol}(D^k, Y)$, it will follow that the corresponding subsequence of $\{f_n\}$ converges to $f = h \circ g^{-1}$. Thus we can assume that $f_n(0) \in L_j$ for some j. It suffices to show that $\{f_n\}$ has a subsequence which converges uniformly on any fixed polycylinder $\bar{D}_r^k = \{(z_1, \cdots, z_k); |z_i| \leqq r\}$, where $0 < r < 1$. But the hypothesis implies that there exists a taut open subset U_j of Y which depends only on $\bar{D}_r^k$ and L_j such that $f_n(\bar{D}_r^k) \subset U_j$ for all n. Since U_j is taut and $f_n(0)$ is in the fixed compact set L_j for all n, the result follows immediately.

(2) s-taut mod $\Delta \rightarrow$ taut mod Δ. The proof is almost identical to that of (1) and hence is omitted.

(3) taut imbd mod $\Delta \rightarrow$ hyp imbd mod Δ. In [12] it was shown that "taut" implies "hyperbolic" and the same proof works here. But for the sake of completeness, we shall give a proof. Let p, q be two distinct points of Y such that $p \notin \Delta$. Let U_p and V_q be neighborhoods of p and q in Y such that $\bar{U}_p \cap \bar{V}_q = \phi$ and $\bar{U}_p \cap \Delta = \phi$. Taking U_p small, we may assume that U_p is hyperbolic. Let W_p be a smaller neighborhood of p such that $\bar{W}_p$ is compact and is contained in U_p. We shall first prove that there is a positive number $r < 1$ such that if $f: D \rightarrow M$ is holomorphic and $f(0) \in \bar{W}_p$, then $f(D_r) \subset U_p$, where D_r denotes the disk of radius r. Assume the contrary. Then for each positive integer n, there exists a map $f_n \in \mathrm{Hol}(D, M)$ such that $f_n(0) \in \bar{W}_p$ and $f_n(D_{1/n}) \not\subset U_p$. Then the sequence $\{f_n\}$ has no subsequence which converges to g in $\mathrm{Hol}(D, Y)$. (For g would have the property that $g(0) \in \bar{W}_p$ on one hand and $g(0) \notin U_p$ on the other hand.) This means that (a) of Definition 4 does not hold. By taking compact sets $K = \{0\}$ and $L = \bar{W}_p$, we see that (b) of Definition 4 does not hold. This contradicts our hypothesis. Hence there exists a number r with the property described above.

Let V_p be a neighborhood of p in Y such that $\bar{V}_p \subset W_p$. We shall show that $d_M(V_p \cap M, V_q \cap M) > 0$. We set

$$b = d_{U_p}(\bar{V}_p, U_p - W_p) > 0 ,$$

and choose a positive constant c such that

$$d_D(0, a) \geqq c \cdot d_{D_r}(0, a) \qquad \text{for } a \in D_{r/2} .$$

Let $x \in V_p \cap M$ and $y \in V_q \cap M$. We shall show that $d_M(x, y) \geqq bc$. Consider a chain of holomorphic disks from x to y, i.e., a chain of points $x = p_0, p_1, p_2, \cdots, p_m = y$ in M, points $a_1, \cdots, a_m$ of D and mappings $f_1, \cdots, f_m \in \mathrm{Hol}(D, M)$ such that $f_i(0) = p_{i-1}$ and $f_i(a_i) = p_i$ for $i = 1, \cdots, m$. By inserting additional points to the chain $p_0, \cdots, p_m$ if necessary, we may assume that $a_i \in D_{r/2}$ and that $p_0, \cdots, p_{k-1} \in W_p$ and $p_k \in U_p - W_p$ for some $k, 1 \leqq k \leqq m-1$. Now

$$\begin{aligned}\textstyle\sum_{i=1}^m d_D(0, a_i) &\geqq \textstyle\sum_{i=1}^k d_D(0, a_i) \geqq c\cdot\sum_{i=1}^k d_{D_r}(0, a_i) \\ &\geqq c\cdot\textstyle\sum_{i=1}^k d_{U_p}(p_{i-1}, p_i) \geqq c\cdot d_{U_p}(p_0, p_k) \geqq cd\ .\end{aligned}$$

Thus, $d_M(x, y) \geqq bc$.

(4) taut mod $\Delta \to$ hyp mod Δ. The proof is identical to that of (3) and hence is omitted.

(5) hyp imbd mod $\Delta \to$ hyp mod Δ. This is clear from Definitions 1 and 2.

(6) s-taut imbd mod $\Delta \to$ s-taut mod Δ under the assumption that M is locally taut in Y.

From Definition 5 it is clear that all we have to prove is that if U is a taut open subset of Y, then $U \cap M$ is also taut. Let $\{f_n\}$ be a sequence in $\mathrm{Hol}(D, U \cap M)$. Assume that $\{f_n\}$ is not compactly divergent, i.e., there exist compact sets $K \subset D$ and $L \subset (U \cap M)$ such that $f_n(K) \cap L \neq \phi$ for infinitely many n. We have to show that $\{f_n\}$ has a subsequence which converges in $\mathrm{Hol}(D, U \cap M)$. Taking a subsequence, we may assume that $f_n(K) \cap L \neq \phi$ for all n. Since U is taut by assumption, $\{f_n\}$ has subsequence which converges to some f in $\mathrm{Hol}(D, U)$ we shall show that $f \in \mathrm{Hol}(D, U \cap M)$. Assume that $f \notin \mathrm{Hol}(D, U \cap M)$. Then the open subset $f^{-1}(U \cap M)$ of D is distinct from D and it is nonempty since $f_n(K) \cap L \neq \phi$ for all n implies that $f(K) \cap L \neq \phi$. Let a be a point of the boundary on $f^{-1}(U \cap M)$ in D and set $p = f(a) \in U$. Let V_p be a neighborhood of p in Y such that $V_p \cap M$ is taut. Let W_a be a small neighborhood of a in D such that $f(\overline{W}_a) \subset V_p$. By taking a subsequence, we may assume that $f_n(\overline{W}_a) \subset V_p$ for all n. Let b be a point in $W_a \cap f^{-1}(U \cap M)$. Consider $\{f_n\}$ as a sequence in $\mathrm{Hol}(W_a, V_p \cap M)$. Since $V_p \cap M$ is taut, it converges in $\mathrm{Hol}(W_a, V_p \cap M)$ or it is compactly divergent. But $\lim f_n(a) = f(a) = p \in U - (U \cap M)$ and $\lim f_n(b) = f(b) \in U \cap M$. This is a contradiction. Hence, $f \in \mathrm{Hol}(D, U \cap M)$.

(7) taut imbd mod $\Delta \to$ taut mod Δ under the assumption that M is locally taut in Y.

Let $\{f_n\}$ be a sequence in $\mathrm{Hol}(D^k, M)$. Assume that (b) of Definition 4 does not hold, i.e., there exist compact sets $K \subset D^k$ and $L \subset M - \Delta$ such that $f_n(K) \cap L \neq \phi$ for infinitely many n. We have to show that $\{f_n\}$ has a subsequence which converges in $\mathrm{Hol}(D^k, M)$. But this can be done exactly in the same way as in (6), replacing U by M and D by D^k.

Q.E.D.

It is not clear whether or not s-taut mod Δ (resp. s-tautly imbd mod Δ) is in general a strictly stronger concept than taut mod Δ (resp. tautly imbd Δ). However, for the cases which we will study in sections 3 and 4, the next theorem shows that they are equivalent.

THEOREM 2. *Let Y be a closed subvariety of $P_n(C)$ and let $H_0, \cdots, H_n$ be $n+1$ hyperplanes in general position in $P_n(C)$. Let M and Δ be as above. If $Z = Y - M$ is a principal hypersurface and if $(H_0 \cap Y) \subset Z$, then M is s-taut mod Δ iff M is taut mod Δ. If $(H_0 \cup \cdots \cup H_n) \cap Y \subset Z$, then M is s-tautly imbd mod Δ in Y iff M is tautly imbd mod Δ in Y.*

Proof. (1) tautly imbd mod $\Delta \to s$-tautly imbd mod Δ under the assumption that $M \cap H_j = \phi$ for $j = 0, \cdots, n$. Let L be a compact subset of $Y - \Delta$ and let K be a compact subset of D. Let $L_0, \cdots, L_n$ be compact subsets of L such that $L_j \cap H_j = \phi$ and $L = \bigcup_{j=0}^{n} L_j$. Let j be fixed and identify $P_n(C) - H_j$ with C^n. Let $B(r)$ denote the ball of radius r centered at the origin in C^n. Assume that for each m there exists a holomorphic map $f_m: D \to M$ with $f_m(0) \in L_j$ and $f_m(K) \not\subset B(m)$. Since $f_m(0) \in L_j$ for all m, (b) of Definition 4 is excluded. Thus we can assume that $\{f_m\}$ converges in $\mathrm{Hol}(D, P_n(C))$ to some mapping f. It is clear that $f(0) \notin H_j$ and that $f(K) \cap H_j \neq \phi$. Proposition 2 given below shows that this is a contradiction. Therefore, there exists an integer m_j such that if $g \in \mathrm{Hol}(D, M)$ and $g(0) \in L_j$, then $g(K) \subset U_j = Y \cap B(m_j)$. Since U_j is a closed analytic subspace of the complete hyperbolic space $B(m_j)$, U_j is complete hyperbolic (and therefore taut).

(2) taut mod $\Delta \to s$-taut mod Δ under the assumption that $M \cap H_0 = \phi$. The proof is the same as (1) except that $L = L_0$ and $U_0 = M \cap B(m_0) = Y \cap B(m_0) - Z$. Since Z is a principal hypersurface, U_0 is locally taut in $Y \cap B(m_0)$. As in step (6) of the proof of Theorem 1, it follows that U_0 is taut. Q.E.D.

PROPOSITION 2. *Let Y be a complex space and let H be a principal* *hypersurface of* Y. *If* $\{f_m\}$ *is a sequence in* $\mathrm{Hol}(D, Y)$ *which converges to* f *and if* $f_m(D) \cap H = \phi$ *for all* m, *then either* $f(D) \cap H = \phi$ *or* $f(D) \subset H$.

Proof. Assume that $f(0) = p \in H$ and that $f(z) \notin H$ for $0 < |z| \leq r$. Let V_p be a neighborhood of p in Y such that $V_p \cap H$ is defined as the zeros of a holomorphic function $\phi: V_p \to D$. Without loss of generality we can assume that $f_m(D_r) \subset V_p$ for all m. Then $g_m = \phi \circ f_m$ is a sequence in $\mathrm{Hol}(D, D^*)$ which converges in $\mathrm{Hol}(D, D)$ to $g = \phi \circ f$. Since $g(z) \neq 0$ for $0 < |z| < r$, it is easy to see that $\{g_m\}$ is not compactly divergent in $\mathrm{Hol}(D, D^*)$. Since D^* is taut, this implies $g \in \mathrm{Hol}(D, D^*)$. This is a contradiction since $g(0) = 0$. Q.E.D.

The previous proposition is an elementary generalization of the classical theorem of Hurwitz which says that the limit of a uniformly convergent sequence of nowhere zero holomorphic functions is either nowhere zero or zero everywhere. It should be noted that the proposition is not true for an arbitrary subvariety H of codimension 1 if Y has singularities.

In [14], a characterization of "hyperbolically imbedded" was given in terms of an hermitian metric on Y. A similar characterization is valid for "hyperbolically imbedded mod Δ", but in order to give this we must define what we mean by an hermitian metric on a complex space. If Y is a complex space with structure sheaf $\mathcal{O}$, the Zariski tangent space is defined as the fibre space over Y whose fibre $T_p(Y)$ over p consists of all derivations $v: \mathcal{O}_p \to \boldsymbol{C}$. By an hermitian metric h on Y we shall mean:

(1) h determines a positive definite hermitian form on each fibre $T_p(Y)$. If $v \in T_p(y)$, then $\|v\|_h$ shall denote the length of v with respect to this form.

(2) If $\phi(t)$ is a continuously differentiable curve in Y, then $\|\phi'(t)\|_h$ is a continuous function of t.

By (2), if $\phi: [a, b] \to Y$ is a piecewise continuously differentiable curve in Y, then the length $L(\phi) = \int_a^L \|\phi'(t)\|_h \, dt$ is well-defined. Thus, the hermitian metric induces a distance on Y in the usual manner. We shall further assume that d_h is a proper distance which induces the topology on Y. If Y is an analytic subvariety of $\boldsymbol{C}^n$, then the usual metric on $\boldsymbol{C}^n$ induces an hermitian metric on Y. By using the standard partition of unity argument, we can always construct an hermitian metric on an hermitian metric on an arbitrary complex space Y.

THEOREM 3. *M is hyperbolically imbedded modulo Δ in Y if and only if, given an hermitian metric h on Y, there exists a continuous nonnegative function ϕ defined on Y such that:*

(a) *ϕ is strictly positive on $Y - \Delta$*

(b) *$f^*(\phi h) \leq ds_D^2$ for all $f \in \mathrm{Hol}(D, M)$ where ds_D^2 denotes the Poincaré-Bergman metric of D.*

Proof. Assume that there exists such a function ϕ. Let $d_{\phi h}$ denote the pseudo-distance on Y defined by the pseudo metric ϕh. Since $f^*(\phi h) \leqq ds_D^2$ for every $f \in \mathrm{Hol}(D, M)$, we have

$$d_{\phi h}(x, y) \leqq d_M(x, y) \qquad \text{for } x, y \in M .$$

Since ϕh is positive definite at points of $Y - \Delta$, it follows that $d_{\phi h}(p, q) > 0$ if $p \in Y - \Delta$, $q \in Y$ and $p \neq q$. For such a pair of points p, q, choose neighborhoods U_p and U_q in Y such that $d_{\phi h}(U_p, U_q) \geq \frac{1}{2} d_{\phi h}(p, q) > 0$. Then $d_M(U_p \cap M, U_q \cap M) \geq d_{\phi h}(U_p \cap M, U_q \cap M) \geq d_{\phi h}(U_p, U_q) > 0$. Hence M is hyperbolically imbedded modulo Δ in Y.

Assume that M is hyperbolically imbedded modulo Δ in Y and let L be a compact of $Y - \Delta$. We shall first show that there exists a constant $c > 0$ such that $f^*(ch) \leq ds_D^2$ in $f^{-1}(L)$ for all $f \in \mathrm{Hol}(D, M)$. If not, then there exists a sequence $\{f_n\}$ in $\mathrm{Hol}(D, M)$ such that $f_n^*((1/n)h) > ds_D^2$ at some point a_n of $f_n^{-1}(L) \subset D$. Since D is homogeneous, we can assume that $a_n = 0$. Let v be a unit vector at the origin 0 of D. Then $\|f_{n*}(v)\|_h^2 > n$. Since $f_n(0) \in L$ and L is compact, we may assume that $\{f_n(0)\}$ converges to a point p of L. Let U be a neighborhood of p in Y which is identified with a closed analytic subset of D^m. Assume that there exists a positive number $r < 1$ such that $f_n(D_r) \subset U$ for $n \geq n_0$. Then $\{f_n | D_r\}$ is a normal family and since $f_0(0) \to p \in U$, there exists a subsequence of $\{f_n | D_r\}$ which converges in $\mathrm{Hol}(D_r, U)$. But this is impossible since $\|f_{n*}(v)\|_h^2 > n$. Thus no such r exists. This means that for each k, there exist $z_k \in D$ and an integer n_k such that $|z_k| < 1/k$ and and $f_{n_k}(z_k) \notin U$. Let $p_k = f_{n_k}(0)$ and $q_k = f_{n_k}(z_k)$. By taking a subsequence if necessary, we may assume that $\{q_k\}$ converges to point $q \notin U$. Since $d_M(p_k, q_k) \leq d_D(0, z_k) \to 0$ as $k \to \infty$, M is not hyperbolically imbedded modulo Δ. Thus, for each compact set $L \subset Y - \Delta$, there exists a constant $c > 0$ with the prescribed properties. Now let $L_1 \subset L_2 \subset L_3 \subset \cdots$

be an increasing sequence of compact subsets of $Y - \Delta$ such that $Y - \Delta = \bigcup_{i=1}^{\infty} L_i$ Then there exists a corresponding sequence of positive constants $c_1 \geq c_2 \geq c_3 \geq \cdots$ with the corresponding properties. Let ϕ be any nonnegative continuous function on Y such that $0 < \phi \leq c_i$ on L_i. Then ϕ satisfies conditions (a) and (b) of the theorem. Q.E.D.

THEOREM 4. *Let M be hyperbolically imbedded modulo Δ in Y. If M is locally complete in Y, then M is complete hyperbolic modulo Δ. In particular, if $Z = Y - M$ is a principal hypersurface of Y, then M is complete hyperbolic modulo Δ.*

Proof. Let ϕ and h be as in Theorem 3. As in the proof above, we have

$$d_{\phi h}(p, q) \leq d_M(p, q) \qquad \text{for all } p, q \in M .$$

If M is not complete hyperbolic modulo Δ, then there exists a sequence $\{p_n\}$ in M which is Cauchy with respect to d_M and such that $p_n \to p \notin M \cup \Delta$. Let V_p be a neighborhood of p in Y such that $V_p \cap M$ is complete hyperbolic. If we can show that $\{p_n\}$ is a Cauchy sequence with respect to $d_{V_p \cap M}$, we will have a contradiction since $V_p \cap M$ is complete and $p_n \to p \notin V_p \cap M$. Let $B(p, r)$ denote the ball of radius r centered at p with respect to the pseudo-distance $d_{\phi h}$. We note that in a neighborhood of $p \in Y - \Delta$, ϕh is positive definite and $d_{\phi h}$ is a distance. Choose $\delta > 0$ such that $B(p, 3\delta) \subset V_p$. Since $\{p_n\}$ is Cauchy with respect to d_M and $p_n \to p$, there is an integer n_0 such that $d_M(p_m, p_n) < \delta/2$ and $p_n \in B(p, \delta)$ for $m, n \geq n_0$. Consider a chain of analytic disks from p_m to p_n, i.e., points $p_m = x_0, x_1, \cdots, x_k = p_n$ in M, points $a_1, \cdots, a_k$ in D and mappings $f_1, \cdots, f_k \in \mathrm{Hol}(D, M)$ such that $f_1(0) = x_{i-1}$ and $f_1(a_i) = x_i$ for $i = 1, \cdots, k$. Since $d_M(p_m, p_n) < \delta/2$, we can assume that $\sum_{i=1}^{k} d_D(0, a_j) < \delta$ and by inserting additional disks if necessary, we may assume that $|a_i| < r/2$ where $\delta = d_D(0, r)$. Since

$$\sum_{i=1}^{k} d_{\phi h}(x_{i-1}, x_i) \leq \sum_{i=1}^{k} d_M(x_{i-1}, x_i) \leq \sum_{i=1}^{k} d_D(0, a_i) < \delta ,$$

it follows that $x_i \in B(p, 2\delta)$ for all i. Since $\delta = d_D(0, r)$, it follows that $f_i(D_r) \subset B(p, 3\delta) \cap M \subset V_p \cap M$ for all i. Choose a constant $c > 0$ such that $d_D(0, z) \geq c d_{D_r}(0, z)$ for $|z| < r/2$. Then

$$\sum_{i=1}^{k} d_D(0, a_i) \geq c \sum_{i=1}^{k} d_{D_r}(0, a_i) \geq c \sum_{i=1}^{k} d_{V_p \cap M}(f_i(0), f_i(a_i)) \geq c d_{V_p \cap M}(p_m, p_n) .$$

Since the last inequality is valid for all such chains, we have

$$d_M(p_m, p_n) \geq c d_{V_p \cap M}(p_m, p_n) \qquad \text{for } m, n \geq n_0 .$$

This shows that $\{p_n\}$ is a Cauchy sequence with respect to the distance $d_{V_p \cap M}$ which is the desired contradiction. Q.E.D.

The following two propositions are immediate from Definition 1.

PROPOSITION 3 (*Little Picard theorem*). *If M is hyperbolic modulo Δ and if $f: C^k \to M$ is holomorphic, then either f is a constant map or $f(C^k) \subset \Delta$. In particular, if Δ is an analytic subvariety of dimension $< k$, then f is degenerate everywhere.*

This follows from the fact that $d_{C^k} \equiv 0$.

PROPOSITION 4. *If M is hyperbolic modulo Δ and* $\dim \Delta = k - 1$, *then M is m-measure hyperbolic for $k \leqq m \leqq \dim M$.*

The concept of m-measure hyperbolic was introduced and studied extensively by D. Pelles (formerly D. Eisenman) in [5]. The definition and basic properties can also be found in [16].

3. Theorems of Bloch and Cartan

We are now in a position to discuss the results of Bloch and Cartan. Throughout this section, we let Y be the n-dimensional complex projective space $P_n(C)$ and let $M = Y - (H_0 \cup \cdots \cup H_{n+1})$ where $H_0, \cdots, H_{n+1}$ are $n+2$ hyperplanes in $P_n(C)$ in general position. Following Cartan, we represent Y and M as follows. Let $(x^0, \cdots, x^{n+1})$ be homogeneous coordinates for $P_{n+1}(C)$ and imbed Y in $P_{n+1}(C)$ as the hyperplane $Y = \{(x^0, \cdots, x^{n+1}) \in P_{n+1}(C); x^0 + \cdots + x^{n+1} = 0\}$. Without loss of generality, we may assume that $H_j = \{(x^0, \cdots, x^{n+1}) \in Y; x^j = 0\}$ and therefore $M = \{(x^0, \cdots, x^{n+1}) \in P_{n+1}(C); x^0 + \cdots + x^{n+1} = 0$ and $x^j \neq 0$ for $j = 0, \cdots, n+1\}$. The advantage of this representation is that it gives equal status to each of the hyperplanes H_j. We now define an analytic subvariety Δ of Y. It will be the union of a particular set of hyperplanes which we shall call diagonal hyperplanes with respect to $H_0, \cdots, H_{n+1}$. Let $\mathscr{I}$ be the set of subsets of $\{0, \cdots, n+1\}$ which consist of at least two elements and not more than n elements. For $I = \{j_1, \cdots, j_k\} \in \mathscr{I}$, we set $\Delta_I = \{x^0, \cdots, x^{n+1}) \in Y; x^{j_1} + \cdots + x^{j_k} = 0\}$ and define $\Delta = \bigcup_{I \in \mathscr{I}} \Delta_I$.

Note that if I' is the subset of $\{0, \cdots, n+1\}$ complementary to I, then $\Delta_{I'} = \Delta_I$.

In terms of homogeneous coordinates for $P_{n+1}(C)$, a holomorphic mapping $f \in \mathrm{Hol}(D, M)$ is equivalent to a set of holomorphic functions $f^j : D \to C - \{0\}$ for $j + 0, \cdots, n+1$ satisfying $f^0(z) + \cdots + f^{n+1}(z) = 0$ for all $z \in D$. If $(w^1, \cdots, w^{n+1})$ are inhomogeneous coordinates with $w^j = x^j/x^0$, then f can be represented by $f(z) = (w^1(z), \cdots, w^{n+1}(z))$ where $w^j(z) = f^j(z)/f^0(z)$ is nowhere zero for each j and $w^1(z) + \cdots + w^{n+1}(z) = -1$ for all $z \in D$. In this section we shall use whichever of these representations is more convenient at the time.

Let $p \in M - \Delta$ and $r < 1$ be fixed. Bloch states (see Theorem VII, page 343 of [2]) that there exists $R > 0$ which depends only on p and r such that if $f \in \mathrm{Hol}(D, M)$ and $f(0) = p$, then $f(D_r) \subset B(R) = \{|w^1|^2 + \cdots + |w^{n+1}|^2 < R^2\}$. However, the proof of the theorem shows that the same R works for all $f \in \mathrm{Hol}(D, M)$ with $f(0) \in U_p$, where U_p is a small relatively compact neighborhood of p with $\overline{U}_p \subset M - \Delta$. Thus, Bloch actually proved:

Let L be a compact subset of $M - \Delta$ and let $r < 1$ be fixed. Then there exists $R > 0$ which depends only on L and r such that if $f \in \mathrm{Hol}(D, M)$ and $f(0) \in L$ then $f(D_r) \subset U = M \cap B(R)$.

Since $Y \cap B(R)$ is a closed analytic subspace of the complete hyperbolic space $B(R)$, it is complete hyperbolic. Let $g : Y \cap B(R) \to C$ be defined by $g(w^1, \cdots, w^{n+1}) = w^1 w^2 \cdots w^{n+1}$ and let $Z = g^{-1}(0)$. Then $U = Y \cap B(R) - Z$ and since Z is the zeros of a bounded holomorphic function on $Y \cap B(R)$, it follows from Theorem 4.10 of [16] that U is complete hyperbolic. Thus Bloch's proof actually shows that M is s-taut mod Δ. Since Y, M, and Δ satisfy the conditions of Theorem 2 in section 2, this is no stronger than showing that M is taut mod Δ. Thus we have:

THEOREM 5 (Bloch). *Let $M = P_n(C) - (H_0 \cup \cdots \cup H_{n+1})$ and Δ be as above. Then M is taut mod Δ.*

The weakness of Bloch's proof is that the estimate of the "size" of $f(D_r)$ blows up as we allow $f(0)$ to approach any of the hyperplanes H_j. In [3], Cartan used Bloch's basic techniques but he was able to overcome this difficulty. Whereas Bloch considered one mapping at a time, Cartan considers sequences of maps. The following statement is contained in Theorem VII, page 312 of [3].

Let $\{f_m\}$ *be a sequence in* $\mathrm{Hol}(D, M)$ *where* $f_m = (f_m^0, \cdots, f_m^{n+1})$. *Then one of the following is satisfied.*

(a) $\{f_m\}$ *has a subsequence which converges in* $\mathrm{Hol}(D, Y)$.

(b) *There exists* $I = (j_1, \cdots, j_k) \in \mathscr{I}$ *such that a subsequence of* $\{(f_m^{j_1} + \cdots + f_m^{j_k})/f_m^{j_1}\}$ *converges uniformly on compact subsets to the zero function. Note that if* $(f_m^{j_1}(z) + \cdots + f_m^{j_k}(z))/f_m^{j_1}(z) \to 0$ *as* $m \to \infty$, *then* $f_m(z)$ *converges to* Δ_I *in the sense that for any open set* V *of* Δ_I *in* Y, *there exists* m_0 *such that* $m \geqq m_0$ *implies* $f_m(z) \in V$. In fact, the weaker assumption that

$$\frac{|f_m^{j_1}(z) + \cdots + f_m^{j_k}(z)|}{\sqrt{|f_m^0(z)|^2 + \cdots + |f_m^{n+1}(z)|^2}} \to 0$$

also implies that $f_m(z)$ converges to Δ_I. It should also be remarked that Cartan's theorem implies a stronger result than that given above. We will discuss this later. However the previous statement is sufficient to prove:

THEOREM 6 (Cartan). *Let* $M = P_n(C) - (H_0 \cup \cdots \cup H_{n+1})$, $Y = P_n(C)$ *and* Δ *be as above. Then* M *is tautly imbedded modulo* Δ *in* Y *and* M *is complete hyperbolic modulo* Δ *in* Y.

Proof. Let $L \subset Y - \Delta$ be compact and choose compact subsets $L_j \subset L$ for $j = 0, \cdots, n+1$ such that $L = \bigcup_{j=0}^{n+1} L_j$ and $L_j \cap H_j = \phi$. We shall find taut open subsets U_j of Y which satisfy the conditions of Definition 5. By symmetry, it suffices to do this for $j = 0$.

Let K be a compact subset of D and choose $r < 1$ such that $K \subset D_r$. Assume that for each positive integer m, there exists a mapping $f_m(z) = (w_m^1(z), \cdots, w_m^{n+1}(z))$ in $\mathrm{Hol}(D, M)$ such that $f_m(0) \in L_0$ $f_m(D_r) \not\subset B(m)$. If $\{f_m\}$ has a subsequence which converges in $\mathrm{Hol}(D, Y)$ to f, then $f(0) \notin H_0$ and $f(\bar{D}_r) \cap H_0 \neq \phi$. By Proposition 2 of Section 2, this is impossible and therefore (b) of Cartan's theorem holds. Thus there exists $I = (j_1, \cdots, j_k) \in \mathscr{I}$ such that after taking a subsequence and relabelling if necessary, we have

$$\frac{f_m^{j_1}(0) + \cdots + f_m^{j_k}(0)}{f_m^{j_1}(0)} \to 0 .$$

This implies that $f_m(0)$ converges to the diagonal hyperplane Δ_I. This is impossible since $f_m(0) \in L_0$ for each m and L_0 is a compact subset of $Y - \Delta$. Therefore there exists $R > 0$ depending only on L_0 and $r < 1$ such that if $f \in \mathrm{Hol}(D, M)$ and $f(0) \in L_0$, then $f(D_r) \subset U = Y \cap B(R)$. Since U is complete hyperbolic, we have shown that M is s-tautly imbedded modulo Δ in Y, which in this case is equivalent to tautly imbedded modulo Δ in Y. Since $Y - M$ is a principal hypersurface, M is complete hyperbolic modulo Δ in Y. Q.E.D.

Remark. Let $\{f_m\}$ be a sequence in $\mathrm{Hol}(D^k, M)$. If $\{f_m\}$ does not have a subsequence which converges in $\mathrm{Hol}(D^k, M)$, then the previous theorem implies that $\{f_m\}$ "converges" to Δ. However from the proof of the last theorem it is easy to see that in the latter case, $\{f_m\}$ has a subsequence which "converges" to a particular diagonal hyperplane Δ_I.

4. Applications

In 1897, E. Borel generalized the little Picard theorem (see [2]) and it was his work which motivated Bloch's paper. It is worthwhile observing that Bloch's theorem implies Borel's result for functions defined on C^m.

COROLLARY 1 (E. Borel). *Let $f^j : C^m \to C - \{0\}$ be holomorphic functions for $j = 0, \cdots, n+1$ satisfying $f^0(z) + \cdots + f^{n+1}(z) = 0$ for all $z \in C^m$. Then there exists a subset $\{j_1, \cdots, j_k\}$ of $\{0, \cdots, n+1\}$ $(k \geqq 2)$ such that $f^{j_1}(z) + \cdots + f^{j_k}(z) = 0$ for all $z \in C^m$ and $f^{j_l}(z)/f^{j_1}(z)$ is a constant function for $l = 1, \cdots, k$.*

Proof. Let $f = (f^0, \cdots, f^{n+1})$. Then $f \in \mathrm{Hol}(D, M)$ where M and Δ are as in Theorem 5. Since M is hyperbolic modulo Δ and since $d_{C^m} \equiv 0$, it follows that f is a constant mapping or $f(C^m) \subset \Delta$. In the first case we are done. If $f(C^m) \subset \Delta$ then it is clear that $f(C^m) \subset \Delta_I \cap M$ for some $I = \{j_1, \cdots, j_k\}$. By induction we can assume that $n = 1$. But for $n = 1$, $\Delta \cap M = \phi$ and therefore f is a constant mapping. (For $n = 1$, M is hyperbolic since $M = P_1(C) - \{p_1, p_2, p_3\}$). Q.E.D.

Corollary 1 was first given by Fujimoto in [7] and Green in [10]. Both authors were able to give a proof which does not depend on Bloch's result and which is much clearer and more elementary. Using Corollary 1 and elementary algebra, they deduced the next theorem. We refer the reader to [10] for a proof based on Corollary 1.

THEOREM 7 (Fujimoto and Green). *Let $H_1, \cdots, H_{n+k}$ be $n+k$ hyperplanes in general position in $P_n(C)$ where $2 \leq k \leq n+1$ and let $M = P_n(C) - (H_1 \cup \cdots \cup H_{n+k})$. If $f: C^m \to M$ is holomorphic, then $f(C^m)$ is contained in a linear subspace of dimension $\leq n/k$. Furthermore, the bound n/k is sharp.*

Let $H_0, \cdots, H_{2n-h}$ be $2n+1-h$ hyperplanes in general position in $P_n(C)$ where $0 \leq h \leq n-1$ and let $M = P_n(C) - (H_0 \cup \cdots \cup H_{2n-h})$. Let $\mathscr{J}$ denote the set of subsets of $\{0, \cdots, 2n-h\}$ consisting of $n+2$ elements. If $J = \{j_1, \cdots, j_{n+2}\} \in \mathscr{J}$, we let Δ^J denote the set of diagonal hyperplanes with respect to $H_{j_1}, \cdots, H_{j_{n+1}}$ and let $\Delta_h = \bigcap_{J \in \mathscr{J}} \Delta^J$. We call Δ_h the diagonal set for the hyperplanes $H_0, \cdots, H_{2n-h}$ and note that if $h = n-1$ then $\Delta_n = \Delta$ where Δ is as Section 3. Since the hyperplanes are in general position, elementary linear algebra shows that Δ_h is the union of a finite number of linear subspaces of $P_n(C)$ and that the dimension of $\Delta_h \leq h$. For $J = (j_1, \cdots, j_{n+2})$, we define $M_J = P_n(C) - (H_{j_1} \cup \cdots \cup H_{j_{n+2}})$. By applying Theorem 6 to $P_n(C), M_j$ and Δ^J for each $J \in \mathscr{J}$, we obtain:

COROLLARY 2. *Let $M = P_n(C) - (H_0 \cup \cdots \cup H_{2n-h})$ and Δ_h be as above. Then M is tautly imbedded mod Δ_h in $P_n(C)$ and M is complete hyperbolic mod Δ_h.*

For emphasis, we restate this corollary for the case $h = 0$. (This is the same as Theorem 3 of [14]).

COROLLARY 3. *Let $M = P_n(C) - (H_0 \cup \cdots \cup H_{2n})$. Then M is tautly imbedded in $P_n(C)$ and M is complete hyperbolic.*

Although Corollary 3 is contained in Corollary 2, it has a special character in the sense that it can be deduced relatively easily from Bloch's theorem, whereas Corollary 3 depends on Cartan's theorem. This fact is probably the reason that the full implications of Cartan's theorem seem to have been overlooked. To be more precise, let $M = P_n(C) - (H_0 \cup \cdots \cup H_{2n-h})$. In [4], Dufresnoy quotes Cartan's result to show that if $h = 0$, then $\mathrm{Hol}(D, M)$ is relatively compact in $\mathrm{Hol}(D, P_n(C))$. In essence, however, Dufresnoy only uses that portion of Cartan's theorem which is already contained in Bloch's paper. This explains why Dufresnoy is only able to prove that M is taut mod Δ_h (for mappings defined on D) when $h > 0$. This oversight is carried over to Fujimoto's papers [6] and [8] where he generalizes Dufresnoy's results to several variables and shows that M is taut mod Δ_h. The papers of Dufresnoy and Fujimoto have the advantage, however, that they expose the geometric nature of Cartan's work. The fact that Cartan's original theorem is not stated in geometric terms may well be the reason that it has been overlooked by many authors in recent years.

The case $h = 0$ is special for the additional reason that it is the only case in which the following strong generalization of the big Picard theorem holds.

Let $M = P_n(C) - (H_0 \cup \cdots \cup H_{2n})$ and let A be a closed analytic subvariety of the complex manifold X. If codimension $A \geq 2$ or if the singularities of A are normal crossings, then any holomorphic map $f: X - A \to M$ extends to a holomorphic map $f: X \to P_n(C)$. This follows from the fact that M is complete hyperbolic and hyperbolically imbedded in $P_n(C)$ (see Theorem 2 of [13] and Theorem 4 of [14]).

If we omit fewer than $2n + 1$ hyperplanes, then the extension theorem given above fails. Furthermore, it does not seem possible to use Cartan's theorem to deduce *any* extension theorem for these cases. However, Fujimoto [7] and Green [11] have used a generalized Borel theorem for punctured domains to show that extensions are possible in certain situations. Green's result is a little more precise than Fujimoto's and since it gives an indication of how families of holomorphic maps might be expected to behave, we quote it here.

THEOREM 8. *Let $f = (f^0, \cdots, f^n) \in \mathrm{Hol}(D^* \times D^{m-1}, M)$ where $M = P_n(C) - (H_1 \cup \cdots \cup H_{n+k})$ and the H_j are $n + k$ hyperplanes in general position with $k \geq 2$. Let $I_1, \cdots, I_s$ be the partition of $\{0, \cdots, n\}$ defined by the equivalence relation $i \sim j$ if f^i/f^j extends meromorphically to D^m. Then*

(a) $s \leq (n + k)/k$

(b) *the image of f lies in a linear subspace of codimension* $\geq (s - 1) \cdot (k - 1)$.

(Note that if $s = 1$, then f extends to a holomorphic map $f: D^m \to P_n(C)$. To see this we can assume that $f^0, \cdots, f^n$ are nowhere zero in $D^* \times D^{m-1}$ and that $f^j(z)/f^0(z) = h^j(z)/g^j(z)$ where $h^j(z)$ and $g^j(z)$ are relatively prime holomorphic functions defined on D^m and $z = (z_1, \cdots, z_m)$. Since the zeros of h^j and g^j are contained in the set $\{z_1 = 0\}$, $h^j(z) = z_1^{\alpha_j}\hat{h}^j(z)$ and $g^j(z) = z_1^{\beta_j}\hat{g}^j(z)$ where $\hat{h}^j$ and $\hat{g}^j$ are nowhere zero in D^m. This implies that

$f^j(z)/f^0(z) = z_1^{l_j}F_j(z)$ where l_j is an integer for $j = 1, \cdots, n$ and F_j is a nowhere zero holomorphic function on D^m. Let $l = \max\{0, -l_1, \cdots, -l_n\}$. Then $f(z) = (z_1^l, z_1^{l+l_1}F_1(z), \cdots, z_1^{l+l_n}F_n(z))$ is a holomorphic extension of f to all of D^m. In particular, Theorem 8 gives a holomorphic extension if f has rank $\geq h = n - k + 2$ at some point $z \in D^* \times D^{m-1}$.)

The number s in Green's theorem is called the degree of irrationality of the mapping f since it measures the extent to which f fails to extend. The theorem shows that the more irrational a map is, the lower the dimension of the smallest linear subspace which contains the image of f. It is reasonable to expect a similar situation to occur for families of holomorphic maps into M. To be more precise, let $\{f_m = (f_m^0, \cdots, f_m^n)\}$ be a sequence in $\mathrm{Hol}(D, M)$ and let $I_1, \cdots, I_s$ be a partition of $\{0, \cdots, n+1\}$ such that, after taking a subsequence of $\{f_m\}$ and relabelling, we have f^i/f^j converges uniformly on compact sets iff $i, j \in I_k$ for some k. Let s be the smallest number which can be obtained in this way. Then we call s the degree of nonconvergence of the sequence $\{f_m\}$.

CONJECTURE. *Let $M = P_n(C) - (H_1 \cup \cdots \cup H_{n+k})$, $\{f_m\}$ and s be as above. Then*

(a) $s \leqq (n+k)/k$

(b) *$\{f_m\}$ has a subsequence which converges to a linear subspace of codimension $\geqq (s-1)(k-1)$.*

To give more credence to the conjecture we now state the original version of Cartan's theorem more precisely. We use the conventions introduced in Section 3 concerning $H_0, \cdots, H_{n+1}$ and $(f^0, \cdots, f^{n+1})$.

Let $M = P_n(C) - (H_0 \cup \cdots \cup H_{n+1})$ and let $\{f_m = (f_m^0, \cdots, f_m^{n+1})\}$ be a sequence in $\mathrm{Hol}(D, M)$. *Then there exists a partition $I_1, \cdots, I_r$ of $\{0, \cdots, n+1\}$ such that, after taking a subsequence and relabelling, we have:*

(a) *$\{f_m^i/f_m^j\}$ converges uniformly on compact sets iff $i, j \in I_q$ for some q.*

(b) *If $I_q = \{j_1, \cdots, j_\alpha\}$ and $q \leqq 2$, then $(f_m^{j_1} + \cdots + f_m^{j_\alpha})/f_m^{j_1} \to 0$ uniformly on compact sets.*

If (b) is satisfied for all $q \leq r$, then each subset of the partition contains at least 2 elements and therefore $r \leq (n+2)/2$. From this it would follow easily that the degree of nonconvergence s of $\{f_m\}$ satisfies

$s \leq (n+2)/2$. Furthermore, $\{f_m\}$ would have a subsequence which converges to a linear subspace of codimension $\geq s-1$. Thus the conjecture for $k=2$ would follow. The general case could probably then be derived from this one in a relatively elementary manner. Although he could not prove that (b) was satisfied for all $j = 1, \cdots, r$, Cartan considered it very likely that this is indeed true.

5. Comments

In [6] and [8], Fujimoto uses the intrinsic distance and normal families of mappings to derive many of the results contained in Section 3 and 4. It was his papers which first called our attention to the papers of Bloch and Cartan.

Finally we shall give two examples and pose a few open problems.

EXAMPLE 1. Let X be a hyperbolic compact complex manifold of dimension ≥ 2. Let Y be obtained from X by blowing up a point $p \in X$ and let Δ be the corresponding projective space. Then Y is taut mod Δ.

EXAMPLE 2. Let M be the quotient of a bounded symmetric domain $\mathscr{D}$ by an arithmetic discrete group Γ and let Y be its compactification in the sense of Satake, Baily and Borel, and Pyatetzki-Shapiro. It was shown in [17] that M is hyperbolically imbedded in Y. Since the proof of Theorem 1 of [14] shows that hyperbolically imbedded implies tautly imbedded if $\Delta = \phi$, we have that M is tautly imbedded in Y.

PROBLEM 1. Determine the relationship between "taut mod Δ" and "complete hyperbolic mod Δ". This is probably difficult. When Δ is empty, it was shown in [12] that "complete hyperbolic" implies "taut", but the converse is not known.

PROBLEM 2. Does "hyperbolically imbedded mod Δ" imply "tautly imbedded mod Δ"? As pointed out in [14], the answer is yes if $\Delta = \phi$.

PROBLEM 3. In Theorem 4, can we weaken the assumption that M is hyperbolically imbedded mod Δ to the assumption that M is hyperbolic mod Δ? The question is open even in the case where $\Delta = \phi$.

PROBLEM 4. Let Y be a complex space which is measure hyperbolic. Does there exist a closed analytic subvariety Δ of Y such that Y is hyperbolic mod Δ?

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