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# SOME RESULTS ON HARMONIC ANALYSIS ON COMPACT QUOTIENTS OF HEISENBERG GROUPS

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Heisenberg group $H_{2g+1}(\boldsymbol{R})$ of dimension $2g+1$ is a real nilpotent group defined on $\boldsymbol{R}\times\boldsymbol{R}^g\times\boldsymbol{R}^g$ by the law of composition

$$(x_0,\hat{x},x)\circ(y_0,\hat{y},y)=(x_0+y_0+\hat{x}^t y,\hat{x}+\hat{y},x+y)\,,$$

which is isomorphic to the unipotent matrix group

$$\left\{\begin{pmatrix}1 & \hat{c}_1,\cdots,\hat{c}_g, & c_0\\ & 1 & & c_1\\ & & \ddots & \vdots\\ & & 1 & c_g\\ & & & 1\end{pmatrix}\right\}\quad(c_0,\hat{c}_i,c_i\in\boldsymbol{R},\,1\leqslant i,j\leqslant g)\,.$$

$H_{2g+1}(\boldsymbol{Z})$ means the discrete subgroup of integral elements, and $L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$ is the $L^2$-space of the quotient space

$$H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})$$

with respect to the invariant measure

$$dx_0d\hat{x}dx=dx_0d\hat{x}_1\cdots d\hat{x}_gdx_1\cdots dx_g\,.$$

The right action of $H_{2g+1}(\boldsymbol{R})$ induces a unitary representation $\rho$ on $L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$:

$$\rho(y_0,\hat{y},y)\phi(x_0,\hat{x},x)=\phi((x_0,\hat{x},x)\circ(y_0,\hat{y},y))\,.$$

For each non-zero real number $\lambda$ $H_{2g+1}(\boldsymbol{R})$ also acts on the usual $L^2$-space $L^2(\boldsymbol{R}^g)$ as follows

$$\chi_\lambda(y_0,\hat{y},y)f(\xi)=\exp(2\pi\lambda\sqrt{-1}(y_0+\hat{y}^t\xi))f(\xi+y)\,,$$
$$(f(\xi)\in L^2(\boldsymbol{R}^g)\,,\qquad(y_0,\hat{y},y)\in H_{2g+1}(\boldsymbol{R}))\,,$$

Since Lebesgue measure is invariant with respect to translations, $\chi_\lambda$ is a

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unitary representation of $H_{2g+1}(\boldsymbol{R})$.

In the present article, for each theta function $\vartheta(\tau|z)$ of level $n$, we shall construct a transformation

$$\phi_\vartheta\colon L^2(\boldsymbol{R}^g) \longrightarrow L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$$

such that

i) $\langle\phi_\vartheta(f_1), \phi_\vartheta(f_2)\rangle = \langle f_1, f_2\rangle \ (f_1, f_2 \in L^2(\boldsymbol{R}^g))$

ii) $\phi_\vartheta \circ \chi_n(y_0, \hat{y}, y) = \rho(y_0, \hat{y}, y) \circ \phi_\vartheta \ ((y_0, \hat{y}, y) \in H_{2g+1}(\boldsymbol{R}))$

$\phi_\vartheta$ is actually given by

$$\begin{aligned}&\phi_\vartheta(\exp(\pi n\sqrt{-1}(\xi\tau^t\xi))\xi^j)(x_0, \hat{x}, x)\\ &\quad = (2\pi n\sqrt{-1})^{-|j|}\exp(\pi n\sqrt{-1}(x\tau^t x + 2x^t x - 2x_0))\\ &\qquad \cdot\left(2\pi n\sqrt{-1} + \frac{\partial}{\partial\hat{x}}\right)^j \vartheta(\tau|\hat{x} + x\tau).\end{aligned}$$

Choosing the canonical basis of theta functions

$$\vartheta^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}(\tau|z) \qquad (a \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g,\ n \geqq 1),$$

we denote by $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}$ the transformation

$$L^2(\boldsymbol{R}^g) \longrightarrow L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$$

associating with theta function $\vartheta^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}(\tau|z)$ and denote

$$\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix} = \phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}(L^2(\boldsymbol{R}^g)), \qquad \overline{\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}} = \overline{\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}(L^2(\boldsymbol{R}^g))}.$$

Then the decomposition of the unitary representation $\rho$ is given by

$$\begin{aligned}L^2\Big(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})\Big) &= \Big(\bigoplus_{\substack{a\in \boldsymbol{Z}^g/n\boldsymbol{Z}^g\\ n\geqq 1}} \boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}\Big)\\ &\oplus\Big(\bigoplus_{\substack{a\in \boldsymbol{Z}^g/n\boldsymbol{Z}^g\\ n\geqq 1}} \overline{\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}}\Big)\oplus\Big(\bigoplus_{(\hat{k},k)\in \boldsymbol{Z}^g\times\boldsymbol{Z}^g} \boldsymbol{C}\exp(2\pi n\sqrt{-1}(\hat{k}^t\hat{x} + k^t x))\Big).\end{aligned}$$

The invariant subspaces $\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}$, $\overline{\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix}}$ $(a \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g,\ n \geqq 1)$ are independent of the choice of $\tau$, and $\boldsymbol{H}^{(n)}\begin{bmatrix} a/n \\ 0\end{bmatrix} = \exp(2\pi a^t\hat{x})\boldsymbol{H}^{(n)}\begin{bmatrix} 0 \\ 0\end{bmatrix}$.

In the next article, we shall be concerned with an application to quantum mechanics.

*Notations.*

$Z_{\geqq 0} = \{\text{non-negative integer}\}$,

$Z^g_{\geqq 0} = \{j = (j_1, \cdots, j_g) | j_i \in Z_{\geqq 0}\}$,

$|j| = j_1 + \cdots + j_g$,

$j \pm \varepsilon_i = (j_1, \cdots, j_{i-1}, j_i \pm 1, j_{i+1}, \cdots, j_g)$,

$\left(2\pi n\sqrt{-1}\, x + \frac{\partial}{\partial \hat{x}}\right)^j = \left(2\pi n\sqrt{-1} x_1 + \frac{\partial}{\partial \hat{x}_1}\right)^{j_1} \cdots \left(2\pi n\sqrt{-1} x_g + \frac{\partial}{\partial \hat{x}_g}\right)^{j_g}$

$\left(x + \ell + \frac{a}{n}\right)^j = \left(x_1 + \ell_1 + \frac{a_1}{n}\right)^{j_1} \cdots \left(x_g + \ell_g + \frac{a_g}{n}\right)^{j_g}$.

## § 1. Equivariant isomorphisms of $L^2(R^g)$ into $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$

**1.1.** First we choose a complex symmetric $g \times g$ matrix $\tau = \tau' + \sqrt{-1}\,\tau''$ with positive definite imaginary part $\tau''$, and fix $\tau$ once for all. A system of complex coordinates is introduced,

$$z = \hat{x} + x\tau\,, \qquad \bar{z} = \hat{x} + x\bar{\tau}\,. \tag{1.1}$$

Real and complex coordinates are related as follows

$$\frac{\partial}{\partial \hat{x}} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\,, \qquad \frac{\partial}{\partial x} = \tau \frac{\partial}{\partial z} + \bar{\tau} \frac{\partial}{\partial z}\,. \tag{1.2}$$

$$\tau \frac{\partial}{\partial \hat{x}} - \frac{\partial}{\partial x} = (\tau - \bar{\tau}) \frac{\partial}{\partial \bar{z}} = 2\sqrt{-1}\,\tau'' \frac{\partial}{\partial z}\,. \tag{1.3}$$

Let us recollect the definition of theta functions. An entire function $f(z)$ in $z$ is called a theta function of level $n$ with respect to $\tau$, if it satisfies

$$f(z + \hat{b} + b\tau) = \exp(-\pi n\sqrt{-1}(b\tau{}^t b + 2z{}^t b))f(z) \qquad ((\hat{b}, b) \in Z^g \times Z^g)\,. \tag{1.4}$$

The space $\Theta_0^{(n)}$ of theta functions of level $n$ is a vector space of dimension $n^g$ with a basis consisting of theta series of level $n$

$$\vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z) = \sum_{\ell \in Z^g} \exp\left(\pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau^t\left(\ell + \frac{a}{n}\right) + 2z^t\left(\ell + \frac{a}{n}\right)\right)\right) \qquad (a \in Z^g/nZ^g)\,. \tag{1.5}$$

A function $\varphi(u, z)$ in $2g$ complex variables $(u, z) = (u_1, \cdots, u_g, z_1, \cdots, z_g)$ is called an auxiliary theta function of level $n$ (with respect to $\tau$), if it satisfies,

i) $\varphi(u, z)$ is a polynomial in $z$ whose coefficient are entire functions in $u$.

ii) $\varphi(u+b, z+\hat{b}+b\tau) = \exp(-\pi n\sqrt{-1}(b\tau^t b + 2z^t b))\varphi(u, z)$

$$((\hat{b}, b) \in \boldsymbol{Z}^g \times \boldsymbol{Z}^g).$$

In the previous article[1] the author proved that the space $\Theta^{(n)}$ of auxiliary theta functions of level $n$ has a basis *consisting of auxiliary theta* series of level $n$,

$$(1.6)\quad \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|u, z) = \left(2\pi n\sqrt{-1}u + \frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z)$$

$$= (2\pi n\sqrt{-1})^{|j|}\sum_{\ell\in\boldsymbol{Z}^g}\left(u+\ell+\frac{a}{n}\right)^j$$

$$\cdot\exp\left(\pi n\sqrt{-1}\left(\left(\ell+\frac{a}{n}\right)\tau^t\left(\ell+\frac{a}{n}\right)+2z^t\left(\ell+\frac{a}{n}\right)\right)\right)$$

$$(a \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g,\ j \in \boldsymbol{Z}^g_{\geq 0}).$$

A mixed theta function of level $n$ (with respect to $\tau$) mean a real analytic function $\varphi(\hat{x}, x)$ in $(\hat{x}, x)$ such that

i) $\varphi(\hat{x}, x)$ is a polynomial in $x$ whose coefficients are entire function in complex variables $z = \hat{x} + x\tau$,

ii) $\varphi(\hat{x}+\hat{b}, x+b) = \exp(-\pi n\sqrt{-1}(b\tau^t b + 2(\hat{x}+x\tau)^t b))\varphi(\hat{x}, x)$

$$((\hat{b}, b) \in \boldsymbol{Z}^g \times \boldsymbol{Z}^g).$$

It will be shown soon that the space $\Theta^{(n)}_{\text{mix}}$ of mixed theta functions of level $n$ (with respect to $\tau$) has a basis consisting of mixed theta series of level $n$,

$$(1.7)\quad \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|x, \hat{x}+x\tau)$$

$$= (2\pi n\sqrt{-1})^{|j|}\sum_{\ell\in\boldsymbol{Z}^g}\left(x+\ell+\frac{a}{n}\right)^j$$

$$\cdot\exp\left(\pi n\sqrt{-1}\left(\left(\ell+\frac{a}{n}\right)\tau^t\left(\ell+\frac{a}{n}\right)+2(\hat{x}+x\tau)^t\left(\ell+\frac{a}{n}\right)\right)\right)$$

$$(a \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g,\ j \in \boldsymbol{Z}^g_{\geq 0},\ n \geq 1),$$

which are the specializations of auxiliary theta series with respect to $(u, z) \mapsto (x, \hat{x}+x\tau)$.

**1.2.** Let us introduced a family of real analytic functions

1) See [3].

$$\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)$$

$$= (2\pi n\sqrt{-1})^{-|j|} \exp(\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x_0))\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x, \hat{x} + x\tau)$$

$$= \exp(-2\pi n\sqrt{-1}\, x_0) \sum_{\ell \in Z^g} \left(x + \ell + \frac{a}{n}\right)$$

$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(x + \ell + \frac{a}{n}\right)\tau^t\left(x + \ell + \frac{a}{n}\right) + 2\hat{x}^t\left(x + \ell + \frac{a}{n}\right)\right)\right)$$

$$(a \in Z^g/nZ^g,\ j \in Z^g,\ n \geqq 1).$$

PROPOSITION 1.1.

$$\text{(1.8)} \quad \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) = \exp(2\pi\sqrt{-1})a^t\hat{x})\phi_j\begin{bmatrix} 0 \\ 0 \end{bmatrix}\left((\tau \mid x_0, \hat{x}, x) \circ \left(0, 0, \frac{a}{n}\right)\right),$$

$$\text{(1.9)} \quad \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid (b_0, \hat{b}, b) \circ (x_0, \hat{x}, x)) = \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)$$

$$((b_0, \hat{b}\ b) \in H_{2g+1}(Z),\ a \in Z^g/nZ^g,\ j \in Z^g_{\geqq 0},\ n \geqq 1).$$

*Proof.* (1.8) is an immediate consequence of the definition of $\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)$. For each $(b_0, \hat{b}, b)$ in $H_{2g+1}(Z)$ we have

$$\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}((b_0, \hat{b}, b) \circ (x_0, \hat{x}, x))$$

$$= \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(b_0 + x_0 + \hat{b}^t x, \hat{b} + \hat{x}, b + x)$$

$$= \exp(2\pi n\sqrt{-1}(b_0 + x_0 + \hat{b}^t x)) \sum_{\ell \in Z^g} \left(x + b + \ell + \frac{a}{n}\right)^j$$

$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(x + b + \ell + \frac{a}{n}\right)\tau^t\left(x + b + \ell + \frac{a}{n}\right)\right.\right.$$

$$\left.\left. + 2(\hat{x} + \hat{b})^t\left(x + b + \ell + \frac{a}{n}\right)\right)\right)$$

$$= \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x).$$

Proposition 1.1 means that $\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)$, $\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid \hat{x}_0, x)$ are real analytic functions on the quotient space $H_{2g+1}(Z)\backslash H_{2g+1}(R)$.

**1.3.** Lie algebra $\mathfrak{h}_{2g+1}(R)$ of left invariant vector fields on $H_{2g+1}(R)$ has a basis

$$D_0 = -\frac{\partial}{\partial x_0}, \quad \hat{D}_i = \frac{\partial}{\partial \hat{x}_i}, \quad D_i = \frac{\partial}{\partial x_i} + \hat{x}_i \frac{\partial}{\partial x_0} \qquad (1 \leqq i \leqq g)$$

such that

$$[D_0, \hat{D}_i] = [D_0, D_i] = [D_i, D_k] = [\hat{D}_i, \hat{D}_k] = 0$$

$$[D_i, \hat{D}_k] = \begin{cases} D_0 & (i = k) \\ 0 & (i \neq k) . \end{cases}$$

THEOREM 1.1.

$$D_0\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x) = 2\pi n\sqrt{-1}\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x), \tag{1.10}$$

$$\hat{D}_i\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x) = 2\pi n\sqrt{-1}\phi_{j+\varepsilon_i}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x), \tag{1.11}$$

$$\begin{aligned} D_i\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x) &= j_i\phi_{j-\varepsilon_1}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x) \\ &\quad + 2\pi n\sqrt{-1}\sum_{p=1}^{g}\tau_{ip}\phi_{j+\varepsilon_p}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x) \\ &\quad (1 \leqq i \leqq g,\ a \in Z^g/nZ^g,\ j \in Z^g_{\geqq 0},\ n \geqq 1) . \end{aligned} \tag{1.12}$$

*Proof.* From the definition of $\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x)$ it follows

$$\begin{aligned} &D_0\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x) \\ &= -\frac{\partial}{\partial x_0}\Big\{(2\pi n\sqrt{-1})^{-|j|} \\ &\quad \cdot \exp(\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x_0))\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x, \hat{x} + x\tau)\Big\} \\ &= 2\pi n\sqrt{-1}\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x), \end{aligned}$$

$$\begin{aligned} &\hat{D}_i\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau | x_0, \hat{x}, x) \\ &= \frac{\partial}{\partial \hat{x}_i}\Big\{\exp(-2\pi n\sqrt{-1}x_0)\sum_{\ell \in Z^g}\left(x + \ell + \frac{a}{n}\right)^j \\ &\quad \cdot \exp\left(\pi n\sqrt{-1}\left(\left(x + \ell + \frac{a}{n}\right)\tau^t\left(x + \ell + \frac{a}{n}\right) + 2\hat{x}^t\left(x + \ell + \frac{a}{n}\right)\right)\right)\Big\} \\ &= 2\pi n\sqrt{-1}\exp(-2\pi n\sqrt{-1}x_0)\sum_{\ell}\left(x + \ell + \frac{a}{n}\right)^{j+\varepsilon_i} \end{aligned}$$

$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(x+\ell+\frac{a}{n}\right)\tau^t\left(x+\ell+\frac{a}{n}\right)+2\hat{x}^t\left(x+\ell+\frac{a}{n}\right)\right)\right)$$

$$= 2\pi n\sqrt{-1}\,\phi^{(n)}_{j+\varepsilon_i}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)\,,$$

$$D_i\phi^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)$$

$$= \left(\frac{\partial}{\partial x_i}+\hat{x}_i\frac{\partial}{\partial x_0}\right)\left\{\exp\left(-2\pi n\sqrt{-1}x_0\right)\sum_{\ell\in Z^g}\left(x+\ell+\frac{a}{n}\right)^j\right.$$

$$\left.\cdot \exp\left(\pi n\sqrt{-1}\left(\left(x+\ell+\frac{a}{n}\right)\tau^t\left(x+\ell+\frac{a}{n}\right)+2\hat{x}^t\left(x+\ell+\frac{a}{n}\right)\right)\right)\right\}$$

$$= \exp\left(-2\pi n\sqrt{-1}x_0\right)\sum_{\ell\in Z^g}\left\{j_i\left(x+\ell+\frac{a}{n}\right)^{j-\varepsilon_i}\right.$$

$$\left. + 2\pi n\sqrt{-1}\left(-\hat{x}_i+\sum_{p=1}^{g}\tau_{ip}\left(x_p+\ell_p+\frac{a_p}{n}\right)+\hat{x}_i\right)\left(x+\ell+\frac{a}{n}\right)^j\right\}$$

$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(x+\ell+\frac{a}{n}\right)\tau^t\left(x+\ell+\frac{a}{n}\right)+2\hat{x}^t\left(x+\ell+\frac{a}{n}\right)\right)\right)$$

$$= j_i\phi^{(n)}_{j-\varepsilon_i}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)+2\pi n\sqrt{-1}\sum_{p=1}^{g}\phi^{(n)}_{j+\varepsilon_p}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)\,.$$

COROLLARY 1.1.1.

$$\left(D_i-\sum_{p=1}^{g}\tau_{ip}\hat{D}_p\right)\phi^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x) = j_i\phi^{(n)}_{j-\varepsilon_i}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x)\,, \tag{1.13}$$

$$\left(\hat{D}_iD_i-\sum_{p=1}^{g}\tau_{ip}\hat{D}_i\hat{D}_p\right)\phi^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x_0, \hat{x}, x) \tag{1.14}$$

$$= 2\pi n\sqrt{-1}\,j_i\phi^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x, \hat{x}, x)\,.$$

$$(1\leqq i\leqq g,\ a\in Z^g/nZ^g,\ j\in Z^g_{\geqq 0}.\ n\geqq 1)\,.$$

These are direct consequences of (1.11), (1.12).

COROLLARY. *$\vartheta^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x, \hat{x}+x\tau)$ $(a\in Z^g/nZ^g,\ j\in Z^g_{\geqq 0},\ n\geqq 1)$ are linearly independent.*

*Proof.* For each $\hat{a}\in Z^g/nZ^g$ we have

$$\vartheta^{(n)}_j\begin{bmatrix} a/n \\ 0 \end{bmatrix}\left(\tau\,|\,x, \hat{x}+\frac{\hat{a}}{n}+x\tau\right)$$

$$= (2\pi n\sqrt{-1})^{|j|}\sum_{\ell\in Z^g}\left(x+\ell+\frac{a}{n}\right)^j$$
$$\cdot\exp\left(\pi n\sqrt{-1}\left(\left(\ell+\frac{a}{n}\right)\tau^t\left(\ell+\frac{a}{n}\right)+2\left(\hat{x}+\frac{\hat{a}}{n}\right)^t\left(\ell+\frac{a}{n}\right)\right)\right)$$
$$= \exp\left(2\pi\sqrt{-1}\,\frac{\hat{a}^t a}{n}\right)\vartheta_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x,\hat{x}+x\tau).$$

Hence by virtue of (1.10), (1.13), (1.14) and the above relation we conclude the linearly independence of

$$\phi_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x_0,\hat{x},x)$$
$$= (2\pi n\sqrt{-1})^{-|j|}\exp(\pi n\sqrt{-1}\,(x\tau^t x+2\hat{x}^t x-2x_0))\vartheta_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x,\hat{x}+x\tau).$$

Denote

$$H_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix} = \text{the completion of the vector space}$$
$$\text{spanned by } \phi_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau|x_0,\hat{x},x)\qquad (j\in Z^g_{\geqq 0})$$
$$\overline{H_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}} = \text{the complex conjugate of } H_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}.$$

THEOREM 1.2. *$H_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}$, $\overline{H_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}}$ are irreducible invariant subspace of $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$ with respect to the unitary representation $\rho$:*

$$\rho(y_0,\hat{y},y)\phi(x_0,\hat{x},x) = \phi((x_0,\hat{x},x)\circ(y_0,\hat{y},y))$$

*such that*

$$H^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix} = \exp(2\pi a^t\hat{x})H^{(n)}\begin{bmatrix}0\\0\end{bmatrix},$$
$$\rho(y_0,0,0)\phi = \exp(-2\pi n\sqrt{-1}\,y_0)\phi\qquad\left(\phi\in H_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\right),$$
$$\rho(y_0\,0,0)\bar{\phi} = \exp(2\pi n\sqrt{-1}\,y_0)\bar{\phi}\qquad\left(\bar{\phi}\in\overline{H_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}}\right).$$

*Proof.* Theorem 1.1 states that the Lie algebra representation $d\rho$ of $\mathfrak{h}_{2g}(R)$ preserves $H_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}$, $\overline{H_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}}$ and

$$d\rho(D_0)\phi = -\frac{\partial}{\partial x_0}\phi = 2\pi n\sqrt{-1}\,\phi \qquad \left(\phi \in H_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\right),$$

$$d\rho(D_0)\bar{\phi} = \frac{\partial}{\partial x_0}\phi = -2\pi n\sqrt{-1}\,\phi \qquad \left(\bar{\phi} \in H_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\right).$$

Since $\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) = \exp(2\pi\sqrt{-1}\, a\, {}^t\hat{x})\phi_j^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau \mid (x_0, \hat{x}, x)\circ(0, 0, a/n))$, we complete the proof of Theorem 1.2.

**1.4.** Let $L^2(\boldsymbol{R}^g, \mu_{\tau''}^{(n)})$ be the $L^2$-space of $\boldsymbol{R}^g$ with respect to the measure

$$\pi_{\tau''}^{(n)}(d\xi) = \exp(-2\pi n\xi\tau''\xi)\,d\xi,$$

where $n \geqq 1$.

LEMMA 1.1. *The transformation*

$$f(\xi) \longmapsto \exp(-\pi n\sqrt{-1}\,\xi\tau\,{}^t\xi)f(\xi) \tag{1.15}$$

*is an isomorphism of Hilbert space* $L^2(\boldsymbol{R}^g, \mu_{\tau''}^{(n)})$ *onto* $L^2(\boldsymbol{R}^g)$.

*Proof.* Since $\tau - \bar{\tau} = 2\sqrt{-1}\,\tau''$, we have

$$\int_{\boldsymbol{R}^g} \overline{\exp(-\pi n\sqrt{-1}\,\xi\tau\,{}^t\xi)}\exp(\pi n\sqrt{-1}\,\xi\tau\,{}^t\xi)f_2(\xi)\,d\xi$$

$$= \int_{\boldsymbol{R}^g} \exp(-2\pi n\xi\tau''\,{}^t\xi)\overline{f_1(\xi)}f_2(\xi)\,d\xi.$$

Hence the transformations (1.15) is an isomorphism of $L^2(\boldsymbol{R}, \mu_{\tau''}^{(n)})$ onto $L^2(\boldsymbol{R}^g)$.

Since the set of monomials $\{\xi^j \mid j \in \boldsymbol{Z}^g_{\geqq 0}\}$ is a basis of $L^2(\boldsymbol{R}^g, \mu_{\tau''}^{(n)})$, the set of functions $\{\exp(\pi n\sqrt{-1}\,\xi\tau\,{}^t\xi)\xi^j \mid j \in \boldsymbol{Z}^g_{\geqq 0}\}$ is a basis of $L^2(\boldsymbol{R}^g)$.

LEMMA 1.2.

$$\int_{H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})} \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\phi_k^{(m)}\begin{bmatrix} c/m \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\,dx_0 d\hat{x} dx \tag{1.16}$$

$$= \begin{cases} \displaystyle\iint_{\boldsymbol{R}^g} y^{j+k}\exp(-2\pi n y\tau''\,{}^t y)\,dy & (n = m,\ a \equiv C \bmod n) \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* Let us first integrate on fibers

$$\left\langle \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x),\ \phi_k^{(m)}\begin{bmatrix} c/m \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\right\rangle$$

$$= \int_{H_{2g+1}(\boldsymbol{Z})\backslash H_{g+1}(\boldsymbol{R})} \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\phi_k^{(m)}\begin{bmatrix} c/m \\ 0 \end{bmatrix}(\tau / x, \hat{x}, x)dx_0 d\hat{x} dx$$

$$= \sum_{\ell, \ell' \in Z^g} \int_{Z^g \backslash R^g} \int_{Z^g \backslash R^g} \left\{ \int_{Z^g \backslash R} \exp(-2\pi\sqrt{-1}(-n+m)x_0) \right.$$

$$\cdot\, dx_0 \left(x + \ell + \frac{a}{n}\right)^j \left(x + \ell' + \frac{c}{m}\right)^k$$

$$\cdot \exp\left(\pi\sqrt{-1}\left(\ell + \frac{a}{n}\right)\tau^t\left(\ell + \frac{a}{n}\right) + m\left(\ell' + \frac{c}{m}\right)\tau^t\left(\ell' + \frac{c}{m}\right)\right)$$

$$\left. \cdot \exp 2\pi\sqrt{-1}\left(\hat{x}\left(-n^t\left(\ell + \frac{a}{n}\right) + m^t\left(\ell' + \frac{c}{m}\right)\right)\right)\right\} d\hat{x} dx .$$

This means that

$$\left\langle \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x),\ \phi_k^{(m)}\begin{bmatrix} c/m \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\right\rangle = 0 \qquad (n \neq m),$$

$$\left\langle \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x),\ \phi_k^{(n)}\begin{bmatrix} c/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\right\rangle = 0 \qquad (a \not\equiv c \bmod n)$$

$$\left\langle \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x),\ \phi_k^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\right\rangle$$

$$= \sum_{\ell \in Z^g} \int_{Z^g \backslash Z^g} \left(x + \ell + \frac{a}{n}\right)^{j+k} \exp -2\pi n\left(x + \ell + \frac{a}{n}\right)\tau''\left(x + \ell + \frac{a}{n}\right) dx$$

$$= \int_{R^g} y^{j+k} \exp(-2\pi n y \tau''{}^t y)\, dy .$$

LEMMA 1.3. *The transformations of* $L^2(R^g, \mu_{\tau''}^{(n)})$ *onto* $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *given by*

$$\xi_j \longmapsto \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0\, \hat{x}, x) \qquad (j \in Z^g_{\geqq 0}) \tag{1.17}$$

*is an isomorphism of Hilbert spaces.*

This is an immediate consequence of Lemma 1.2.

We define unitary action of $H_{2g+1}(R)$ on $L^2(R)$ by

$$\chi_n(x_0, \hat{x}, x) f(\xi) = \exp(-2\pi n\sqrt{-1}(x_0 + x^t\xi)) f(\xi + \hat{x}) . \tag{1.18}$$

THEOREM 1.3. *Let* $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *be the transformation of* $L^2(R^g)$ *onto* $H_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *given by*

$$\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\,\xi^j)(x_0, \hat{x}, x) \tag{1.19}$$

$$= \phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) \qquad (j \in Z^g_{\geqq 0}) .$$

*Then* $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *is an isomorphism of Hilbert space* $L^2(\boldsymbol{R}^g)$ *to Hilbert space* $\boldsymbol{H}_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *such that*

$$\rho(y_0, \hat{y}, y) \circ \phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} \circ \chi_n(y_0, \hat{y}, y) \quad ((y_0, \hat{y}, y) \in H_{2g+1}(\boldsymbol{R})), \tag{1.20}$$

$$\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \exp(2\pi\sqrt{-1}\, a^t\hat{x})\rho\left(0, 0, \frac{a}{n}\right)\phi_\tau^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.21}$$

*Proof.* By virtue of Lemmas 1.2 and 1.3 $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ is an isomorphism of Hilbert spaces. Let us prove the equivariance of $\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$. From the action

$$\begin{aligned} &\chi_n(x_0, \hat{x}, x)(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j) \\ &\quad = \exp(-2\pi n\sqrt{-1}\,(x_0 + x^t\xi)) \exp(\pi n\sqrt{-1}(\xi + \hat{x})\tau^t(\xi + \hat{x}))(\xi + \hat{x})^j, \end{aligned}$$

we have

$$\begin{aligned} &d\chi_n(D_0)(\exp(\pi n\sqrt{-1}\,\xi\tau^t)\xi^j) \\ &\quad = -\frac{\partial}{\partial x_0}\bigg|_{x_0=0}(\chi_n(x_0, \hat{x}, x) - 1)(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j) \\ &\quad = -2\pi n\sqrt{-1}\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j, \\ &d\chi_n(D_i)(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j) \\ &\quad = -2\pi n\sqrt{-1}\,\xi_i \exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j \\ &\quad = -2\pi n\sqrt{-1}\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^{j+\varepsilon_i}, \\ &d\chi_n(\hat{D}_i)(\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^j) \\ &\quad = \exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\left(2\pi n\sqrt{-1}\left(\sum_{p=1}^{g}\tau_{ip}\xi_p\right)\xi^j + j_i\xi^{j-\varepsilon_i}\right) \\ &\quad = 2\pi\sqrt{-1}\sum_{p=1}^{g}\tau_{ip}\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^{j+\varepsilon}p + j_i\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi)\xi^{j-\varepsilon_i}. \end{aligned}$$

Hence by virtue of (1.10), (1.11), (1.12) we have

$$\begin{aligned} &\left(\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} \circ d\chi_n(D_0)\right)(\xi^j \exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0, \hat{x}, x) \\ &\quad = 2\pi n\sqrt{-1}\,\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,x, \hat{x}, x) \\ &\quad = d\rho(D_0)\phi_\tau^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\xi^j \exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0, \hat{x}, x) \end{aligned}$$

$$\left(\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\circ d\chi_n(-D_i)\right)(\xi^j\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0,\hat{x},x)$$

$$=2\pi n\sqrt{-1}\,\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\xi^{j+\varepsilon_i}\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0,\hat{x},x)$$

$$=2\pi n\sqrt{-1}\,\phi_{j+\varepsilon_i}^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0,\hat{x},x)=d\rho(\hat{D}_i)\phi_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0,\hat{x},x)$$

$$=\left(d\rho(\hat{D}_i)\circ\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\right)(\xi^j\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0,\hat{x},x),$$

$$\left(\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\circ d\chi_n(\hat{D}_i)\right)(\xi_j\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0,\hat{x},x)$$

$$=j_i\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\xi^{j-\varepsilon_i}\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0,\hat{x},x)+2\pi n\sqrt{-1}\sum_{p=1}^{g}\tau_{ip}\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}$$

$$\cdot(\xi^{j+\varepsilon_p}\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0,\hat{x},x)$$

$$=j_i\phi_{j-\varepsilon_i}^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0,\hat{x},x)+2\pi n\sqrt{-1}\sum_{p=1}^{g}\tau_{ip}\phi_{j+\varepsilon_p}^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0,\hat{x},x)$$

$$=d\rho(D_i)\phi_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,x_0,\hat{x},x)$$

$$=d\rho(D_i)\circ\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\xi^j\exp(\pi n\sqrt{-1}\,\xi\tau^t\xi))(x_0,\hat{x},x)\,.$$

This means

$$d\rho^{\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)}\circ\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}=\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\circ d\chi_n\,,$$

and thus

$$\rho^{\left(\begin{smallmatrix}0&1\\-1&1\end{smallmatrix}\right)}\circ\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}=\phi_\tau^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}\circ\chi_n\,,$$

where

$$\rho^{\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)}(x_0,\hat{x},x)=\rho\left((x_0,\hat{x},x)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right).$$

## § 2. Decomposition of unitary representation $\rho$

**2.1.** An irreducible unitary representation $\rho_\lambda$ of $H_{2g+1}(\boldsymbol{R})$ is characterized by a real number $\lambda$ such that

$$\rho_\lambda(y_0,0,0)\phi=\exp(-2\pi\lambda\sqrt{-1}\,y_0)\phi$$

provided $\lambda\neq0$. If $\lambda=0$, then it is characterized by a pair $(\hat{k},k)\in\boldsymbol{R}^g\times\boldsymbol{R}^g$ of vectors as follows

$$\rho_{\hat{k},k}(y_0, \hat{y}, y)\phi = \exp(2\pi\sqrt{-1}(\hat{k}^t\hat{y} + k^t y)\phi .$$

For each integer $b_0$ and $\phi(x_0, \hat{x}, x)$ in $L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$

$$\begin{aligned}\phi(x_0, \hat{x}, x) &= \phi((b_0, 0, 0)\circ(x_0, \hat{x}, x)) = \phi((x_0, \hat{x}, x)\circ(b_0, 0, 0))\\ &= \rho_\lambda(b_0, 0, 0)\phi(x_0, \hat{x}, x),\end{aligned}$$

hence for every irreducible factor $\rho_\lambda$ of $\rho$, $\lambda$ must be an integer.

LEMMA 2.1. *Let $\phi(x_0, \hat{x}, x)$ be a real analytic function on $H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})$ such that*

i) $\exp(2\pi n\sqrt{-1}\,x_0)\phi(x_0, \hat{x}, x)$ *is independent on* $x_0$,

ii) $\left(D_i - \sum_{p=1}^{g} \tau_{ip}\hat{D}_p\right)\phi(x_0, \hat{x}, x) = 0 \qquad (1 \leqq i \leqq g).$

*Then*

$$\psi(\hat{x}, x) = \exp(-\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x))\phi(x_0, \hat{x}, x)$$

*is a theta function of level $n$ in $z = \hat{x} + x\tau$ with respect to $\tau$.*

*Proof.* For each $(b_0, \hat{b}, b)$ in $H_{2g+}(\boldsymbol{Z})$ we have

$$\begin{aligned}&\exp(-\pi n\sqrt{-1}((x+b)\tau^t(x+b) + 2(\hat{x}+\hat{b})^t(x+b) - 2b_0 - 2x - 2\hat{b}^t x))\\ &\quad\cdot\phi((b_0, \hat{b}, b)\circ(x_0, \hat{x}, x))\\ &\quad= \exp(-\pi n\sqrt{-1}(b\tau^t b + 2(\hat{x}+x\tau)^t b))\\ &\quad\cdot\exp(-\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x_0))\phi(x_0, \hat{x}, x).\end{aligned}$$

Hence we have the difference relation:

$$\psi(\hat{x}+\hat{b}, x+b) = \exp(-\pi n\sqrt{-1}(b\tau^t b + 2(\hat{x}+x\tau)^t b)\psi(\hat{x}, x).$$

From the relation

$$\tau\frac{\partial}{\partial\hat{x}} - \frac{\partial}{\partial x} = (\tau - \bar{\tau})\frac{\partial}{\partial\bar{z}},$$

in order to prove $(\partial/\partial\bar{z}_i)\psi(\hat{x}, x) = 0$ $(1 \leqq i \leqq g)$, it is sufficient to show

$$\left(\tau\frac{\partial}{\partial\hat{x}} - \frac{\partial}{\partial x}\right)\psi(\hat{x}, x) = 0.$$

$$\begin{aligned}\left(\sum_{p=1}^{g}\tau_{ip}\frac{\partial}{\partial\hat{x}_p} - \frac{\partial}{\partial x_i}\right)\psi(\hat{x}, x) &= \left(\sum_{p=1}^{g}\tau_{ip}\frac{\partial}{\partial\hat{x}_p} - \frac{\partial}{\partial x_i} - \hat{x}_i\frac{\partial}{\partial x_0}\right)\psi(\hat{x}, x)\\ = \left(\sum_{p=1}^{g}\tau_{ip}\frac{\partial}{\partial\hat{x}_p} - \frac{\partial}{\partial x_i} - \hat{x}_i\frac{\partial}{\partial x_0}\right)&\\ \cdot(\exp(-\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x_0)))\phi(x_0, \hat{x}, x)&\end{aligned}$$

$$+\exp(-\pi n\sqrt{-1}(x\tau^t x + 2\hat{x}^t x - 2x_0))\Big(\sum_{p=1}^{g} \tau_{ip}\hat{D}_p - D_i\Big)\phi(x_0, \hat{x}, x)$$
$$= 0\,.$$

Hence $\psi(\hat{x}, x)$ is an entire function in $z = \hat{x} + x\tau$ satisfying the difference equation for theta function of level $n$, and thus $\psi(\hat{x}, x)$ must be a theta function of level $n$.

THEOREM 2.1. *Let* $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *be the completion of the vector space spanned by* $\left\{\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) \mid j \in Z^g_{\geqq 0}\right\}$ *and* $\overline{H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}$ *be the complex conjugate of* $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$. *Then* $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ *and* $\overline{H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}$ *are irreducible invariant subspaces of* $L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$ *such that*

$$\rho(y_0, 0, 0)\phi(x_0, \hat{x}, x) = \exp(-2\pi n\sqrt{-1}\, y_0)\phi(x_0, \hat{x}, x) \tag{2.1}$$
$$\rho(y_0, 0, 0)\overline{\phi(x_0, \hat{x}, x)} = \exp(-2\pi n\sqrt{-1}\, y_0)\overline{\phi(x_0, \hat{x}, x)}$$
$$\left(\phi \in H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\right),$$

*and the decomposition of* $\rho$ *is given by*

$$\begin{aligned} &L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})) \\ &= \Big(\bigoplus_{\substack{a \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g \\ n \geqq 1}} H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\Big) \oplus \Big(\bigoplus_{\substack{a \in \boldsymbol{Z}^g/n\boldsymbol{Z}^g \\ n \geqq 1}} \overline{H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}}\Big) \\ &\Big(\bigoplus_{(\hat{k}, k) \in \boldsymbol{Z}^g \times \boldsymbol{Z}^g} \boldsymbol{C}\exp(2\pi\sqrt{-1}(\hat{k}^t\hat{x} + k^t x))\Big)\,. \end{aligned} \tag{2.2}$$

*The invariant subspaces*

$$H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \exp(2\pi a^t\hat{x})H^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \overline{H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}} = \exp(-2\pi a^t\hat{x})\overline{H^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

*are independent of the choice of* $\tau$.

*Proof.* Since the space $A$ of real an analytic functions on $H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R})$ is dense in $L^2(H_{2g+1}(\boldsymbol{Z})\backslash H_{2g+1}(\boldsymbol{R}))$ and $A$ is invariant for $D_0, \hat{D}_i, D_i$ $(1 \leqq i \leqq g)$, i.e. for the action of $H_{2g+1}(\boldsymbol{R})$, it is sufficient to decompose $A$. Let $W$ be an irreducible invariant subspace of $A$ such that $\rho(y_0, 0, 0)\phi(x_0, \hat{x}, x) = \exp(-2\pi n\sqrt{-1}\, y_0)\phi(x_0, \hat{x}, x)$ $(y_0 \in \boldsymbol{R}, \phi \in \overline{W})$. If $n = 0$, then there exists $(\hat{k}, k) \in \boldsymbol{Z}^g \times \boldsymbol{Z}^g$ such that

$$W = \boldsymbol{C}\exp(2\pi\sqrt{-1}(\hat{k}^t\hat{x} + k^t x))\,.$$

If $n$ is negative, we replace $W$ by $\overline{W}$. So we may assume that $n$ is a positive integer. Let $\nu$ be a $\mathfrak{h}_{2g+1}(\boldsymbol{R})$-module isomorphism of $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} \cap A$ onto $\overline{W}$, i.e., a linear isomorphism satisfying

$$D_0 \circ \nu = \nu \circ D_0, \quad \hat{D}_i \circ \nu = \nu \circ \hat{D}_i, \quad D_i \circ \nu = \nu \circ D_i \quad (1 \leqq i \leqq g).$$

Since $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} \cap A$ contains the element $\phi_0^{(n)}$ satisfying

$$\Big(D_i - \sum_{d=1}^{g} \tau_{ip}\hat{D}_p\Big)\phi_0^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \,|\, x_0, \hat{x}, x) = 0 \qquad (1 \leqq i \leqq g).$$

There exists an element $\phi_0(x_0, \hat{x}, x)$ in $\overline{W}$ such that

$$\Big(D_i - \sum_{p=1}^{g} \tau_{ip}\hat{D}_p\Big)\phi_0(x_0, \hat{x}, x) = 0 \qquad (1 \leqq i \leqq g).$$

From $\phi_0(x_0, \hat{x}, x) = \rho(x_0, 0, 0)\phi_0(0, \hat{x}, x) = \exp(-2\pi n\sqrt{-1}\,x_0)\phi_0(0, \hat{x}, x)$ we see that $\phi_0(x_0, \hat{x}, x)$ satisfies the conditions in Lemma 2.1, and thus

$$\phi_0(x_0, \hat{x}, x) = \exp(\pi n\sqrt{-1}\,(x\tau^t x + 2\hat{x}^t x - 2x_0)) \sum_{b \in Z^g/nZ^g} \alpha_b \vartheta(n)\begin{bmatrix} b/n \\ 0 \end{bmatrix}(\tau \,|\, \hat{x} + x\tau)$$

with constants $\alpha_b$. Hence

$$\phi_0(x_0, \hat{x}, x) \in \bigoplus_{b \in Z^g/nZ^g} H^{(n)}\begin{bmatrix} b/n \\ 0 \end{bmatrix} = H^{(n)}.$$

On the other hand $W$ is spanned by $\hat{D}^j\phi_0 (j \in Z^g_{\geqq 0})$, and thus

$$W \subset \bigoplus_{b \in Z^g/nZ^g} H^{(n)}\begin{bmatrix} b/n \\ 0 \end{bmatrix}.$$

From the relation

$$\begin{aligned} &\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\Big(\tau \,|\, x_0, \hat{x} + \frac{\hat{a}}{n}, x\Big) \\ &\quad = \exp\Big(2\pi\sqrt{-1}\Big(\hat{a}^t x + \frac{1}{n}\hat{a}^t a\Big)\Big)\phi_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \,|\, x_0, \hat{x}, x) \\ &\qquad\qquad (\hat{a} \in Z^g/nZ^g,\ j \in Z^g_{\geqq 0}) \end{aligned}$$

we observe that an element $\phi$ in

$$\bigoplus_{b \in Z^g/nZ^g} H^{(n)}\begin{bmatrix} b/n \\ 0 \end{bmatrix} \quad \text{belongs to} \quad H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$$

if and only if

$$\phi\left(x_0, \hat{x} + \frac{\hat{a}}{n}, x\right) = \exp\left(2\pi\sqrt{-1}\left(\hat{a}^t x + \frac{1}{n}\hat{a}^t a\right)\right)\phi(x_0, \hat{x}, x)$$

$$(\hat{a} \in Z^g/nZ^g).$$

On the other hand $\bigoplus_{b\in Z^g/nZ^g} H^{(n)}\begin{bmatrix} b/n \\ 0 \end{bmatrix}$ are independent on the choice of $\tau$, hence each $H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}$ is independent on the choice of $\tau$. From (1.18) we have

$$H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix} = \exp(2\pi\sqrt{-1}\, a^t\hat{x}) H^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

COROLLARY 2.2.1. *For a non-zero integer $n$, $\rho_n$ means the irreducible unitary representation such that*

$$\rho_n(y_0, 0, 0)\phi = \exp(-2\pi n\sqrt{-1}\, y_0)\phi \qquad (y_0 \in R),$$

*then the multiplicity $m_{\rho:\rho_n}$ of $\rho_n$ in $\rho$ is given by*

$$m_{\rho:\rho_n} = |n|^g. \tag{2.3}$$

*Proof.* The space of theta functions of level $n$ is a vector space of dimension $n^g$, hence by virtue of Theorem 2.1 we have (2.3).

**2.2.** Using Hermitian polynomials we shall first construct an orthogonal basis of $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$ associating with $\tau$, and define a natural unitary representation of $Sp_{2g}(R)$ on $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$.

Hermitian polynomials $H_n(v)$ in one variable $v$ are defined by the generating function

$$\exp(-(s^2 - 2sv)) = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(v), \tag{2.4}$$

which satisfy the orthogonal relation:

$$\int_{-\infty}^{\infty} H_n(v)H_m(v)e^{-v^2}dv = \begin{cases} 2^n n!\sqrt{\pi} & (n = m) \\ 0 & (n \neq m). \end{cases} \tag{2.5}$$

For $j = (j_1, \cdots, j_g) \in Z^g_{\geqq 0}$ and $x = (x_1, \cdots, x_g)$ we denote

$$H_j(x) = H_{j_1}(x_1) \cdots H_{j_g}(x_g) \tag{2.6}$$

then Hermitian polynomials in many variables satisfy the orthogonal relation

$$\int_{R^g} H_j(x)H_k(x)\exp(-x^t x)dx = \begin{cases} 2^{|j|}j!\pi^{g/2} & (j = k) \\ 0 & (j \neq k). \end{cases} \tag{2.7}$$

Since $\tau'' = (1/2\sqrt{-1})(\tau - \bar{\tau})$ is positive definite, we can define the unique square root $\sqrt{\tau''}$. The composite functions

$$H_j(x\sqrt{2\pi\tau''}) \qquad (j \in Z^g_{\geqq 0})$$

satisfy the orthogonal relation:

$$\int_{R^g} H_j(x\sqrt{2\pi\tau''})H_k(x\sqrt{2\pi\tau''})\exp(-2\pi n x\tau''^t x)dx = \begin{cases} \dfrac{2^{|j|}j!}{2^{g/2}\sqrt{\det\tau''}} & (j = k) \\ 0 & (j \neq k). \end{cases} \tag{2.8}$$

THEOREM 2.2. *Putting*

$$H_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) \tag{2.9}$$
$$= \frac{2^{g/4}(\det\tau'')^{1/4}}{2^{|j|/2}\sqrt{j!}} \exp(-2\pi n\sqrt{-1}\,x_0) \sum_{\ell \in Z^g} H_j\left(\left(x + \ell + \frac{a}{n}\right)\sqrt{2\pi\tau''}\right)$$
$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(x + \ell + \frac{a}{n}\right)\tau^t\left(x + \ell + \frac{a}{n}\right) + 2\hat{x}^t\left(x + \ell + \frac{a}{n}\right)\right)\right)$$
$$(a \in Z^g/nZ^g,\ j \in Z^g_{\geqq 0},\ n \geqq 1),$$

*we obtain an orthonormal basis*

$$\left\{H_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) \,\middle|\, j \in Z^g_{\geqq 0}\right\} \quad \textit{of} \quad H^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}.$$

*Proof.* From the orthogonal relation for $H_j(x\sqrt{2\pi\tau''})$ and Lemma 1.2 it follows the orthogonal relation

$$\int_{H_{2g+1}(Z)\backslash H_{2g+1}(R)} \overline{H_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)}\, H_k^{n}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)\, dx_0 d\hat{x} dx$$
$$= \begin{cases} 1 & (j = k) \\ 0 & (j \neq k). \end{cases}$$

COROLLARY 2.2.1. $L^2(H_{2g+1}(Z)\backslash H_{2g+1}(R))$ has an orthonormal basis

$$\left\{H_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x),\ \overline{H_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x)},\right. \tag{2.12}$$
$$\exp(2\pi\sqrt{-1}\,(\hat{k}^t\hat{x} + k^t x)) \mid a \in Z^g/nZ^g,$$
$$\left. j \in Z^g_{\geqq 0},\ n \geqq 1,\ (\hat{k}, k) \in Z^g \times Z^g\right\},$$

such that

$$H_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) = \exp(2\pi\sqrt{-1}\,\hat{a})\,H_j^{(n)}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x) \circ \left(0, 0, \frac{a}{n}\right)\biggr). \tag{2.13}$$

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