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THE SEMI-BALAYABILITY OF REAL CONVOLUTION KERNELS

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Dedicated to Professor Yukio Kusunoki on his 60th birthday

§1.

Let X be a locally compact, σ-compact and non-compact abelian group. Throughout this paper, we shall denote by ξ a fixed Haar measure on X and by δ the Alexandroff point of X.

A real convolution kernel (i.e., a real Radon measure) N on X is said to be semi-balayable if N satisfies the semi-balayage principle on all open sets (see Definition 6). We know that every convolution kernel N of logarithmic type is semi-balayable (see [8]). Here N is said to be of logarithmic type if, with a vaguely continuous, markovian, semi-transient and recurrent convolution semi-group $(\alpha_t)_{t\geqq 0}$ of non-negative Radon measures on X,

$$N * \mu = \int_0^\infty \alpha_t * \mu dt \left(= \lim_{t\to\infty} \int_0^t \alpha_s * \mu ds\ ^{1)}\right)$$

for all real Radon measure μ on X with compact support and $\int d\mu = 0$.

In this paper, we shall show that the semi-balayability is an essential property to characterize convolution kernels of logarithmic type. More precisely, we shall establish the following theorems.

THEOREM 1. *Let N be a real convolution kernel on X. If $X \approx R \times F$ or $X \approx Z \times F$, we suppose an additional condition: $N = o(|x|)$ at the infinity[2]. Then N is of logarithmic type if and only if N is semi-balayable, non-periodic and satisfies $\inf_{x\in X} N * f(x) \leqq 0$ for any finite continuous function f on X with compact support and $\int f d\xi = 0$.*

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1) For a net $(\mu_\alpha)_{\alpha\in A}$ of real Radon measures and a real Radon measure μ, we write $\mu = \lim_{\alpha\in A} \mu_\alpha$ if $(\mu_\alpha)_{\alpha\in A}$ converges vaguely to μ along A.

2) If $X = R \times F$ or $X = Z \times F$, $N = o(|x|)$ at the infinity means that for any $f \in C_K^+(X)$, $N*f((x,y)) = o(|x|)$ as $|x| \to \infty$, where $(x,y) \in R \times F$ or $\in Z \times F$. In the case of $X \approx R \times F$ or $X \approx Z \times F$, the definition $N = o(|x|)$ at the infinity follows naturally from the above definition.

Here R, Z and F denote the additive group of real numbers, the additive group of integers and a certain compact abelian group, respectively.

By virtue of the main theorems in [8] (see Théorèmes 52 and 52′), Theorem 1 follows immediately from the following

THEOREM 2. *Let N be a non-periodic real convolution kernel on X satisfying* $\inf_{x\in X} N*f(x) \leqq 0$ *for any finite continuous function f on X with compact support and* $\int f d\xi = 0$. *Then N is semi-balayable if and only if N satisfies the semi-complete maximum principle and $\eta_{N,\delta} = -\infty$, i.e., for any exhaustion $(K_n)_{n=1}^{\infty}$ of X*[3] *and any non-negative continuous function $f \neq 0$ on X with compact support,* $\lim_{n\to\infty} \int f d\eta_{N,CK_n} = -\infty$, *where η_{N,CK_n} is the N-reduced measure of N on CK_n.*

The "if" part is already known (see Proposition 28 in [8]), so that this paper will be devoted principally to the proof of the "only if" part.

It is interesting to compare Theorem 1 with the Choquet-Deny theorem for Hunt convolution kernels[4] (see [3]).

Contrary to a conjecture in [8] (see Problème 29), Theorem 1 shows that, under some additional conditions, non-periodic and semi-balayable real convolution kernels are of logarithmic type.

§ 2.

We denote by:

$C(X)$ the usual Fréchet space of finite continuous functions on X;

$C_K(X)$ the usual topological vector space of finite continuous functions on X with compact support;

$M(X) = C_K(X)^*$ the topological vector space of real Radon measures on X with the vague (weak*) topology;

$M_K(X) = C(X)^*$ the usual topological vector space of real Radon measures on X with compact support;

$C^+(X)$, $C_K^+(X)$, $M^+(X)$ and $M_K^+(X)$ their subsets of non-negative elements.

Furthermore, we put

$$C_K^0(X) = \left\{f \in C_K(X); \int f d\xi = 0\right\} \quad \text{and} \quad M_K^0(X) = \left\{\mu \in M_K(X); \int d\mu = 0\right\}.$$

3) An exhaustion $(K_n)_{n=1}^{\infty}$ of X means a sequence of compact sets satisfying $K_n \subset$ the interior of K_{n+1} and $\cup_{n=1}^{\infty} K_n = X$.

4) A non-negative convolution kernel N_0 on X is a Hunt convolution kernel if and only if N_0 is balayable (see Remark 14 (3)) and not pseudo-periodic.

DEFINITION 3. A real convolution kernel N on X is said to satisfy the semi-complete maximum principle (denoted by $N \in$ (SMP)) if for any $f, g \in C_K^+(X)$ with $\int f d\xi = \int g d\xi$ and any $a \in R$, we have the implication:

$$N * f(x) \leqq N * g(x) + a \quad \text{on supp}(f) \Longrightarrow N * f(x) \leqq N * g(x) + a \quad \text{on } X,$$

where supp(f) denotes the support of f.

DEFINITION 4. A real convolution N on X is said to satisfy the transitive semi-complete maximum principle with respect to ξ (denoted by $(N, \xi) \in$ (TSMP)) if for any $f, g \in C_K^+(X)$ with $\int f d\xi = \int g d\xi$ and any $a \in R$, we have the implication:

$$N * f(x) \leqq N * g(x) + a \quad \text{on supp}(f) \Longrightarrow a \geqq 0 .$$

We can describe the above principles by the term of non-negative Radon measures.

Remark 5. $N \in$ (SMP) (resp. $(N, \xi) \in$ (TSMP)) if and only if for any $\mu, \nu \in M_K^+(X)$ with $\int d\mu = \int d\nu$ and any $a \in R$, we have the implication:

$$\begin{aligned} N * \mu \leqq N * \nu + a\xi \quad & \text{in a certain open set} \supset \text{supp}(\mu) \\ \Longrightarrow N * \mu \leqq N * \nu + a\xi \quad & \text{on } X \text{ (resp. } \Longrightarrow a \geqq 0), \end{aligned}$$

where supp(μ) denotes also the support of μ.

For a real convolution kernel N on X, we put

$$D^+(N) = \{\mu \in M^+(X);\ N * \mu \text{ is defined in } M(X)\} .$$

Let $\mu \in M^+(X)$. Evidently $\mu \in D^+(N)$ if and only if for any $f \in C_K^+(X)$, $\int |\check{N} * f| d\mu < \infty$. Here $\check{N}$ denotes the real convolution kernel on X defined by $\int f d\check{N} = \int \check{f} dN$ for all $f \in C_K(X)$, where $\check{f}(x) = f(-x)$.

DEFINITION 6. A real convolution kernel N on X is said to satisfy the semi-balayage principle (resp. the semi-balayage principle on all open sets) (denoted by $N \in$ (SBP) (resp. denoted by $N \in (\text{SBP}_g)$)) if for any $\mu \in M_K^+(X)$, any $a \in R$ and any relatively compact open set (resp. any open set) $\omega \neq \phi$ in X, there exists an element $(\mu', a') \in M^+(X) \times R$ such that:

(B.1) $\int d\mu' = \int d\mu$.

(B.2) supp$(\mu') \subset \bar{\omega}$.

(B.3) $\mu' \in D^+(N)$ and $N * \mu' + a'\xi = N * \mu + a\xi$ in ω.

(B.4) $N * \mu' + a'\xi \leqq N * \mu + a\xi$ on X.

In this case, we call (μ', a') a semi-balayaged couple of (μ, a) on ω with respect to N and denote by $\mathrm{SB}_N((\mu, a); \omega)$ the totality of such couples. If $N \in (\mathrm{SBP}_g)$, we say that N is semi-balayable.

We set

$$\underline{\mathrm{SB}}_N((\mu, a); \omega) = \{(\mu', a') \in \mathrm{SB}_N((\mu, a); \omega); N * \mu + a'\xi = \inf\{N * \mu'' + a''\xi; (\mu'', a'') \in \mathrm{SB}_N((\mu, a); \omega)\}^{5)}\}.$$

When $\bar{\omega}$ is non-compact, it is not easy to examine directly whether $\underline{\mathrm{SB}}_N((\mu, a); \omega) \neq \phi$ or $= \phi$.

Let $N \in (\mathrm{SBP})$ (resp. $N \in (\mathrm{SBP}_g)$). For $\mu \in D^+(N)$ with $\int d\mu < \infty$, $a \in R$ and a relatively compact open set (resp. an open set) $\omega \neq \phi$ in X, we can define $\mathrm{SB}_N((\mu, a); \omega)$ and $\underline{\mathrm{SB}}_N((\mu, a); \omega)$ analogously.

We shall use known results concerning potential theoretic principles for a real convolution kernel N on X (see Remarques 2, 7, Proposition 11 and Corollaire 14 in [8]).

Remark 7. (1) $N \in (\mathrm{SMP})$ and $N \in (\mathrm{SBP})$ are equivalent.

(2) Assume that $N \in (\mathrm{SMP})$. Then $(N, \xi) \in (\mathrm{TSMP})$ is equivalent to $\inf_{x \in X} N * f(x) \leqq 0$ for any $f \in C_K^0(X)$.

(3) Assume that $(N, \xi) \in (\mathrm{TSMP})$. Then N and $\check{N}$ satisfy the maximum principle, that is, for any $f \in C_K^+(X)$, we have $N * f(x) \leqq \sup_{y \in \mathrm{supp}(f)} N * f(y)$ on X and $\check{N} * f(x) \leqq \sup_{y \in \mathrm{supp}(f)} \check{N} * f(y)$ on X.

LEMMA 8. *Let $N \in (\mathrm{SMP})$ and $\omega \neq \phi$ be a relatively compact open set in X. Then we have:*

(1) *For any $\mu \in D^+(N)$ with $\int d\mu < \infty$ and any $a \in R$, we have $\underline{\mathrm{SB}}_N((\mu, a); \omega) \neq \phi$, and for any $(\mu', a') \in \underline{\mathrm{SB}}_N((\mu, a); \omega)$, there exist nets $(\mu_\alpha)_{\alpha \in A}$ in $M_K^+(X)$ and $(a_\alpha)_{\alpha \in A}$ in R such that $\mathrm{supp}(\mu_\alpha) \subset \omega$ and $(N * \mu_\alpha + a_\alpha \xi)_{\alpha \in A}$ converges increasingly to $N * \mu' + a'\xi$ on X along A.*

(2) *For $0 < c \in R$, we denote by $\mathrm{SP}_c(N)$ the vague closure of*

$$\left\{N * \nu + a\xi; \nu \in M_K^+(X), \int d\nu = c, a \in R\right\}.$$

For any $\eta \in \mathrm{SP}_c(N)$, there exists an element $(\mu', a') \in M_K^+(X) \times R$ such that

5) This means that $\inf\{N*\mu'' + a''\xi; (\mu'', a'') \in \mathrm{SB}_N((\mu, a); \omega)\}$ exists as a real Radon measure on X and it is equal to $N*\mu' + a'\xi$.

$\int d\mu' = c$, $\operatorname{supp}(\mu') \subset \bar{\omega}$, $N*\mu' + a'\xi = \eta$ *in* ω *and* $N*\mu' + a'\xi \leqq \eta$ *on* X.

Proof. The assertion (1) is proved in the same manner as in [8] (see Corollaire 12). We shall show the assertion (2). We choose nets $(\mu_\alpha)_{\alpha\in A}$ in $M_K^+(X)$ with $\int d\mu_\alpha = c$ and $(a_\alpha)_{\alpha\in A}$ in R such that $\lim_{\alpha\in A}(N*\mu_\alpha + a_\alpha\xi) = \eta$. Let $(\mu'_\alpha, a'_\alpha) \in \mathrm{SB}_N((\mu_\alpha, a_\alpha); \omega)$. Since $\int d\mu'_\alpha = \int d\mu_\alpha = c$, we may assume that $(\mu'_\alpha)_{\alpha\in A}$ converges vaguely. Put $\mu' = \lim_{\alpha\in A}\mu'_\alpha$. All μ'_α being supported by the compact set $\bar{\omega}$, we have $N*\mu' = \lim_{\alpha\in A} N*\mu'_\alpha$. This implies that $(a'_\alpha)_{\alpha\in A}$ converges. Putting $a' = \lim_{\alpha\in A} a'$, we see that (μ', a') is a required element.

We shall use a more general form of the semi-complete maximum principle.

PROPOSITION 9. *Let* $N \in (\mathrm{SMP})$, $(N, \xi) \in (\mathrm{TSMP})$, $\mu \in D^+(N)$ *with* $c = \int d\mu < \infty$, $a \in R$ *and let* $\eta \in \mathrm{SP}_c(N)$. *If* $N*\mu + a\xi \leqq \eta$ *in a certain open set containing* $\operatorname{supp}(\mu)$, *then the same inequality holds on* X.

For the proof of this proposition, we shall use the following known lemma.

LEMMA 10 (see Lemme 21 in [8]). *Let* $N \in (\mathrm{SMP})$ *and* $(\mu_\alpha)_{\alpha\in A}$ *be a net in* $M_K^+(X)$. *If* $\lim_{\alpha\in A}\int d\mu_\alpha = 0$ *and* $(N*\mu_\alpha)_{\alpha\in A}$ *converges vaguely, then there exists* $b \in R$ *such that* $\lim_{\alpha\in A} N*\mu_\alpha = b\xi$. *Furthermore, if* $(N, \xi) \in (\mathrm{TSMP})$, *then* $b \leqq 0$.

Proof of Proposition 9. If $\mu \in M_K^+(X)$, then our assertion follows from Remark 5 and Lemma 8. In general case, we choose an open exhaustion $(\omega_n)_{n=1}^\infty$ of X[6]. Let ω be an open set in X satisfying $\omega \supset \operatorname{supp}(\mu)$ and $N*\mu + a\xi \leqq \eta$ in ω. We may assume that $\omega \cap \omega_1 \neq \phi$. Put $\mu_n = \mu|_{\omega_n}$[7] and $\lambda_n = \mu - \mu_n$. Let $(\lambda'_n, a'_n) \in \underline{\mathrm{SB}}_N((\lambda_n, a); \omega \cap \omega_n)$. Then $(\mu_n + \lambda'_n, a'_n) \in \underline{\mathrm{SB}}_N((\mu, a); \omega \cap \omega_n)$, and Lemma 8 (1) gives

$$N*(\mu_n + \lambda'_n) + a'_n\xi \leqq \eta \qquad \text{on } X.$$

Hence it suffices to show that $\lim_{n\to\infty}(N*(\mu_n + \lambda'_n) + a'_n\xi) = N*\mu + a\xi$.

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6) An open exhaustion $(\omega_n)_{n=1}^\infty$ of X means a sequence of relatively compact open sets $\neq \phi$ in X satisfying $\omega_{n+1} \supset \bar{\omega}_n$ and $\cup_{n=1}^\infty \omega_n = X$.

7) Far $\mu \in M(X)$ and a universally measurable set E in X, $\mu|_E$ denotes the real Radon measure on X defined by $\mu|_E = \mu$ on E and $\mu|_E = 0$ on CE.

From $(N, \xi) \in (\mathrm{TSMP})$, we see that $a'_n \leqq a'_{n+1} \leqq a$ for all $n \geqq 1$, so that $(N * \lambda'_n)_{n=1}^{\infty}$ converges vaguely. By Lemma 10 and $\lim_{n\to\infty} \int d\lambda'_n = 0$, there exists $0 \geqq b \in R$ such that $\lim_{n\to\infty} N * \lambda'_n = b\xi$. Since

$$\lim_{n\to\infty} N * (\mu_n + \lambda'_n) + (\lim_{n\to\infty} a'_n)\xi = N * \mu + a\xi \qquad \text{in } \omega\,,$$

$\lim_{n\to\infty} a'_n = a$ and $b = 0$. Thus $N * (\mu_n + \lambda'_n) + a'_n\xi$ converges increasingly to $N * \mu + a\xi$ as $n \uparrow \infty$, which completes the proof.

Similarly we obtain the following

PROPOSITION 11. *Let $N \in (\mathrm{SBP}_g)$ and $(N, \xi) \in (\mathrm{TSMP})$. Then, for any $\mu \in M_K^+(X)$, any $a \in R$, any open set $\omega \neq \phi$ in X and any $(\mu', a') \in \mathrm{SB}_N((\mu, a); \omega)$, we have $a' \leqq a$. Furthermore, if $C\omega$ is compact, $a' = a$.*

Proof. Let $(\omega_n)_{n=1}^{\infty}$ be an open exhaustion of X. Put $\mu'_n = \mu'|_{\omega_n}$ and $\lambda_n = \mu' - \mu'_n$. Choose $(\lambda'_n, a'_n) \in \underline{\mathrm{SB}}_N((\lambda_n, a'); \omega_n)$; then $(\mu'_n + \lambda'_n, a'_n) \in \underline{\mathrm{SB}}_N((\mu', a'); \omega_n)$. Then $(N, \xi) \in (\mathrm{TSMP})$ gives $a'_n \leqq a$. From the above proof, we see that $\lim_{n\to\infty} a'_n = a'$, that is, $a' \leqq a$.

The latter part is shown in the same manner as in Proposition 28 (2) in [8].

It is a question when $a' = a$ holds.

§ 3.

In this paragraph, we shall prepare some potential theoretic results concerning shift-bounded Hunt convolution kernels.

DEFINITION 12. A non-negative convolution kernel N_0 on X is said to be a Hunt convolution kernel if it is of form

$$N_0 = \int_0^\infty \alpha_t dt \left(\text{i.e., for any } f \in C_K(X),\ \int f dN_0 = \int_0^\infty dt \int f d\alpha_t\right), \tag{3.1}$$

where $(\alpha_t)_{t\geqq 0}$ is a vaguely continuous convolution semi-group (of positive Radon measures) on X, i.e., $\alpha_0 =$ the unit measure ε at the origin 0, $\alpha_t * \alpha_s = \alpha_{t+s}$ for all $t \geqq 0$, $s \geqq 0$ and $t \to \alpha_t$ is vaguely continuous.

In this case, $(\alpha_t)_{t\geqq 0}$ is uniquely determined (see [5]) and called the convolution semi-group of N_0.

A vaguely continuous convolution semi-group $(\alpha_t)_{t\geqq 0}$ is said to be sub-markovian (resp. markovian) if $\int d\alpha_t \leqq 1$ $\left(\text{resp. } \int d\alpha_t = 1\right)$ for all $t \geqq 0$.

DEFINITION 13. A family $(N_p)_{p>0}$ of non-negative convolution kernels on X is said to be a resolvent if for any $p>0$ and $q>0$,

$$N_p - N_q = (q-p)N_p * N_q \quad \text{(The resolvent equation)}. \tag{3.2}$$

A non-negative convolution kernel N_0 on X possesses the resolvent if there exists a resolvent $(N_p)_{p>0}$ with $N_0 = \lim_{p\downarrow 0} N_p$.

In this case, $N_0 - N_p = pN_0 * N_p$ and $\operatorname{supp}(N_0) = \operatorname{supp}(N_p)$ $(p>0)$ hold, and $(N_p)_{p>0}$ is uniquely determined (see [5]). We call it the resolvent of N_0.

A resolvent $(N_p)_{p>0}$ is said to be sub-markovian (resp. markovian) if for any $p>0$, $p\int dN_p \leqq 1$ $\left(\text{resp. } p\int dN_p = 1\right)$.

The following results are fundamental for Hunt convolution kernels (see [1], [3], [5], [6] and [7]).

Remark 14. (1) A non-negative convolution kernel N_0 on X is a Hunt convolution kernel if and only if its resolvent exists and N_0 is non-periodic, i.e., for any $x \in X$, $N_0 \neq N_0 * \varepsilon_x$ provided with $x \neq 0$, where ε_x denotes the unit measure at x.

(2) Let N_0 be a Hunt convolution kernel on X. Then the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) hold:

(a) The convolution semi-group of N_0 is sub-markovian (resp. markovian).

(b) The resolvent of N_0 is sub-markovian (resp. markovian).

(c) N_0 is shift-bounded, i.e., for any $f \in C_K(X)$, $N * f$ is bounded on X $\left(\text{resp. shift-bounded and } \int dN_0 = \infty\right)$.

(3) Let N_0 be a shift-bounded Hunt convolution kernel on X. Then we have:

(a) (The balayability). For any $\mu \in M_K^+(X)$ and any open set ω in X, there exists $\mu' \in D^+(N_0)$ such that $\operatorname{supp}(\mu') \subset \bar{\omega}$, $N_0 * \mu' = N_0 * \mu$ in ω and $N_0 * \mu' \leqq N_0 * \mu$ on X.

In this case, μ' is called an N_0-balayaged measure of μ on ω, and $\int d\mu' \leqq d\mu$ holds. We have $\int dN_0 = \infty$ if and only if, for any $\mu \in M_K^+(X)$, any open set ω in X whose complement is compact and any N_0-balayaged measure μ' of μ on ω, $\int d\mu' = \int d\mu$.

(b) (The complete maximum principle). For any $\mu, \nu \in M_K^+(X)$ and any $0 \leqq c \in R$, $N_0 * \mu \leqq N_0 * \nu + c\xi$ in a certain neighborhood of $\operatorname{supp}(\mu)$

implies that the same inequality holds on X.

(c) (The equilibrium principle). For any relatively compact open set ω in X, there exists $\gamma \in M_K^+(X)$ such that $\operatorname{supp}(\gamma) \subset \bar{\omega}$, $N_0 * \gamma = \xi$ in ω and $N_0 * \gamma \leqq \xi$ on X.

In this case, γ is called an N_0-equilibrium measure of ω.

(d) (The positive mass principle). For any $\mu, \nu \in M_K^+(X)$, $N_0 * \mu \leqq N_0 * \nu$ on X implies $\int d\mu \leqq \int d\nu$.

(e) (The dominated convergence property). Let $(\mu_\alpha)_{\alpha \in A}$ be a net in $D^+(N_0)$ and $\mu \in M^+(X)$. If $\lim_{\alpha \in A} \mu_\alpha = \mu$ and there exists $\nu \in D^+(N_0)$ satisfying $N_0 * \mu_\alpha \leqq N_0 * \nu$ on X for all $\alpha \in A$, then $\lim_{\alpha \in A} N_0 * \mu_\alpha = N_0 * \mu$.

(f) (The injectivity). For any $\mu, \nu \in D^+(N_0)$, $N_0 * \mu = N_0 * \nu$ on X implies $\mu = \nu$.

For $\mu \in D^+(N_0)$ and an open set ω in X, we can define analogously N_0-balayaged measures of μ on ω and denote by $\mathrm{B}_{N_0}(\mu; \omega)$ their totality. It is well-known that $\mathrm{B}_{N_0}(\mu; \omega) \neq \phi$. Put

$$\underline{\mathrm{B}}_{N_0}(\mu; \omega) = \{\mu' \in \mathrm{B}_{N_0}(\mu; \omega); N_0 * \mu' = \inf\{N_0 * \mu''; \mu'' \in \mathrm{B}_{N_0}(\mu; \omega)\}\} \quad (\text{see}^{5)})$$

and

$$\bar{\mathrm{B}}_{N_0}(\mu; \omega) = \{\mu' \in \mathrm{B}_{N_0}(\mu; \omega); N_0 * \mu' = \sup\{N_0 * \mu''; \mu'' \in \mathrm{B}_{N_0}(\mu; \omega)\}\} \quad (\text{see}^{5)}) .$$

For an open set ω in X, we can define analogously N_0-equilibrium measures of ω and denote by $\mathrm{E}_{N_0}(\omega)$ their totality. Put

$$\underline{\mathrm{E}}_{N_0}(\omega) = \{\gamma \in \mathrm{E}_{N_0}(\omega); N_0 * \gamma = \inf\{N_0 * \gamma'; \gamma' \in \mathrm{E}_{N_0}(\omega)\}\} \quad (\text{see}^{5)})$$

provided with $\mathrm{E}_{N_0}(\omega) \neq \phi$.

LEMMA 15. *Let N_0 be a shift-bounded Hunt convolution kernel on X. Then we have:*

(1) *For any $\mu \in D^+(N_0)$ and any open set ω in X, $\underline{\mathrm{B}}_{N_0}(\mu; \omega) \neq \phi$ and $\bar{\mathrm{B}}_{N_0}(\mu; \omega) \neq \phi$. Moreover, $\underline{\mathrm{B}}_{N_0}(\mu; \omega)$ and $\bar{\mathrm{B}}_{N_0}(\mu; \omega)$ form only one element.*

(2) *For any $\mu \in D^+(N_0)$ and any two open sets ω_1, ω_2 in X with $\omega_1 \subset \omega_2$, we have $N_0 * \mu_1' \leqq N_0 * \mu_2'$ and $N_0 * \mu_1'' \leqq N_0 * \mu_2''$ on X, where $\mu_i' \in \underline{\mathrm{B}}_{N_0}(\mu; \omega_i)$ and $\mu_i'' \in \bar{\mathrm{B}}_{N_0}(\mu: \omega_i)$ $(i = 1, 2)$.*

(3) *Put $P(N_0) = \overline{\{\mathrm{N}_0 * \mu; \mu \in D^+(N_0)\}}$, where the closure is in the sense of the vague topology. For any $\mu \in D^+(N_0)$ and any $\eta \in P(N_0)$, $N_0 * \mu \leqq \eta$ in a certain open set $\supset \operatorname{supp}(\mu)$ implies that the same inequality holds on X.*

(4) *For an open set* ω *in* X, $\mathrm{E}_{N_0}(\mu) \neq \phi$ *implies* $\underline{\mathrm{E}}_{N_0}(\mu) \neq \phi$. *In this case,* $\underline{\mathrm{E}}_{N_0}(\mu)$ *forms only one element.*

(5) *For* $0 < c \in R$, *we put* $P_c(N_0) = \left\{N_0 * \mu;\ \mu \in D^+(N_0),\ \int d\mu \leqq c\right\}$. *For any* $\mu \in D^+(N_0)$ *and any* $\eta \in P_c(N_0)$, $N_0 * \mu \leqq \eta$ *on* X *implies* $\int d\mu \leqq c$.

(6) *Let* $(\mu_\alpha)_{\alpha \in A}$ *be a net in* $D^+(N_0)$ *and* $0 \neq \lambda_1$, $0 \neq \lambda_2 \in M_K^+(X)$. *If there exist* $\nu \in D^+(N_0)$ *and a relatively compact net* $(x_\alpha)_{\alpha \in A}$ *in* X *such that* $N_0 * \mu_\alpha * \lambda_1 \leqq N_0 * \nu * \varepsilon_{x_\alpha} * \lambda_2$ *on* X, *then* $(\mu_\alpha)_{\alpha \in A}$ *is vaguely bounded. If* $\mu_\alpha \to \mu \in M^+(X)$, *then* $\lim_{\alpha \in A} N_0 * \mu_\alpha = N_0 * \mu$.

Proof. (1) Let $(\omega_\alpha)_{\alpha \in A}$ be a net of open sets in X with $\bar{\omega}_\alpha \subset \omega_\beta (\alpha \lneqq \beta)$ and $\bigcup_{\alpha \in A} \omega_\alpha = \omega$. We choose $\mu'_\alpha \in B_{N_0}(\mu; \omega_\alpha)$. Then the complete maximum principle of N_0 implies that for any $\mu'' \in \mathrm{B}_{N_0}(\mu; \omega)$, $N_0 * \mu'_\alpha \leqq N_0 * \mu''$ on X. This and the dominated convergence property of N_0 show that $(\mu')_{\alpha \in A}$ is vaguely bounded and every vaguely accumulation point of $(\mu'_\alpha)_{\alpha \in A}$ as $\omega_\alpha \uparrow \omega$ is contained in $\underline{\mathrm{B}}_{N_0}(\mu; \omega)$, which gives $\underline{\mathrm{B}}_{N_0}(\mu; \omega) \neq \phi$. Let $(\omega'_{\alpha'})_{\alpha' \in A'}$ be a net of open sets in X with $\omega'_{\alpha'} \supset \bar{\omega}'_{\beta'}$ $(\alpha' \lneqq \beta')$ and $\bigcap_{\alpha' \in A'} \omega'_{\alpha'} = \bar{\omega}$. We choose $\mu''_{\alpha'} \in \mathrm{B}_{N_0}(\mu; \omega'_{\alpha'})$. Similarly as above, $(\mu''_{\alpha'})_{\alpha' \in A'}$ as $\omega'_{\alpha'} \downarrow \bar{\omega}$ is contained in $\bar{\mathrm{B}}_{N_0}(\mu; \omega)$, that is, $\bar{\mathrm{B}}_{N_0}(\mu; \omega) \neq \phi$. The injectivity of N_0 shows that $\underline{\mathrm{B}}_{N_0}(\mu; \omega)$ and $\bar{\mathrm{B}}_{N_0}(\mu; \omega)$ form only one element.

Consequently, let $\mu'_\alpha \in \underline{\mathrm{B}}_{N_0}(\mu; \omega_\alpha)$, $\mu' \in \underline{\mathrm{B}}_{N_0}(\mu; \omega)$, $\mu''_{\alpha'} \in \bar{\mathrm{B}}_{N_0}(\mu; \omega'_{\alpha'})$ and $\mu'' \in \bar{\mathrm{B}}_{N_0}(\mu; \omega)$; then $\lim_{\alpha \in A} \mu'_\alpha = \mu'$ and $\lim_{\alpha' \in A'} \mu''_{\alpha'} = \mu''$.

(2) Using the complete maximum principle of N_0 and noting the above proof, we see easily (2).

(3) Let $\nu \in M_K^+(X)$ with $\nu \leqq \mu$. We choose a relatively compact open set ω in X such that $\omega \supset \operatorname{supp}(\nu)$ and $N_0 * \nu \leqq \eta$ in ω. By virtue of the balayability of N_0, we can choose $\lambda \in M_K^+(X)$ such that $\operatorname{supp}(\lambda) \subset \bar{\omega}$, $N_0 * \lambda = \eta$ in ω and $N_0 * \lambda \leqq \eta$ on X. This shows that $N_0 * \nu \leqq N_0 * \lambda \leqq \eta$ on X, and ν being arbitrary, we have $N_0 * \mu \leqq \eta$ on X.

(4) In the same manner as in the proof of $\underline{\mathrm{B}}_{N_0}(\mu; \omega) \neq \phi$ in (1), we see that $\mathrm{E}_{N_0}(\omega) \neq \phi$ implies $\underline{\mathrm{E}}_{N_0}(\omega) \neq \phi$. For any $\gamma \in \mathrm{E}_{N_0}(\omega)$, $\underline{\mathrm{E}}_{N_0}(\omega) = \underline{\mathrm{B}}_{N_0}(\gamma; \omega)$. If $\underline{\mathrm{E}}_{N_0}(\omega) \neq \phi$, the injectivity of N_0 shows that $\underline{\mathrm{E}}_{N_0}(\omega)$ forms only one element.

(5) By using the positive mass principle of N_0 and the similar method to (3), we obtain (5).

(6) Evidently $(\mu_\alpha)_{\alpha \in A}$ is vaguely bounded. We shall show only the latter half part. Let $(K_n)_{n=1}^\infty$ be an exhaustion of X. We choose $\varepsilon'_n \in \mathrm{B}_{N_0}(\varepsilon, CK_n)$. The dominated convergence property of N_0 gives $\lim_{n \to \infty} N_0 * \varepsilon'_n = 0$. Let

$f \in C_K^+(X)$. Since $(x_\alpha)_{\alpha \in A}$ is relatively compact, $\int f dN_0 * \varepsilon_n' * \varepsilon_{x_\alpha} * \nu * \lambda_2$ converges uniformly to 0 on $(x_\alpha)_{\alpha \in A}$ as $n \to \infty$. Hence

$$\overline{\lim_{\alpha \in A}} \int f dN_0 * \mu_\alpha * \lambda_1 \leqq \int f dN_0 * \mu * \lambda_1 .$$

Using the lower semi-continuity of convolutions of non-negative Radon measures, we have $\underline{\lim}_{\alpha \in A} \int f dN_0 * \mu_\alpha \geqq \int f dN_0 * \mu$. Thus $\mu \in D^+(N_0)$ and $\lim_{\alpha \in A} N_0 * \mu_\alpha = N_0 * \mu$.

From Lemma 15 and its proof, we see the following

LEMMA 16. *Let N_0 be a shift-bounded Hunt convolution kernel on X, $(\Omega_j)_{j=1}^m$ and $(\omega_k)_{k=1}^n$ two finite families of open sets in X and let $(\mu_j)_{j=1}^m \subset D^+(N_0)$. Assume that $\mathrm{E}_{N_0}(\omega_k) \neq \phi$ $(k = 1, 2, \cdots, n)$. Let $\mu_j' \in \underline{\mathrm{B}}_{N_0}(\mu_j; \Omega_j)$, $\gamma_k \in \underline{\mathrm{E}}_{N_0}(\omega_k)$ $(j = 1, 2, \cdots, m; k = 1, 2, \cdots, n)$ and let $\eta \in P(N_0)$. If $\sum_{j=1}^m \sum_{k=1}^n N_0 * (\mu_j' + \gamma_k) \leqq \eta$ in $(\cup_{j=1}^m \Omega_j) \cup (\cup_{k=1}^n \omega_k)$, then the same inequality holds on X.*

LEMMA 17. *Let N_0 be the same as above, $\mu \in D^+(N_0)$ and let ω be an open set in X. For $x \in X$, we denote by μ_x' and μ_x'' the unique element in $\underline{\mathrm{B}}_{N_0}(\mu * \varepsilon_x; \omega)$ and that in $\overline{\mathrm{B}}_{N_0}(\mu * \varepsilon_x; \omega)$, respectively. Then we have:*

(1) *The mapping $x \to \mu_x'$ and $x \to \mu_x''$ are universally measurable, that is, for any $f \in C_K(X)$, the functions $\int f d\mu_x'$ and $\int f d\mu_x''$ of x are universally measurable on X.*

(2) *For any $\nu \in M_K^+(X)$, $(\mu * \nu)' \in \underline{\mathrm{B}}_{N_0}(\mu * \nu; \omega)$ and $(\mu * \nu)'' \in \overline{\mathrm{B}}_{N_0}(\mu * \nu; \omega)$ are of form*

$$(3.3) \qquad (\mu * \nu)' = \int \mu_x' d\nu(x)^{8)} \quad \textit{and} \quad (\mu * \nu)'' = \int \mu_x'' d\nu(x) .$$

Proof. Let $x \in X$ and $(x_\alpha)_{\alpha \in A}$ be a net in X with $x_\alpha \to x$. Then Lemma 15 (6) shows that $(\mu_{x_\alpha}')_{\alpha \in A}$ and $(\mu_{x_\alpha}'')_{\alpha \in A}$ are vaguely bounded and that every vaguely accumulation point of $(\mu_{x_\alpha}')_{\alpha \in A}$ and that of $(\mu_{x_\alpha}'')_{\alpha \in A}$ as $x_\alpha \to x$ belong to $\mathrm{B}_{N_0}(\mu * \varepsilon_x; \omega)$. This implies that the mapping $x \to N_0 * \mu_x'$ is lower semi-continuous (i.e., for any $f \in C_K^+(X)$, the function $\int f dN_0 * \mu_x'$ is lower semi-continuous) and the mapping $x \to N_0 * \mu_x''$ is upper semi-continuous. Let $(N_p)_{p>0}$ be the resolvent of N_0. Then, for any $p > 0$, $x \to N_0 * N_p * \mu_x'$

[8] This means that for any $f \in C_K(X)$, $\int f d(\mu * \nu)' = \iint f d\mu_x' d\nu(x)$

is also lower semi-continuous and $x \to N_0 * N_p * \mu''_x$ is also upper semi-continuous, because N_p is also a Hunt convolution kernel on X, so that N_p possesses the dominated convergence property. Hence, for any $f \in C_K(X)$ and any $p > 0$, the resolvent equation shows that $\int f dN_p * \mu'_x$ and $\int f dN_p * \mu''_x$ are universally measurable functions of x on X. Since $\lim_{p\to\infty} pN_p = \varepsilon$ and there exists $g \in C_K^+(X)$ such that $|p\check{N}_p * f| \leqq \check{N}_0 * g$ on X for all $p > 0$, the Lebesgue dominated convergence theorem gives $\int f d\mu'_x = \lim_{p\to\infty} p \int f dN_p * \mu'_x$ and $\int f d\mu''_x = \lim_{p\to\infty} p \int f dN_p * \mu''_x$, which show that $x \to \mu'_x$ and $x \to \mu''_x$ are universally measurable.

We shall show the assertion (2). For any $f \in C_K^+(X)$, $\iint f d\mu'_x d\nu(X)$ and $\iint f d\mu''_x d\nu(x)$ are defined and

$$\iint \check{N}_0 * f d\mu'_x d\nu(x) \leqq \iint \check{N}_0 * f d\mu''_x d\nu(x) \leqq \int f dN_0 * (\mu * \nu) ,$$

so that $\int \mu'_x d\nu(x)$ and $\int \mu''_x d\nu(x)$ belong to $D^+(N_0)$. We see easily that $\int \mu'_x d\nu(x)$, $\int \mu''_x d\nu(x) \in \mathrm{B}_{N_0}(\mu * \nu; \omega)$. Let $(\omega_\alpha)_{\alpha \in A}$ be a net of open sets in X satisfying $\bar{\omega}_\alpha \subset \omega_\beta (\alpha \lneqq \beta)$ and $\bigcup_{\alpha \in A} \omega_\alpha = \omega$. We choose $\mu'_{x,\alpha} \in \underline{\mathrm{B}}_{N_0}(\mu * \varepsilon_x; \omega_\alpha)$. Then Lemma 15 (1), (3) show that $N_0 * \mu'_{x,\alpha} \uparrow N_0 * \mu'_x$ as $\omega_\alpha \uparrow \omega$, that is,

$$N_0 * \left(\int \mu'_{x,\alpha} d\nu(x)\right) \uparrow N_0 * \left(\int \mu'_x d\nu(x)\right) \qquad \text{as} \quad \omega_\alpha \uparrow \omega .$$

This shows that $\int \mu'_x d\nu(x) \in \underline{\mathrm{B}}_{N0}(\mu * \nu; \omega)$, and Lemma 15 (1) gives the first equality in (3.3). Let $(\omega'_{\alpha'})_{\alpha' \in A'}$ be a net of open sets in X satisfying $\omega'_{\alpha'} \supset \bar{\omega}'_{\beta'}$ $(\alpha' \lneqq \beta')$ and $\bigcap_{\alpha' \in A'} \omega'_{\alpha'} = \bar{\omega}$. We choose $\mu''_{x,\alpha'} \in \bar{\mathrm{B}}_{N_0}(\mu * \varepsilon_x; \omega'_{\alpha'})$. Similarly as above, we have

$$N_0 * \left(\int \mu''_{x,\alpha'} d\nu(x)\right) \downarrow N_0 * \left(\int \mu''_x d\nu(x)\right) \qquad \text{as} \quad \omega'_{\alpha'} \downarrow \bar{\omega} ,$$

and hence the second equality in (3.3) holds. Thus Lemma 17 is shown.

The following proposition will play an important role to prove our main theorem.

PROPOSITION 18. *Let N_0 be a shift-bounded Hunt convolution kernel on X and assume that the closed subgroup generated by* supp(N_0) *is equal to X. Then, for any $0 \neq \mu \in M_K^+(X)$, there exist an open set $\omega \neq \phi$ in X*

and an open neighborhood V of the origin such that:

(1) *For any* $\mu' \in \mathrm{B}_{N_0}(\mu, \omega + V)$[9], $\int d\mu' < \int d\mu$.

(2) N_0*-equlibrium measures of* ω *with finite total mass do not exist.*

For the poof of this proposition, we use the following result:

LEMMA 19 (see [2], [4]). *Let* $\sigma \in M^+(X)$ *with* $\int d\sigma = 1$. *If a shift-bounded real Radon measure* μ *on* X *satisfies* $\mu = \mu * \sigma$, *then, for any* x *in the closed subgroup* Γ *generated by* $\mathrm{supp}(\sigma)$, *we have* $\mu = \mu * \varepsilon_x$, *that is, each* x *in* Γ *is a period of* μ.

Proof of Proposition 18. It suffices to show the following assertion: Let $0 \neq f \in C_K^+(X)$. Then there exist an open set $\omega \neq \phi$ in X and open neighborhood V of the origin such that:

(1′) For $(f\xi)'' \in \bar{\mathrm{B}}_{N_0}(f\xi; \omega + V)$, $\int d(f\xi)'' < \int f d\xi$.

(2′) $\mathrm{E}_{N_0}(\omega) = \phi$, or $\mathrm{E}_{N_0}(\omega) \neq \phi$ and for $\gamma \in \underline{\mathrm{E}}_{N_0}(\omega)$, $\int d\gamma = \infty$.

In fact, admit this assertion and let $0 \neq \mu \in M_K^+(X)$. Choose $\varphi \in C_K^+(X)$ with $\int \varphi d\xi = 1$. Then there exist an open set $\omega \neq \phi$ in X and an open neighborhood V of the origin such that, for $f = \mu * \varphi$, (1′) and (2′) are verified. Since $\int d\mu = \int \mu * \varphi d\xi$, Lemma 17 (2) shows that there exists $x \in \mathrm{supp}(\varphi)$ such that for $(\mu * \varepsilon_x)'' \in \bar{\mathrm{B}}_{N_0}(\mu * \varepsilon_x; \omega + V)$, $\int d(\mu * \varepsilon_x)'' < \int d\mu * \varepsilon_x = \int d\mu$. We remark here that $(\mu * \varphi)\xi = \int \mu * \varepsilon_x \varphi(x) d\xi(x)$ and for any $y \in X$, $\int d(\mu * \varepsilon_y)'' \leqq \int d\mu * \varepsilon_y$. Put $\omega_x = \omega - \{x\}$ and $\mu_x'' \in \bar{\mathrm{B}}_{N_0}(\mu; \omega_x + V)$. Then we see easily that $(\mu * \varepsilon_x)'' = \mu_x'' * \varepsilon_x$, which implies $\int d\mu_x'' < \int d\mu$. We remark that $\mathrm{E}_{N_0}(\omega) = \phi$ and $\mathrm{E}_{N_0}(\omega_x) = \phi$ are equivalent and if $\mathrm{E}_{N_0}(\omega) \neq \phi$, then, for $\gamma \in \underline{\mathrm{E}}_{N_0}(\omega)$ and $\gamma_x \in \underline{\mathrm{E}}_{N_0}(\omega_x)$, $\gamma = \gamma_x * \varepsilon_x$. By the positive mass principle and Lemma 15 (5), we see that ω_x and V are our required open set and open neighborhood of the origin.

Dividing into the following two cases, we shall show our required assertion.

(a) Assume that there exists $0 \neq g \in C_X^+(X)$ with $\overline{\lim}_{x \to \delta} N_0 * g(x) > 0$. Then $\int dN_0 = \infty$. Noting that $(N_0 * \varepsilon_x)_{x \in X}$ is vaguely bounded, we can

[9] For subsets A, B of X, $A + B = \{x + y; x \in A, y \in B\}$, $-B = \{-x; x \in B\}$.

choose a net $(x_\alpha)_{\alpha\in A}$ in X with $x_\alpha \to \delta$ such that $(N_0 * \varepsilon_{x_\alpha})_{\alpha\in A}$ converges vaguely and $\lim_{\alpha\in A} N_0 * \varepsilon_{x_\alpha} * g(0) = \overline{\lim}_{x\to\delta} N_0 * g(x)$. Put $\eta = \lim_{\alpha\in A} N_0 * \varepsilon_{x_\alpha}$; then $\eta \neq 0$. Let $(N_p)_{p>0}$ be the resolvent of N_0. By the resolvent equation and $p\int dN_p = 1$ $(p > 0)$, we have

$$\eta = pN_p * \eta \,(p > 0)\,.$$

Since $\operatorname{supp}(N_p) = \operatorname{supp}(N_0)$ $(p > 0)$ and η is shift-bounded, Lemma 19 gives $\eta = c\xi$ with some constant $c > 0$. We may assume that $\int f d\xi = 1$. Let Ω be a relatively compact open set with $\Omega \supset \operatorname{supp}(f)$. Since $(N_0 * \varepsilon_x * f)_{x\in X}$ converges uniformly to $N_0 * f$ on $\bar{\Omega}$ as $x \to 0$, there exists an open neighborhood V of the origin such that $V = -V$, $\operatorname{supp}(f) + \bar{V} \subset \Omega$ and for any $x \in \bar{V}$, $|N_0 * \varepsilon_x * f - N_0 * f| < \frac{1}{3}c$ on $\bar{\Omega}$. By virtue of the complete maximum principle of N_0, we have $|N_0 * \varepsilon_x * f - N_0 * f| < \frac{1}{3}c$ on X for all $x \in \bar{V}$. Put $\omega = \{x \in X; N_0 * f(x) < \frac{1}{3}c\}$ and $\omega = \{x \in X; N_0 * f(x) < \frac{2}{3}c\}$. Then $\bar{\omega} + \bar{V} \subset \omega'$. We shall show that ω and V are our required open set and open neighborhood of the origin. First we see that $\mathrm{E}_{N_0}(\omega) = \phi$, because, if there exists $\gamma \in \mathrm{E}_{N_0}(\omega)$, then $N_0 * (\frac{1}{3}c\gamma + f\xi) \geqq \frac{1}{3}c\xi$ on X, which contradicts $p\int dN_p = 1$ for all $p > 0$ and $pN_p * N_0 \downarrow 0$ as $p \downarrow 0$. It remains to prove that (1′) is verified. By Lemma 15 (2), it suffices to show that for any $(f\xi)' \in B_{N_0}(f\xi;\, \omega')$, $\int d(f\xi)' < \int f d\xi = 1$. For any integer $m \geqq 1$, $N_0 * (f\xi)' \leqq (\frac{2}{3} + 1/m)\eta$ in a certain open set $\supset \operatorname{supp}((f\xi)')$, so that Lemma 15 (3) gives $N_0 * (f\xi)' \leqq (\frac{2}{3} + 1/m)\eta$ on X. Letting $m \uparrow \infty$ and using Lemma 15 (5), we obtain $\int \mathrm{d}(f\xi)' \leqq \frac{2}{3}$. Thus ω and V are our required open set and open neighborhood of the origin.

(b) Assume that N_0 vanishes at the infinity (i.e., for any $g \in C_K(X)$, $\lim_{x\to\delta} N_0 * g(x) = 0$). Let U_0 be a relatively compact open set $\neq \phi$ in X with $\bar{U}_0 \subset \{x \in X; f(x) > 0\}$. Since $\operatorname{supp}(N_0) \ni 0$, we may assume that $N_0 * f(x) > 1$ on $\bar{U}_0$. We choose an open set $\omega_0 \neq \phi$ and an open neighborhood V of the origin such that $\bar{\omega}_0 + \bar{V} \subset U_0$. Since $\lim_{x\to\delta} N_0 * \varepsilon_x = 0$, we can choose inductively a sequence $(x_n)_{n=0}^{\infty}$ in X with $x_0 = 0$ and $x_n \to \delta$ $(n \to \infty)$ such that, for any $n \geqq 0$ and $m \geqq 0$ with $n \neq m$,

$$N_0 * \varepsilon_{x_n} * f \leqq \frac{1}{2^{|n-m|+1}} \qquad \text{on } \{x_m\} + U_0\,.$$

Put $U_n = \{x_n\} + U_0$ $(n = 1, 2, \cdots)$ and $U = \bigcup_{n=1}^{\infty} U_n$. Evidently $\bar{U}_m \cap \bar{U}_n = \phi$

if $n \neq m$. Put $\omega_n = \{x_n\} + \omega_0$ $(n = 1, 2, \cdots)$ and $\omega = \bigcup_{n=1}^{\infty} \omega_n$. Then $\omega + V \subset U$. For any $(f\xi)' \in \mathrm{B}_{N_0}(f\xi; U)$, we set $(f\xi)'_n = (f\xi)'|_{\bar{U}_n}$ $(n \geqq 1)$. Then, by virtue of the complete maximum priciple of N_0,

$$N_0 * (f\xi)'_n \leqq \frac{1}{2^{n+1}} (N_0 * \varepsilon_{x_n} * f) \qquad \text{on } X,$$

and hence $\int d(f\xi)'_n \leqq (1/2^{n+1}) \int f d\xi$. Consequently, $\int d(f\xi)' \leqq \frac{1}{2} \int f d\xi$. From Lemma 15 (2), it follows that for $(f\xi)'' \in \bar{\mathrm{B}}_{N_0}(f\xi; \omega + V)$, $\int d(f\xi)'' \leqq \frac{1}{2} \int f d\xi$.

Let $\gamma'_n \in \underline{\mathrm{E}}_{N_0}(\omega_n)$. Then $N_0 * \gamma'_n \leqq (N_0 * \varepsilon_{x_n} * f)\xi$ on X. For any $n \geqq 1$ and any k with $1 \leqq k \leqq n$, we have, in ω_k,

$$N_0 * (\sum_{j=1}^{n} \gamma'_j) \leqq \xi + \sum_{j=1}^{k-1} (N_0 * \varepsilon_{x_j} * f)\xi + \sum_{j=k+1}^{n} (N_0 * \varepsilon_{x_j} * f)\xi \leqq 2\xi,$$

that is, $N_0 * (\sum_{j=1}^{n} \gamma'_j) \leqq 2\xi$ in $\bigcup_{j=1}^{n} \omega_j$. This and Lemma 16 show that the same inequality holds on X. Thus $\sum_{n=1}^{\infty} \gamma'_n$ converges vaguely. Put $\gamma' = \sum_{n=1}^{\infty} \gamma'_n$; then $N_0 * \gamma' \geqq \xi$ in ω and $N_0 * \gamma' \leqq 2\xi$ on X. Let $\gamma_n \in \underline{\mathrm{E}}_{N_0}(\bigcup_{k=1}^{n} \omega_k)$. Then $N_0 * \gamma' \geqq N_0 * \gamma_n$ and $\sum_{k=1}^{n} N_0 * \gamma'_k \leqq 2N_0 * \gamma_n$ on X. By virtue of the dominated covergence property of N_0, we have $\underline{\mathrm{E}}_{N_0}(\omega) \neq \phi$. Let $\gamma \in \underline{\mathrm{E}}_{N_0}(\omega)$; then $\lim_{n\to\infty} \gamma_n = \gamma$. This implies that

$$N_0 * \gamma \leqq N_0 * \gamma' \leqq 2N_0 * \gamma \qquad \text{on } X.$$

Evidently $\int d\gamma'_n = \int d\gamma'_m$ for all $n \geqq 1$, $m \geqq 1$ and $\gamma' \neq 0$, so that $\int d\gamma' = \infty$. The positive mass principle of N_0 gives $\int d\gamma = \infty$. Thus ω and V are our required open set and open neighborhood of the origin.

It is a question if there exist an open set $\omega \neq \phi$ in X and an open neighborhhod V of the origin such that for any $0 \neq \mu \in M_K^+(X)$ with $\mathrm{supp}(\mu) \subset C(\overline{\omega + V})$ and any $\mu' \in \mathrm{B}_{N_0}(\mu; \omega + V)$, $\int d\mu' < \int d\mu$ and N_0-equilibrium measures γ of ω with $\int d\gamma < \infty$ do not exist.

§4.

We return to the argument of real convolution kernels. We begin with the following

DEFINITION 20. For a real convolution kernel N on X and an open set $\omega \neq \phi$ in X, we denote by $\mathrm{SP}_1(N; \omega)$ the vague closure of

$$\left\{N * \mu + a\xi;\ \mu \in M_K^+(X),\ \int d\mu = 1,\ \mathrm{supp}(\mu) \subset \omega,\ a \in R\right\}$$

and put

$$\eta_{N,\omega} = \sup\{\eta \in \mathrm{SP}_1(N;\omega);\ \eta \leqq N \text{ on } X\}$$

provided that the right hand exists in $M(X)$. If $\eta_{N,\omega}$ exists, we call it the N-reduced measure of N on ω.

Assume that $N \in$ (SMP). Then $\eta_{N,\omega}$ always exists and satisfies $\eta_{N,\omega} = N$ in ω, $\eta_{N,\omega} \leqq N$ on X (see Remarque 19 in [8]). Let $(K_n)_{n=1}^{\infty}$ be an exhaustion of X. Then $(\eta_{N,CK_n})_{n=1}^{\infty}$ is decreasing and $\lim_{n\to\infty} \eta_{N,CK_n}$ is independent of the choice of $(K_n)_{n=1}^{\infty}$ (see § 3 in [8]). Put $\eta_{N,\delta} = \lim_{n\to\infty} \eta_{N,CK_n}$. Then $\eta_{N,\delta} = -\infty$, i.e., for any $0 \neq f \in C_K^+(X)$, $\lim_{n\to\infty} \int f d\eta_{N,CK_n} = -\infty$, or $\eta_{N,\delta} \in M(X)$ (see Remarque 19 in [8]).

Proposition 9 gives immediately the following

Remark 21. Let $N \in (\mathrm{SMP}_g)$, $(N, \xi) \in$ (TSMP), $(K_n)_{n=1}^{\infty}$ be an exhaustion of X and let $(\varepsilon'_{CK_n}, 0) \in \mathrm{SB}_N((\varepsilon, 0); CK_n)$ (see Proposition 11). Then, for any $n \geqq 2$,

$$\eta_{N,CK_n} \leqq N * \varepsilon'_{CK_n} \leqq \eta_{N,CK_{n-1}} \qquad \text{on } X .$$

The following proposition is shown in [8] (see Théorème 20).

PROPOSITION 22. *Let $N \in$ (SMP), $(N, \xi) \in$ (TSMP) and let $(\omega_n)_{n=1}^{\infty}$ be an open exhaustion of X. Then we have:*

(1) *For any $0 < p \in R$ and any $n \geqq 1$, there exists a uniquely determined $(\varepsilon'_{p,n}, a_{p,n}) \in M_K^+(X) \times R$ such that $\int d\xi'_{p,n} = 1$, $\mathrm{supp}(\varepsilon'_{p,n}) \subset \bar{\omega}_n$, $(N + (1/p)\varepsilon) * \varepsilon'_{p,n} + a_{p,n}\xi = N$ in ω_n, $(N + (1/p)\varepsilon) * \varepsilon'_{p,n} + a_{p,n}\xi \leqq N$ on X and for any $\nu \in M_K^+(X)$ with $\int d\nu = 1$ and any $a \in R$, $(N + (1/p)\varepsilon) * \nu + a\xi \geqq (N + (1/p)\varepsilon) * \varepsilon'_{p,n} + a_{p,n}\xi$ on X whenever $(N + (1/p)\varepsilon) * \nu + a\xi \geqq N$ in ω_n.*

(2) *Put $V_{p,\omega_n}\varepsilon = (1/p)\varepsilon'_{p,n}$. Then $V_{p,\omega_n}\varepsilon \geqq V_{p,\omega_{n+1}}\varepsilon$ in ω_n and $\lim_{n\to\infty} V_{p,\omega_n}\varepsilon$ exists.*

(3) *Put*

$$N_p = \lim_{n\to\infty} V_{p,\omega_n}\varepsilon (\in M^+(X)) , \tag{4.1}$$

then $(N_p)_{p>0}$ is a sub-markovian resolvent and independent of the choice of $(\omega_n)_{n=1}^{\infty}$.

By using Proposition 22, we have the following

LEMMA 23. *Let* $N \in (\mathrm{SBP}_g)$, $(N, \xi) \in (\mathrm{TSMP})$ *and assume that* N *is non-periodic. Then there exists a uniquely determined resolvent* $(N_p)_{p>0}$ *such that*

$$N = pN * N_p + N_p \,. \tag{4.2}$$

Proof. First we remark that $N \in (\mathrm{SBP})$ and $N \in (\mathrm{SMP})$ are equivalent. Let $V_{p,\omega_n}\varepsilon$, N_p and $a_{p,n}$ be the same as in Proposition 22. Then, for any $p > 0$,

$$\lim_{n\to\infty} ((pN + \varepsilon) * V_{p,\omega_n}\varepsilon + a_{p,n}\xi) = N \,. \tag{4.3}$$

Let $(K_m)_{m=1}^{\infty}$ be an exhaustion of X with $K_1 \ni 0$. We shall show that for any $m \geqq 2$, $N \neq \eta_{N,CK_m}$. Assume contrary that for an $m \geqq 2$,$N = \eta_{N,CK_m}$. Then Remark 21 gives $N = N * \varepsilon'_{CK_m}$, where $(\varepsilon'_{CK_m}, 0) \in \mathrm{SB}_N((\varepsilon, 0); CK_m)$. Let Γ be the closed subgroup generated by $\mathrm{supp}\,(\xi'_{CK_m})$; then $\Gamma \neq \{0\}$. For any $x \in X$, $N * (\varepsilon - \varepsilon_x)$ is shift-bounded (see Remarque 4 in [8]), and Lemma 19 shows that for any $y \in \Gamma$, $N * (\varepsilon - \varepsilon_x) * \varepsilon_y = N * (\varepsilon - \varepsilon_x)$. This implies that for any $x \in \Gamma$ and any integer $n \geqq 1$, $N - N * \varepsilon_{nx} = n(N - N * \varepsilon_x)$. Since for any $f \in C_K^+(X)$, $\check{N} * f$ is upper bounded (see Remark 7 (3)), we have $\int f d(N - N * \varepsilon_x) \geqq 0$, and Γ being a subgroup of X, we see that $N = N * \varepsilon_x$ for all $x \in \Gamma$. This contradicts the non-periodicity of N. Thus $N \neq \eta_{N,CK_m}$ for all $m \geqq 2$. Next we shall show that $(N_p)_{p>0}$ is markovian. From (4.1), $\int d\varepsilon'_{CK_m} = 1$ and $(pN + \varepsilon) * V_{p,\omega_n}\varepsilon + a_{p,n}\xi \uparrow N$ as $n \uparrow \infty$, it follows that

$$N - N * \varepsilon'_{CK_m} = p(N - N * \varepsilon'_{CK_m}) * N_p + N_p * (\varepsilon - \varepsilon'_{CK_m}) \,. \tag{4.4}$$

Assume that $(N_p)_{p>0}$ is not markovian. Then, for any $p > 0$, $p\int dN_p < 1$. From (4.4), it follows that for any $p > 0$, any $n \geqq 1$ and any $m \geqq 1$,

$$N - N * \varepsilon'_{CK_m} = (N - N * \varepsilon'_{CK_m}) * (pN_p)^n + \frac{1}{p}\sum_{k=1}^{n} (pN_p)^k * (\varepsilon - \varepsilon'_{CK_m}) \,,$$

where $(pN_p)^1 = pN_p$ and $(pN_p)^n = (pN_p)^{n-1} * (pN_p)$ $(n \geqq 2)$. Letting $n \uparrow \infty$, we have

$$N - N * \varepsilon'_{CK_m} = \frac{1}{p}\sum_{k=1}^{\infty} (pN_p)^k * (\varepsilon - \varepsilon'_{CK_m}) \,.$$

Since $\int d(\sum_{k=1}^{\infty}(pN_p)^k) < \infty$ and $\int d\varepsilon'_{CK_m} = 1$, we have $\int d(N - N * \varepsilon'_{CK_m}) = 0$, so that $N = N * \varepsilon'_{CK_m}$. This contradicts $N \neq \eta_{N,CK_m}$ and $\eta_{N,CK_{m-1}} \geqq N * \varepsilon'_{CK_m}$ $(m \geqq 2)$. Thus $(N_p)_{p>0}$ is markovian. In the same manner as in [8] (see Théorème 20 and Remarque 24), we see the rest of the proof.

DEFINITION 24. Let $N \in$ (SMP). If a sub-markovian resolvent $(N_p)_{p>0}$ satisfying (4.2) exists, then $(N_p)_{p>0}$ is called the resolvent associated with N.

The resolvent associated with N is uniquely determined if it exists (see Remarque 24 in [8]).

LEMMA 25. *Let $N \in$ (SMP) and $(N, \xi) \in$ (TSMP). Assume that $\eta_{N,\delta} \neq -\infty$, N is non-periodic and that the resolvent $(N_p)_{p>0}$ associated with N exists and is markovian. Put $N' = \eta_{N,\delta}$ and $N_0 = N - N'$. Then N_0 is a shift-bounded Hunt convolution kernel on X, $N_0 = \lim_{p\to 0} N_p$ and every point in the closed subgroup generated by* supp (N_0) *is a period on N'.*

Proof. Let $(K_n)_{n=1}^{\infty}$ and $(\omega_m)_{m=1}^{\infty}$ be an exhaustion of X and an open exhaustion of X, respectively. We choose $(\varepsilon'_{n,m}, a_{n,m})$ $\underline{\mathrm{SB}}_N((\varepsilon,0); CK_n \cap \omega_m)$ whenever $CK_n \cap \omega_m \neq \phi$. Then $N * \varepsilon'_{n,m} + a'_{n,m}\xi \uparrow \eta_{N,CK_n}$ as $m \uparrow \infty$ (see Remarque 19 in [8]). Here we may assume that $(\varepsilon'_{n,m})_{m=1}^{\infty}$ converges vaguely as $m \to \infty$. Put $\varepsilon'_n = \lim_{m\to\infty} \varepsilon'_{n,m}$; then $\int d\varepsilon'_n \leqq 1$. Since $\int dN_p = 1/p$ $(p > 0)$, we have, for any $p > 0$ and any $n \geqq 1$,

$$
\begin{aligned}
(4.5) \qquad p(N - \eta_{N,CK_n}) * N_p &= \lim_{m\to\infty} p(N - N * \varepsilon'_{n,m} - a_{n,m}\xi) * N_p \\
&= \lim_{m\to\infty} (N - N * \varepsilon'_{n,m} - N_p + N_p * \varepsilon'_{n,m} - a_{n,m}\xi) \\
&= N - \eta_{N,CK_n} - N_p + N_p * \varepsilon'_n .
\end{aligned}
$$

Letting $n \uparrow \infty$, we have $pN_0 * N_p = N_0 - N_p$. Letting $p \downarrow 0$ in (4.5), we have $\lim_{p\downarrow 0} N_p \geqq N - \eta_{N,CK_n}$. Thus we see $\lim_{p\to 0} N_p = N_0$, that is, $(N_p)_{p>0}$ is the resolvent of N_0. Since N is non-periodic, (4.2) shows that N_p is also non-periodic $(p > 0)$, which implies that N_0 is also non-periodic. Remark 14 (1), (2) show that N_0 is a shift-bounded Hunt convolution kernel. On the other hand, we have $pN' * N_p = N'$ for all $p > 0$. Let Γ be the closed subgroup generated by supp (N_0). For any $x \in X$, $N - N * \varepsilon_x$ is shift-bounded (see Remarque 4 in [8]), and $N' \in \mathrm{SP}_1(N)$ gives the shift-boundedness of $N' - N' * \varepsilon_x$. Lemma 19 and supp $(N_0) =$ supp (N_p) $(p > 0)$ show that for any $y \in \Gamma$, $(N' - N' * \varepsilon_x) * \varepsilon_y = N' - N' * \varepsilon_x$. This implies that for any $x \in \Gamma$ and any integer $n \geqq 1$, $N' - N' * \varepsilon_{nx} = n(N' - N' * \varepsilon_x)$. For any $f \in C_K^+(X)$, we have $\check{N}' * f(x) \leqq \check{N} * f(x) \leqq \sup_{y \in \mathrm{supp}(f)} \check{N} * f(y)$ on X. Similarly as in Lemma 23, we have $N' = N' * \varepsilon_x$ for all $x \in \Gamma$. Thus every point in Γ is a period of N'.

We shall give the proof of the "only if" part in Theorem 2. By Remark 7, it suffices to show the following

Proposition 26. *If a real convolution kernel N on X is semi-balayable, non-periodic and satisfies $(N, \xi) \in$ (TSMP), then $\eta_{N,\delta} = -\infty$.*

Proof. Assume contrary that $\eta_{N,\delta} \neq -\infty$. Then $\eta_{N,\delta} \in M(X)$. Put $N' = \eta_{N,\delta}$ and $N_0 = N - N'$. We denote by Γ the closed subgroup generated by $\mathrm{supp}\,(N_0)$. First we shall show that $N' \in$ (SMP). Let $\mu, \nu \in M_K^+(X)$ with $\int d\mu = \int d\nu \neq 0$ and $a \in R$. Assume that $N' * \mu \leqq N' * \nu + a\xi$ in a certain open set $\omega' \supset \mathrm{supp}\,(\mu)$. By Lemma 23 and Lemma 25, we have $N' * \mu \leqq N' * \nu + a\xi$ in $\omega' + \Gamma$. We choose a relatively compact open set ω in X such that $\omega' \supset \bar{\omega} \supset \omega \supset \mathrm{supp}\,(\mu)$. Let $(\mu', a') \in \mathrm{SB}_N(\mu, 0)$; $C(\bar{\omega} + \Gamma)$). Then $N * \mu' + a'\xi \leqq N' * \mu$ in $C(\mathrm{supp}\,(\mu) + \Gamma)$. Put $c = \int d\mu$. Then $N * \mu \in \mathrm{SP}_c(N)$. Hence Proposition 9 gives $N * \mu' + a'\xi \leqq N' * \mu$ on X. Evidently $N * \mu' + a'\xi = N' * \mu$ in $C(\bar{\omega} + \Gamma)$. For an exhaustion $(K_n)_{n=1}^\infty$ of X, we choose $\varepsilon'_{CK_n} \in \mathrm{B}_{N_0}(\varepsilon; CK_n)$. Then $\mathrm{supp}\,(\varepsilon'_{CK_n}) \subset \Gamma$ and $\int d\varepsilon'_{CK_n} = 1$ (see Remark 14 (2) and Lemmas 23, 25), so that

$$N * \mu' * \varepsilon'_{CK_n} + a'\xi \leqq N' * \mu * \varepsilon'_{CK_n} = N' * \mu \qquad \text{on } X$$

and

$$N * \mu' * \varepsilon'_{CK_n} + a'\xi = N' * \mu \qquad \text{in } C(\bar{\omega} + \Gamma)\,.$$

Letting $n \uparrow \infty$, we obtain that

$$N' * \mu' + a'\xi \leqq N' * \mu \ \text{ on } X \text{ and } N' * \mu' + a'\xi = N' * \mu \ \text{ in } C(\bar{\omega} + \Gamma)\,,$$

because $\lim_{n \to \infty} N_0 * \varepsilon'_{CK_n} = 0$. Hence $N' * \mu' = N * \mu'$ in $C(\bar{\omega} + \Gamma)$, which shows that $\mathrm{supp}\,(N_0 * \mu') \subset \bar{\omega} + \Gamma$. This implies $\mathrm{supp}\,(\mu') \subset \bar{\omega} + \Gamma$. On the other hand, $\mathrm{supp}\,(\mu') \subset \overline{C(\bar{\omega} + \Gamma)}$, that is, $\mathrm{supp}\,(\mu')$ is contained in the boundary $\partial(\bar{\omega} + \Gamma)$ of $\bar{\omega} + \Gamma$. Thus $N * \mu' + a'\xi \leqq N' * \mu \leqq N' * \nu + a\xi$ in $\omega' + \Gamma \supset \mathrm{supp}\,(\mu')$, and Proposition 9 gives $N * \mu' + a'\xi \leqq N' * \nu + a\xi$ on X. This implies $N' * \mu \leqq N' * \nu + a\xi$ in $C(\bar{\omega} + \Gamma)$, that is, $N' * \mu \leqq N' * \nu + a\xi$ on X, which shows that $N' \in$ (SMP). From $(N, \xi) \in$ (TSMP) and $N' \in$ (SMP), we see also $(N', \xi) \in$ (TSMP).

Evidently N_0 may be considered as a shift-bounded Hunt convolution kernel on Γ. We denote by ξ_Γ a fixed Haar measure on Γ. Proposition 18 shows that, for any positive Radon measure $\mu \neq 0$ on Γ with compact support (i.e., $\mu \in M_K^+(\Gamma)$), there exist an open set $\omega_\Gamma \neq \phi$ in Γ and a relatively compact open neighborhood V_Γ of the origin in Γ such that:

(A) For any $\mu'' \in \mathrm{B}_{N_0,\Gamma}(\mu; \omega_\Gamma + V_\Gamma)$, $\int d\mu'' < \int d\mu$.

(B) $\mathrm{E}_{N_0,\Gamma}(\omega_\Gamma) = \phi$, or $\mathrm{E}_{N_0,\Gamma}(\omega_\Gamma) \neq \phi$ and for any $\gamma \in \mathrm{E}_{N_0,\Gamma}(\omega_\Gamma)$, $\int d\gamma = \infty$,

where N_0 being considered as a shift-bounded Hunt convolution kernel on Γ, $\mathrm{B}_{N_0,\Gamma}(\mu; \omega_\Gamma + V_\Gamma)$ denotes the totality of N_0-balayaged measures of μ on $\omega_\Gamma + V_\Gamma$ and $\mathrm{E}_{N_0,\Gamma}(\omega_\Gamma)$ denotes the totality of N_0-equilibrium measures of ω_Γ[10]. Let V be a relatively compact open neighborhood of the origin in X with $\overline{V} \cap \Gamma = \overline{V}_\Gamma$. Put $\omega_V = \omega_\Gamma + V$; then ω_V is open in X. We choose another open neighborhood U of the origin in X such that $U = -U$ and $U + U \subset V$. We may consider $M_K^+(\Gamma)$ as a subset of $M_K^+(X)$. Choose $(\mu', a') \in \mathrm{SB}_N((\mu, 0); \omega_V)$. Then $N * \mu \geqq N * \mu' + a'\xi$ on X implies $N' * \mu \geqq N' * \mu' + a'\xi$ on X. Assume that $N' * \mu - N' * \mu' - a'\xi = 0$. Then $N_0 * \mu' = N_0 * \mu$ in ω_V and $N_0 * \mu' \leqq N_0 * \mu$ on X. Hence $\operatorname{supp}(\mu') = \overline{\omega}_V \cap \Gamma = (\overline{\omega}_\Gamma + \overline{V}) \cap \Gamma = \overline{\omega}_\Gamma + \overline{V}_\Gamma$. Thus we may consider μ' as in $M^+(\Gamma)$. This shows that $\mu' \in \mathrm{B}_{N_0,\Gamma}(\mu; \omega_\Gamma + V_\Gamma)$ and $\int d\mu' = \int d\mu$, which contradicts (A). Therefore $N' * \mu - N' * \mu' - a'\xi \neq 0$. By $N' \in$ (SMP) and Proposition 9, we have $\operatorname{supp}(N' * \mu - N' * \mu' - a'\xi) \cap \operatorname{supp}(\mu) \neq \phi$, which implies $\operatorname{supp}(N' * \mu - N' * \mu' - a'\xi) \supset \Gamma$. Let $f \in C_K^+(X)$ with $\operatorname{supp}(f) \subset U$ and $f(0) > 0$. Then there eixsts $g \in C_K^+(X)$ such that $g \leqq f$, $g(0) > 0$ and

$$(N' * \mu - N' * \mu' - a'\xi) * f \geqq \xi_\Gamma * g \quad \text{on } X. \tag{4.6}$$

Since $N_0 * \mu' = N_0 * \mu + (N' * \mu - N' * \mu' - a'\xi)$ in ω_V, we obtain that

$$N_0 * \mu' * f = N_0 * \mu * f + (N' * \mu - N' * \mu' - a'\xi) * f \quad \text{in } \omega_\Gamma + U. \tag{4.7}$$

Let $(\omega_{\Gamma,\alpha})_{\alpha \in A}$ be a net of relatively compact open sets in Γ with $\overline{\omega}_{\Gamma,\alpha} \subset \omega_{\Gamma,\beta}$ $(\alpha \lneqq \beta)$ and $\bigcup_{\alpha \in A} \omega_{\Gamma,\alpha} = \omega_\Gamma$, $\gamma_\alpha \in \mathrm{E}_{N_0,\Gamma}(\omega_{\Gamma,\alpha})$ $(\alpha \in A)$ and let $\mu''_{\omega_\Gamma} \in \underline{\mathrm{B}}_{N_0,\Gamma}(\mu; \omega_\Gamma)$[11]. Then, by (4.6) and (4.7), we have

---

[10] In the case of $\mathrm{E}_{N_0,\Gamma}(\omega_\Gamma) \neq \phi$, each $\gamma \in \mathrm{E}_{N_0,\Gamma}(\omega_\Gamma)$ satisfies $\operatorname{supp}(\gamma) \subset \overline{\omega}_\Gamma$, $N_0 * \gamma \leqq \xi_\Gamma$ and $N_0 * \gamma = \xi_\Gamma$ on ω_Γ.

[11] Similarly as in the definition of $\underline{\mathrm{B}}_{N_0}(\mu; \omega)$, we define $\underline{\mathrm{B}}_{N_0,\Gamma}(\mu; \omega_\Gamma)$ from $\mathrm{B}_{N_0,\Gamma}(\mu; \omega_\Gamma)$.

$$
\begin{aligned}
N_0 * (\mu''_{\omega_\Gamma} + \gamma_\alpha) * g &\leqq N_0 * \mu * g + \xi_\Gamma * g \\
&\leqq N_0 * \mu * f + (N' * \mu - N' * \mu' - a'\xi) * f \\
&= N_0 * \mu' * f \qquad \text{in } \omega_\Gamma + U .
\end{aligned}
$$

Since $\operatorname{supp}((\mu''_{\omega_\Gamma} + \gamma_\alpha) * g) \subset \omega_\Gamma + U$, the complete maximum principle of N_0 gives $N_0 * (\mu''_{\omega_\Gamma} + \gamma_\alpha) * g \leqq N_0 * \mu' * f$ on X. Letting $\omega_{\Gamma,\alpha} \uparrow \omega_\Gamma$, we see, from the dominated convergence property of N_0, that there exists $\gamma \in \mathrm{E}_{N_0,\Gamma}(\omega_\Gamma)$ such that

$$
N_0 * (\mu''_{\omega_\Gamma} + \gamma) * g \leqq N_0 * \mu' * f \qquad \text{on } X
$$

(see also Lemma 15 (6)). By the positive mass principle of N_0 (see also Lemma 15 (5)), we have $\left(\int d\mu''_{\omega_\Gamma} + \int d\gamma\right) \cdot \int g d\xi \leqq \left(\int d\mu'\right) \cdot \int f d\xi$, which implies $\int d\gamma < \infty$. This contradicts (B). The assumption $\eta_{N,\delta} \neq -\infty$ leads to this contradiction. Consequently, $\eta_{N,\delta} = -\infty$. This completes the proof.

Let $(\alpha_t)_{t \geqq 0}$ be a vaguely continuous convolution semi-group on X. It is said to be recurrent if there exists $0 \neq f \in C_K^+(X)$ with $\lim_{t\to\infty} \int_0^t \int f d\alpha_s ds = \infty$, and it is said to be semi-transient if $\lim_{t\to\infty} \alpha_t = 0$ and $\mu \in M_K^0(X)$, $\left(\int_0^t \alpha_s * \mu ds\right)_{t>0}$ is vaguely bounded.

As we mentioned in Section 1, Theorem 2 and main theorems in [8] (Théorèmes 52 and 52′) imply Theorem 1. By Theorem 2 and a result in [8] (see Théorème 25), it can be also stated as follows:

THEOREM 27. *If a real convolution kernel N on X is semi-balayable, non-periodic and satisfies $\inf_{x\in X} N * f(x) \leqq 0$ for all $f \in C_K^0(X)$, then there exists a uniquely determined vaguely continuous, markovian, semi-transient and recurrent convolution semi-group $(\alpha_t)_{t\geqq 0}$ on X such that for any $t > 0$, $N \geqq N * \alpha_t$ and*

$$
\lim_{t\to 0} \frac{N - N * \alpha_t}{t} = \varepsilon .
$$

In Theorem 2, it is a question if the condition $\inf_{x\in X} N * f(x) \leqq 0$ for all $f \in C_K^0(X)$ can be removed. By Theorem 2 and Proposition 28 in [8], we have the following

Remark 28. Assume that a real convolution kernel N on X satisfies the same conditions as in Theorem 27. Then, for any $\mu \in D^+(N)$ with

$\int d\mu < \infty$ and any open set $\omega \neq \phi$ in X, $\underline{\mathrm{SB}}_N((\mu, 0); \omega) \neq \phi$ and it forms only one element.

In fact, it is known that if $\mu \in M_K^+(X)$, $\underline{\mathrm{SB}}_N((\mu, 0); \omega) \neq \phi$ (see Proposition 28 in [8]). Assume that $\mathrm{supp}(\mu)$ is non-compact. Then we write $\mu = \sum_{n=1}^{\infty} \mu_n$, where $\mu_n \in M_K^+(X)$. Let $(\mu_n', a_n') \in \underline{\mathrm{SB}}_N((\mu_n, 0); \omega)$. Then $a_n' \leqq 0$. Let ω' be a relatively compact open set $\neq \phi$ in X with $\bar{\omega}' \subset \omega$ and $(\nu, b) \in \underline{\mathrm{SB}}_N((\mu, 0); \omega')$ (see Lemma 8). Then $\sum_{n=1}^{\infty} a_n' \geqq b$, that is, $\sum_{n=1}^{\infty} a_n' > -\infty$. This implies that $\sum_{n=1}^{\infty} \mu_n' \in D^+(N)$. Hence we see easily that $(\sum_{n=1}^{\infty} \mu_n', \sum_{n=1}^{\infty} a_n') \in \underline{\mathrm{SB}}_N((\mu, 0); \omega)$, that is, $\underline{\mathrm{SB}}_N((\mu, 0); \omega) \neq \phi$. Let (μ', a') and (μ'', a'') be in $\underline{\mathrm{SB}}_N((\mu, 0); \omega)$. Then $N * \mu' + a'\xi = N * \mu'' + a''\xi$. Let $(N_p)_{p>0}$ be the resolvent associated with N and $x \in X$. Since $N * \mu' * (\varepsilon - \varepsilon_x)$ and $N * \mu'' * (\varepsilon - \varepsilon_x)$ are shift-bounded, the above equality and (4.2) give

$$N_p * (\mu' * (\varepsilon - \varepsilon_x)) = N_p * (\mu'' * (\varepsilon - \varepsilon_x)) \qquad \text{for all } p > 0 ,$$

which implies $\mu' - \mu' * \varepsilon_x = \mu'' - \mu'' * \varepsilon_x$. Letting $x \to \delta$, we have $\mu' = \mu''$, because $\int d\mu' = \int d\mu'' = \int d\mu < \infty$, so that $a' = a''$. Thus $\underline{\mathrm{SB}}_N((\mu, 0); \omega)$ forms only one element.

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