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CHARACTERIZATION OF RELATIVE DOMINATION PRINCIPLE

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1. Introduction

Let X be a locally compact and σ-compact Abelian group and ξ be the Haar measure of X. A positive Radon measure N on X is called a convolution kernel when we regard it as a kernel of potentials of convolution type. M. Itô [4], [6] characterized the convolution kernel which satisfies the domination principle. The purpose of this paper is to characterize the relative domination principle for the convolution kernels. We call $x \in X$ a period of a real Radon measure μ on X if $\mu * \varepsilon_x = \mu$ holds, where ε_x is the unit mass at x, and denote by $p(\mu)$ the set of all periods of μ. We shall prove the following result:

Let N_1 be a convolution kernel of Hunt on X and N_2 ($\neq 0$) be a bounded convolution kernel on X. Then N_1 satisfies the relative domination principle with respect to N_2 if and only if one of the following conditions is satisfied.

(1) There exist a positive measure $\mu (\neq 0)$ and a positive measure H on X such that

$$N_2 = N_1 * \mu + H$$

and $p(H)$ contains the support S_{N_1} of N_1.

(2) N_1 is bounded and $p(N_2)$ contains S_{N_1}.

By virtue of this theorem, we shall obtain that the relative domination principle defines an order on the totality of bounded convolution kernels of Hunt on X.

2. Preliminaries

We denote by L_{loc} the family of real valued locally ξ-summable functions on X, by M_K the family of bounded functions of L_{loc} with

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compact support and by C_K the family of continuous functions of M_K. L^+_{loc}, M^+_K and C^+_K are their subfamilies constituted by non-negative functions.

For a real Radon measure μ on X, $N*\mu$ is called a N-potential of μ when the convolution has a sense. If $N*\mu$ is ξ-absolutely continuous, we denote its density by $N\mu$. Particularly we write $N*\mu = N*f$ and $N\mu = Nf$ when $\mu = f\xi$ for $f \in L_{loc}$.

DEFINITION 1. Let N_1 and N_2 be convolution kernels on X. We say that N_1 satisfies the relative domination principle with respect to N_2 and write $N_1 \prec N_2$, when the following statement is true. If f and g are in M^+_K and $N_1 f \leqq N_2 g$ ξ-a.e. on $k(f) = \{x \in X; f(x) > 0\}$, then $N_1 f \leqq N_2 g$ ξ-a.e. on X. We say, simply, that N satisfies the domination principle when $N \prec N$.

Remark 1. Let N be a convolution kernel on X satisfying the domination principle. Suppose that $N(f + g)$ has a sense for f and g in L^+_{loc} and that $Nf + cf \leqq Ng + cg$ ξ-a.e. on $k(f)$ for some constant $c > 0$. Then $Nf + c'f \leqq Ng + c'g$ ξ-a.e. on X for any constant c' such that $0 \leqq c' \leqq c$ (cf. [5]).

DEFINITION 2. A convolution kernel N is said to be bounded if $N*\varphi(x)$ is bounded on X for any $\varphi \in C_K$ and it is said to be of positive type if $N*\varphi*\check{\varphi}(0) \geqq 0$ for any $\varphi \in C_K$.

DEFINITION 3. A family $(\mu_t)_{t \geqq 0}$ of positive measures is said to be a vaguely continuous semi-group if

(1) $\mu_t * \mu_s = \mu_{t+s}$, $\forall t \geqq 0$, $\forall s \geqq 0$,

(2) $\mu_0 = \varepsilon$ (the Dirac measure),

(3) $t \to \mu_t$ is vaguely continuous.

A convolution kernel $N (\neq 0)$ is called a Hunt kernel if there exists a vaguely continuous semi-group $(\mu_t)_{t \geqq 0}$ such that $N = \int_0^\infty \mu_t dt$.

Remark 2. For a convolution kernel of Hunt, there exists a unique system $(N_p)_{p \geqq 0}$ called the resolvent of N such that $N_0 = N$ and that

$$N_p - N_q = (q - p) N_p * N_q , \qquad p \geqq 0, \ q > 0 .$$

By the above resolvent equation, we have

$$N + \frac{1}{p}\varepsilon = \frac{1}{p} \sum_{n=0}^{\infty} (pN_p)^n$$

for any $p > 0$ and hence N satisfies the domination principle.

DEFINITION 4. A convolution kernel N is said to be associated with the fundamental family Σ if there exists a fundamental system $V(0)$ of compact neighbourhoods of 0 such that with every $v \in V(0)$, we can associate a positive measure $\sigma_v \in \Sigma$ satisfying

(1) $N \geqq N*\sigma_v$ and $N \neq N*\sigma_v$,

(2) $N = N*\sigma_v$ as a measure on Cv,

(3) $\lim_{n\to\infty} N*(\sigma_v)^n = 0$.

Remark 3. Let N be a convolution kernel of Hunt and $V(0)$ be the family of all compact neighbourhoods of 0. J. Deny proved in [3] that for any $v \in V(0)$, there exists a balayaged measure σ_{cv} of ε on Cv with respect to N and that if we put $\Sigma = \{\sigma_{cv}\,; v \in V(0)\}$, then N is associated with the fundamental family Σ.

3. Relative balayaged measure

LEMMA 1. *Let N_1 and $N_2 (\neq 0)$ be convolution kernels on X such that $N_1 \prec N_2$. Suppose that N_2 is bounded on X. Then N_1 is bounded on X.*

Proof. For any $\varphi \in C_K^+$, $N_1*\varphi(x)$ is bounded on S_φ and hence there exists $\psi \in C_K^+$ such that $N_1*\varphi \leqq N_2*\psi$ on S_φ. The assumption $N_1 \prec N_2$ implies that $N_1*\varphi \leqq N_2*\psi$ on X. This means that N_1 is bounded if N_2 is bounded.

Remark 4. Let N be a convolution kernel satisfying the domination principle. M. Itô [5] proved that the following conditions are equivalent:

(1) N is bounded.

(2) N is of positive type.

(3) For any positive measure ν with compact support and for any relatively compact open set ω, we denote by ν'_ω a balayaged measure of ν on ω with respect to N. Then $\int d\nu'_\omega \leqq \int d\nu$.

Remark 5. To construct a relative balayaged measure, we use here the following existence theorem of M. Itô (see [6]).

Let N be a convolution kernel of positive type and u be a locally bounded ξ-measurable function on X. Then, for any compact set K and for any $c > 0$, there exists a unique element f_u of M_K^+ supported by K

such that

(1) $Nf_u + cf_u \geqq u$ ξ-a.e. on K,

(2) $Nf_u + cf_u = u$ ξ-a.e. on $k(f_u) = \{x \in X; f_u(x) > 0\}$.

LEMMA 2. *Let N_1 and N_2 be convolution kernels such that $N_1 \prec N_2$ and that N_1 is of positive type. Then, for any positive measure μ with compact support and for any relatively compact open set ω, there exists a positive measure μ''_ω supported by $\bar{\omega}$ such that*

(1) $N_1 * \mu''_\omega = N_2 * \mu$ as a measure in ω,

(2) $N_1 * \mu''_\omega \leqq N_2 * \mu$ as a measure in X,

(3) If ν is a positive measure supported by $\bar{\omega}$ such that $N_1 * \nu \geqq N_2 * \mu$ in ω, then $N_1 * \nu \geqq N_1 * \mu''_\omega$ in X.

Proof. If $f \in M_K^+$, $N_2 f$ is locally bounded and ξ-measurable and hence, by the above existence theorem, there exists $f'' \in M_K^+$ supported by $\bar{\omega}$ such that

(1) $N_1 f'' + cf'' \geqq N_2 f$ ξ-a.e. on $\bar{\omega}$,

(2) $N_1 f'' + cf'' = N_2 f$ ξ-a.e. on $k(f'')$.

It is known that $N_1 \prec N_2$ if and only if $N_1 + c\varepsilon \prec N_2$ for any $c > 0$ (see [5]).

Therefore (1) and (2) imply that

$$N_1 f'' + cf'' \leqq N_2 f \ \ \xi\text{-a.e. on } X,$$
$$N_1 f'' + cf'' = N_2 f \ \ \xi\text{-a.e. on } \bar{\omega}.$$

By the ordinary limit process, we obtain a positive measure μ''_ω for μ having the desired properties (cf. [5]).

DEFINITION 5. In the above lemma, $N_1 * \mu''_\omega$ is uniquely determined but μ''_ω is not always uniquely determined. We call μ''_ω a relative balayaged measure of μ on ω with respect to (N_1, N_2).

4. Characterization of relative domination principle

LEMMA 3. *Let N be a bounded convolution kernel of Hunt and σ_{cv} be a balayaged measure of ε on Cv for $v \in V(0)$ with respect to N. Then $\int dN < +\infty$ $\left(resp. \int dN = +\infty\right)$ if and only if $\int d\sigma_{cv} < 1$ $\left(resp. \int d\sigma_{cv} = 1\right)$ for every $v \in V(0)$.*

Proof. By Remark 3, N is associated with the fundamental family $\Sigma = \{\sigma_{cv}; v \in V(0)\}$. The boundedness of N means, by virtue of Remark 4, that $\int d\sigma_{cv} \leqq 1$ for every $v \in V(0)$. On the other hand, J. Deny [2] proved this lemma for the associated kernel with a fundamental family under the hypothesis that $\int d\sigma_{cv} \leqq 1$ for every $v \in V(0)$. Therefore our assertion is true.

LEMMA 4. *Let N be a convolution kernel of Hunt. Then we have*

$$S_N = \overline{\bigcup\{S_{(\sigma_{cv})^n}\,;\, v \in V(0),\ n = 1, 2, 3, \cdots\}}$$

Proof. By the definition of σ_{cv}, we have

$$N \geqq N*\sigma_{cv} \geqq N*(\sigma_{cv})^2 \geqq \cdots \geqq N*(\sigma_{cv})^n .$$

On the other hand, the fact that N satisfies the domination principle asserts that $0 \in S_N$. Accordingly, $S_N \supset S_{(\sigma_{cv})^n}$ for any v and for any integer $n > 0$ and hence

$$S_N \supset \overline{\bigcup \{S_{(\sigma_{cv})^n}\}} .$$

Next, we shall prove the inverse inclusion. Le $(v_\alpha)_{\alpha \in A}$ be a decreasing net of compact neighbourhoods of 0 such that $\bigcap_{\alpha \in A} v_\alpha = \{0\}$. For any positive integer n, we have

$$N*(\varepsilon - \sigma_{cv_\alpha})*\sum_{p=0}^{n-1} (\sigma_{cv_\alpha})^p = N - N*(\sigma_{cv_\alpha})^n .$$

By Remark 3 and by the property of fundamental family, we have

$$\lim_{n \to \infty} N*(\sigma_{cv_\alpha})^n = 0$$

and hence

$$N = N*(\varepsilon - \sigma_{cv_\alpha})*\sum_{p=0}^{\infty} (\sigma_{cv_\alpha})^p .$$

This means that

$$S_N \subset v_\alpha + \bigcup_n \{S_{(\sigma_{cv\alpha})^n}\} \subset v_\alpha + \overline{\bigcup_{n,v} \{S_{(\sigma_{cv})^n}\}}$$

and hence

$$S_N \subset \overline{\bigcup_{n,v} \{S_{(\sigma_{cv})^n}\}} ,$$

because $\bigcap_{\alpha \in A} v_\alpha = \{0\}$.

Consequently the equality holds.

THEOREM. *Let N_1 be a convolution kernel of Hunt on X and $N_2(\neq 0)$ be a bounded convolution kernel on X. Then N_1 satisfies the relative domination principle with respect to N_2 if and only if one of the following conditions is satisfied.*

(1) *There exist a positive measure $\mu(\neq 0)$ and a positive measure H on X such that*

$$N_2 = N_1 * \mu + H$$

and that $p(H)$ contains the support S_{N_1} of N_1.

(2) *N_1 is bounded and $p(N_2)$ contains S_{N_1}.*

Proof. Necessity. For any relatively compact open set ω, we write μ_ω a relative balayaged measure of ε on ω with respect to (N_1, N_2). The inequality $N_1 * \mu_\omega \leqq N_2$ for any ω implies that $\{\mu_\omega\}$ is vaguely bounded as $\omega \uparrow X$ and hence there exists a positive measure μ such that $\mu_\omega \to \mu$ vaguely as $\omega \uparrow X$. If we put

$$H = N_2 - N_1 * \mu = \lim_{\omega \uparrow x} N_1 * \mu_\omega - N_1 * \mu ,$$

then H is a positive measure on X. Therefore it is sufficient to prove the periodicity of H.

For any $v \in V(0)$, we denote by σ_{cv} a balayaged measure of ε on Cv with respect to the kernel N_1 (cf. Remark 3). Then we have

$$H * (\varepsilon - \sigma_{cv}) = \lim_{\omega \uparrow X} N_1 * (\mu_\omega - \mu) * (\varepsilon - \sigma_{cv}) = \lim_{\omega \uparrow X} N_1 * (\varepsilon - \sigma_{cv}) * (\mu_\omega - \mu) = 0 ,$$

and hence $H = H * \sigma_{cv} = H * (\sigma_{cv})^n$ for every $v \in V(0)$ and for every integer $n > 0$.

If $\int dN < +\infty$, then $\int d\sigma_{cv} < 1$ for every v (cf. Lemma 3) and hence $H = 0$, because $H = H * (\sigma_{cv})^n$ for every n.

If $\int dN = +\infty$, then $\int d\sigma_{cv} = 1$. Therefore, by virtue of the theorem of G. Choquet and J. Deny (see [1]), $p(H)$, the set of all periods of H, contains the support $S_{\sigma_{cv}}$ of σ_{cv} for every v. On the other hand, we have, by Lemma 4,

$$S_{N_1} = \overline{\bigcup \{S_{(\sigma_{cv})^n} ; v \in V(0),\ n = 1, 2, 3, \cdots\}} .$$

consequently $p(H)$ contains S_{N_1}.

Sufficiency. If the condition (1) holds, N_1 and H are bounded, because N_2 is bounded and hence it is sufficient to prove that $N_1 \prec N_2$ under the following hypothesis:

N_1 is bounded and there exist positive measures μ and H such that

$$N_2 = N_1 * \mu + H$$

and that $p(H)$ contains S_{N_1}.

N_1 being bounded, there exists a system $(N_p^{(1)})_{p \geqq 0}$ of resolvent satisfying

$$\int p dN_p^{(1)} \leqq 1 \quad (\forall p > 0) , \qquad N_0^{(1)} = N_1$$

and

$$N_p^{(1)} - N_q^{(1)} = (q-p) N_p^{(1)} * N_q^{(1)} \qquad (\forall p \geqq 0, \forall q > 0) .$$

By the resolvent equation, we have

$$N_p^{(1)} + \frac{1}{q-p}\varepsilon = \frac{1}{q-p} \sum_{n=0}^{\infty} ((q-p) N_q^{(1)})^n .$$

Accordingly, for any $p > 0$ and for $c > 0$, there exists a positive measure $\sigma_{p,c}$ such that

$$\int d\sigma_{p,c} < 1 , \qquad S_{\sigma_{p,c}} = S_{(N_p^{(1)} + c\varepsilon)} = S_{N_1}$$

and that

$$N_p^{(1)} + c\varepsilon = c \sum_{n=1}^{\infty} (\sigma_{p,c})^n .$$

By the periodicity of H, we have

$$\frac{1}{c}(\varepsilon - \sigma_{p,c}) * H = \frac{1}{c}\left(1 - \int d\sigma_{p,c}\right) H \geqq 0$$

and hence there exists a positive measure α satisfying

$$H = (N_p^{(1)} + c\varepsilon) * \alpha .$$

By the resolvent equation, there exists a positive measure β satisfying

$$N_1 = (N_p^{(1)} + c\varepsilon) * \beta .$$

Therefore, for some positive measure ν, N_2 can be written in the following form

$$N_2 = (N_p^{(1)} + c\varepsilon) * \nu .$$

We suppose, for f and g in M_K^+, that

$$(N_p^{(1)} + c\varepsilon)f \leqq N_2 g = (N_p^{(1)} + c\varepsilon)(\nu * g) \quad \xi\text{-a.e.} \qquad \text{on } k(f)\,.^{1)}$$

Then we have

$$(N_p^{(1)} + c\varepsilon)f \leqq N_2 g \quad \xi\text{-a.e.} \qquad \text{on } X\,,$$

because $(N_p^{(1)} + c\varepsilon)$ satisfies the domination principle (cf. Remark 1). Therefore

$$(N_p^{(1)} + c\varepsilon) \prec N_2\,.$$

p and c being arbitrary, we may conclude that $N_1 \prec N_2$ by the ordinary limit process.

Consequently the theorem is proved.

Let $\mathscr{H}_b$ be the totality of bounded convolution kernels of Hunt on X. We denote $N_1 \sim N_2$ when N_1 is proportional to N_2 and $\dot{\mathscr{H}}_b = \mathscr{H}_b / \sim$.

COROLLARY. *The relation $\prec$ is an order on $\dot{\mathscr{H}}_b$.*

Proof. The reflexive law follows by the domination principle. Assume that $N_1 \prec N_2$ and $N_2 \prec N_1$ for $N_1, N_2 \in \mathscr{H}_b$. By our theorem, N_1 and N_2 can be written in the following forms

$$N_2 = N_1 * \mu + H_1\,,$$
$$N_1 = N_2 * \nu + H_2\,,$$

where μ, ν, H_1 and H_2 are positive measures on X and $p(H_1) \supset S_{N_1}$, $p(H_2) \supset S_{N_2}$. If $\int dN_1 < +\infty$, then we may clearly choose a non-zero measure as μ. If $\int dN_1 = +\infty$ and $\mu = 0$, then $N_2 = N_2 * \varepsilon'_{1,cv}$ for any $v \in V(0)$, where $\varepsilon'_{1,cv}$ is a balayaged measure of ε on Cv with respect to N_1. This contradicts to the unicity principle for N_2.[2] Similarly, we may suppose $\nu \neq 0$. Therefore

$$N_1 = (N_1 * \mu) * \nu + H_1 * \nu + H_2\,.$$

It is known that $\lim_{v \uparrow X} N_1 * \varepsilon'_{1,cv} = 0$ (cf. [6]) and hence

$$\lim_{v \uparrow X} H_1 * \nu * \varepsilon'_{1,cv} = 0\,.$$

By $p(H_1) \supset S_{N_1}$, $H_1 * \nu = 0$ and hence $H_1 = 0$. Similarly $H_2 = 0$. Consequently

1) In this case $\nu * g$ means the density of $\nu * (g\xi)$.

2) This means that $\mu = \nu$ whenever $N * \mu = N * \nu$.

$$N_1 = (N_1*\mu)*\nu \ .$$

For a compact set K in X, we denote by μ_K and ν_K the restrictions of μ and ν to K, respectively. Then

$$N_1 \geqq (N_1*\mu_K)*\nu_K = N_1*(\mu_K*\nu_K)$$

and hence $\int d(\mu_K*\nu_K) \leqq 1$, that is, $\int d\mu_K \int d\nu_K \leqq 1$. K being arbitrary, $\int d\mu < +\infty$ and $\int d\nu < +\infty$. Consequently

$$N_1 = (N_1*\mu)*\nu = N_1*(\mu*\nu) \ .$$

By the unicity principle for N_1, $\mu*\nu = \varepsilon$ and hence $\mu = c\varepsilon$ and $\nu = (1/c)\varepsilon$, where c is a positive constant. This means $N_1 \sim N_2$ (asymmetric law).

Let $N_1, N_2, N_3 \in \mathscr{H}_b$ and suppose that $N_1 \prec N_2$, $N_2 \prec N_3$ and that for $f, g \in M_K^+$,

$$N_1 f \leqq N_3 g \ \xi\text{-a.e.} \qquad \text{on } S_{(f\xi)} \ .$$

By Lemma 2, there exists $g'_n \in M_K^+$ satisfying

$$N_2 g'_n + \frac{1}{n} g'_n = N_3 g \ \xi\text{-a.e.} \qquad \text{on } S_{(f\xi)} \ ,$$

$$N_2 g'_n + \frac{1}{n} g'_n \leqq N_3 g \ \xi\text{-a.e.} \qquad \text{on } X \ .$$

Put

$$F_n = \{x \in S_{(f\xi)}\, ; N_1 f(x) \leqq N_2 g'_n(x)\}$$

and let f_n be the restriction of f to F_n. Then

$$N_1 f_n \leqq N_2 g'_n \ \xi\text{-a.e.} \qquad \text{on } k(f_n)$$

and hence the same inequality holds ξ-a.e. on X, that is,

$$N_1 f_n \leqq N_3 g \ \xi\text{-a.e.} \qquad \text{on } X \ .$$

$\{(1/n)g'_n\}$ converging to 0 ξ-a.e. on X as $n \to \infty$, $f_n \to f$ ξ-a.e. on X. Consequently $N_1 f \leqq N_3 g$ ξ-a.e. on X, that is, $N_1 \prec N_3$ (transitive law). This completes the proof.

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