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COUNTABLE J_a^S-ADMISSIBLE ORDINALS

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§ 0. Introduction

In [3], Platek constructs a hierarchy of jumps J_a^S indexed by elements a of a set $\mathcal{O}^S$ of ordinal notations. He asserts that a real $X \subseteq \omega$ is recursive in the superjump S if and only if it is recursive in some J_a^S. Unfortunately, his assertion is not correct as is shown in [1]. In [1], it also has been shown that an ordinal $> \omega$ is J_a^S-admissible if it is $|a|_S$-recursively inaccessible, where $|a|_S$ is the ordinal denoted by a.

Let A be an arbitrary set. We say that an oridinal α is A-admissible if the structure $\langle L_\alpha[A], \in, A \cap L_\alpha[A]\rangle$, which we will denote by $L_\alpha[A]$ for simplicity, is admissible, a model of the Kripke-Platek set theory KP, where $L_\alpha[A]$ is the sets constructible relative to A in fewer than α steps. We use ω_1^A or $\omega_1(A)$ to denote the first A-admissible ordinal $> \omega$, and use $\omega_1(A_1, \cdots, A_n)$ for $\omega_1(\langle A_1, \cdots, A_n\rangle)$.

The purpose of this paper is to prove the following theorem.

THEOREM 1. *Suppose $a \in \mathcal{O}^S$ and $\alpha > \omega$ is a countable $|a|_S$-recursively inaccessible ordinal. Then, there exists a real $X \subseteq \omega$ such that $\alpha = \omega_1(J_a^S, X)$.*

In the case $|a|_S = 0$, $J_a^S = {}^2E$, the Kleene object of type 2, and $\omega_1({}^2E, X) = \omega_1^X$ for all reals $X \subseteq \omega$. α is an admissible oridnal if and only if it is 0-recursively inaccessible. Therefore, Theorem 1 is an extension of the following theorem of Sacks.

THEOREM 2 (Sacks [4]). *If $\alpha > \omega$ is a countable admissible ordinal, then there exists a real X such that $\alpha = \omega_1^X$.*

Sacks also showed that the real X mentioned in Theorem 2 can be taken to have the minimality property:

$$\omega_1^Y < \alpha \text{ for every } Y \text{ of lower hyperdegree than } X.$$

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Likewise, we can show that for every countable $|a|_S$-recursively inaccessible $\alpha > \omega$ there is a real X such that:

$$\alpha = \omega_1(J^S_\alpha, X) ;$$

and

$$\omega_1(J^S_\alpha, Y) < \alpha \quad \text{for every } Y \text{ of lower } J^S_\alpha\text{-degree than } X.^{1)}$$

Theorem 1 will be proved by the forcing with J^S_α-pointed perfect trees. Let $\alpha > \omega$ be a countable $|a|_S$-recursively inaccessible ordinal and X be a generic real with respect to this forcing relation. Then $L_\alpha[X]$ is admissible and $\alpha \leqq \omega_1(J^S_\alpha, X)$. To see $\omega_1(J^S_\alpha, X) \leqq \alpha$, we must show that X preserves sufficiently many admissible ordinals below α to make α to be $\langle J^S_\alpha, X\rangle$-admissible.

§ 1. $|a|_S$-recursively inaccessible ordinals

A normal type 2 object is a total function F from ω^ω to ω such that the Kleene object ${}^2E$ of type 2:

$${}^2E(f) = \begin{cases} 0 & \text{if } (\exists n) f(n) = 0\,, \\ 1 & \text{otherwise,} \end{cases}$$

is recursive in F. The superjump $S(F)$ of F is a type 2 object defined by:

$$S(F)(\langle n, f\rangle) = \begin{cases} 0 & \text{if } \{n\}^F(f) \text{ is defined}\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Platek [3] defines a hierarchy J^S_a of type 2 objects along with a set $\mathcal{O}^S$ of ordinal notations, starting from ${}^2E$ and iterating the superjump operation transfinitely.

An ordinal α is 0-recursively inaccessible if it is admissible. α is $(\sigma+1)$-recursively inaccessible if it is σ-recursively inaccessible and a limit of σ-recursively inaccessible ordinals. For limit λ, α is said to be λ-recursively inaccessible if it is σ-recursively inaccessible for all $\sigma < \lambda$. Let X be an arbitrary set. σ-recursively-in-X inaccessible ordinals are defined in the same way starting from X-admissible ordinals. By $RI(\sigma, X)$, we denote the class of all σ-recursively-in-X inaccessible ordinals. In the case $X=\varnothing$, $RI(\sigma, \varnothing)$ is the class of all σ-recursively inaccessible ordinals.

The following lemma, due to Aczel and Hinman, gives a characterization of $\omega_1(J^S_a, X)$ for $X \subseteq \omega$.

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1) For J^S_a-degrees, the reader may refer to [5].

LEMMA 3 (Aczel and Hinman [1]). *Suppose* $a \in \mathcal{O}^S$ *and* $\sigma = |a|_S$, *the ordinal denoted by a. Then* $\sigma < \omega_1(J^S_a)$, *and for any ordinal* $\alpha > \omega$ *and* $X \subseteq \omega$:

$$\alpha \in RI(\sigma, X) \rightarrow \alpha \quad \text{is } \langle J^S_a, X\rangle\text{-admissible},$$

and $\omega_1(J^S_a, X)$ *is the least ordinal in* $RI(\sigma, X)$.

Let λ_0 be the least ordinal λ such that λ is λ-recursively inaccessible. Lemma 3 shows that $|\mathcal{O}^S| = \sup\{|a|_S\colon a \in \mathcal{O}^S\} \leqq \lambda_0$. In [1], it has been shown that $|\mathcal{O}^S| = \lambda_0$.

Let $\alpha > \omega$ be a countable admissible ordinal. Using the unbounded Levy forcing over L_α, we can add to L_α a generic function $K\colon (\alpha - \omega) \times \omega \rightarrow \alpha$ such that if $\omega \leqq \beta < \alpha$ then the function $\lambda n K(\beta, n)$ is a bijection from ω onto β. Therefore, in $L_\alpha[K]$ all sets are countable. It has been shown in [4] that $\langle L_\alpha[K], \in, K\rangle$ is an admissible structure in which Σ_1-DC (Σ_1-Dependent Choice) holds.

Suppose $a \in 0^S$. For any $X, Y \subseteq \omega$, $X \leq_{J^S_a} Y$ means X is recursive in $\langle J^S_a, Y\rangle$, which is equivalent to that $X \in L_\rho[J^S_a, Y]$, where $\rho = \omega_1(J^S_a, Y)$. X and Y have the same J^S_a-degree, $X \equiv_{J^S_a} Y$, if $X \leq_{J^S_a} Y$ and $Y \leq_{J^S_a} X$. $X <_{J^S_a} Y$ if $X \leq_{J^S_a} Y$ but $X \not\equiv_{J^S_a} Y$.

LEMMA 4. *Suppose* $\alpha > \omega$ *is a countable* $|a|_S$*-recursively inaccessible ordinal and* K *is a generic function with respect to the unbounded Levy forcing over* L_α. *Then for any* $X, Y \subseteq \omega$:

$$X \leq_{J^S_a} Y \quad \text{and} \quad Y \in L_\alpha[K] \longrightarrow X \in L_\alpha[K].$$

Proof. The unbounded Levy forcing preserves admissible ordinals. That is, if $\beta < \alpha$ is an admissible ordinal then β is K-admissible. This is because for admissible β, $K \upharpoonright (\beta - \omega) \times \omega$ is generic with respect to the unbounded Levy forcing over L_β. Therefore, if $Y \in L_\alpha[K]$ then α is $|a|_S$-recursively-in-Y inaccessible, so $L_\rho[Y] \subseteq L_\alpha[K]$, where $\rho = \omega_1(J^S_a, Y)$. Thus we have the lemma. □

§ 2. J^S_a-pointed perfect trees

Let a be an element of $\mathcal{O}^S$ such that $|a|_S > 0$. We put $J = J^S_a$ for simplicity.

A perfect tree is a set P of finite sequences of 0's and 1's such that:

(1) $$p \in P \quad \text{and} \quad q \subseteq p \longrightarrow q \in P;$$

and

(2) $(\forall p \in P)(\exists q, r \in P)$ (q and r are incompatible extensions of p),

where $q \subseteq p$ denotes that p is an extension of q. For a perfect tree P, $[P]$ denotes the set of all infinite paths through P:

$$[P] = \{f \in 2^\omega \colon (\forall n)\bar{f}(n) \in P\}.$$

We say that P is J-pointed if:

(3) $(\forall f \in [P])(\omega_1(J, P) \leqq \omega_1(J, f)$ and $P \in L_{\omega_1(J,P)}[f])$.

Note that if P is J-pointed then it is $\leq_J$-pointed in the sense of Sacks [4:2.1], but not vice versa.

LEMMA 5. *Suppose P is J-pointed. If $X \subseteq \omega$ and $P \leq_J X$, then there exists a J-pointed $Q \subseteq P$ such that $Q \equiv_J X$.*

Proof. In [4: 2.3], Sacks constructed a perfect subtree Q of P such that:

(4) Q is recursive in P and f for every $f \in [Q]$;

and

(5) $$Q \equiv_J X.$$

To see Q is J-pointed in our sense, fix $f \in [Q]$. Since P is J-pointed and $f \in [P]$, by (3), we have:

(6) $$P \in L_{\omega_1(J,P)}[f].$$

Clearly:

(7) $$f \in L_{\omega_1(J,P)}[f].$$

From (4), (6) and (7), we obtain:

(8) $$Q \in L_{\omega_1(J,P)}[f].$$

From (5) and the assumption $P \leq_J X$, we see:

(9) $$\omega_1(J, P) \leqq \omega_1(J, Q).$$

From (8) and (9), we obtain $Q \in L_{\omega_1(J,Q)}[f]$. □

For any ordinal δ, $\{\delta\}^f$ denotes the δ-th element of $L[f]$ in the canonical wellordering on $L[f]$. A perfect tree P is said to be uniformly J-pointed if there exists an ordinal δ such that:

$$(\forall f \in [P])(P = \{\delta\}^f \quad \text{and} \quad \delta < \omega_1(J, f)) . \tag{10}$$

Obviously, uniformly J-pointed perfect trees are J-pointed. Let $\alpha > \omega$ be a countable $|a|_S$-recursively inaccessible ordinal and K a generic function over L_α in the sense of the unbounded Levy forcing. Observe that if P is uniformly J-pointed and $P \in L_\alpha[K]$ then there exists a $\delta < \alpha$ which satisfies (10) since the leftmost path f_P through P is recursive in P and so $\omega_1(J, f_P) \leqq \omega_1(J, P) < \alpha$.

Let M be a countable admissible set and P be a perfect tree in M. Then P becomes a partially ordered set as usual. The forcing with P as the set of conditions is called the local Cohen forcing over M and denoted by $\Vdash^{P}_{M}$, or simply by $\Vdash^{P}$. If $f \in [P]$ is generic with respect to $\Vdash^{P}$, then $M[f]$ is an admissible set, and so is $L_\mu[f]$, where $\mu = M \cap On$.

Lemma 6. *For any* $\xi < \alpha$ *and any J-pointed perfect tree P in* $L_\alpha[K]$, *there exists a uniformly J-pointed perfect tree* $Q \subseteq P$ *such that* $\xi < \omega_1(J, Q)$ *and* $Q \in L_\alpha[K]$.

Proof. Since ξ is countable in $L_\alpha[K]$, there is a real $X \in L_\alpha[K]$ such that ξ is recursive in X. By Lemma 5, there is a J-pointed perfect subtree P_1 of P such that $P_1 \equiv_J X$. Then we see $\xi < \omega_1(J, P_1)$, and $P_1 \in L_\alpha[K]$ by Lemma 4. Thus, we may assume $\xi < \omega_1(J, P)$ from the beginning. Put $M = L_{\omega_1(J,P)}[P]$. Consider the local Cohen forcing relation $\Vdash^{P}_{M}$ over M. Since P is J-pointed, we have:

$$(\forall f \in [P])\omega_1(J, P) \leqq \omega_1(J, f) ; \tag{11}$$

and

$$(\forall f \in [P])(\exists \gamma < \omega_1(J, P))\{\gamma\}^f = P . \tag{12}$$

By (12), there exists a $p_0 \in P$ and $\gamma < \omega_1(J, P)$ such that:

$$p_0 \Vdash^{P}_{M} \{\check{\gamma}\}^{\mathscr{T}} = \check{P} , \tag{13}$$

where $\mathscr{T}$ is the canonical name which denotes the generic reals. As in [4: 2.10], we can construct a perfect tree $Q \subseteq P$ such that:

$$Q \in L_{\omega_1(J,P)}[P] ; \tag{14}$$

and

$$(15) \qquad (\forall f \in [Q])\{\gamma\}^f = P\,.$$

From (14), we can find a $\delta < \omega_1(J, P)$ such that $\{\delta\}^P = Q$. So, by (15), there is an $\varepsilon < \omega_1(J, P)$ such that:

$$(16) \qquad (\forall f \in [Q])\{\varepsilon\}^f = Q\,.$$

Let f_Q be the leftmost branch of Q. Then, by (11):

$$(17) \qquad \omega_1(J, P) \leqq \omega_1(J, f_Q) \leqq \omega_1(J, Q)\,.$$

Hence, from (16), we see that Q is uniformly J-pointed. By (17), we also see $\xi < \omega_1(J, Q)$. Since $P \in L_\alpha[K]$, we have $\omega_1(J, P) \leqq \alpha$, and so $Q \in L_{\omega_1(J,P)}[P] \subseteq L_\alpha[K]$. □

Let $\mathscr{L}$ be a first-order language. A Π^1_1 formula in $\mathscr{L}$ is a second-order formula of the form:

$$(\forall S_1) \cdots (\forall S_m)\psi\,,$$

where $S_1, \cdots, S_m$ are predicate variables and ψ is a first-order formula in the expanded language $\mathscr{L} \cup \{S_1, \cdots, S_m\}$.

LEMMA 7. *Suppose A is a countable admissible set such that $\omega \in A$ and $\mathscr{L} \in A$ is a first-order language. Let $\theta(x_1, \cdots, x_n)$ be a Π^1_1 formula in $\mathscr{L}$. Then there exists a Σ_1 formula $\Phi(x_1, \cdots, x_n, y)$ such that for any structure $\mathscr{M} = \langle M, \cdots \rangle \in A$ for $\mathscr{L}$ and any $a_1, \cdots, a_n \in M$:*

$$A \models \Phi(a_1, \cdots, a_n, \mathscr{M}) \longleftrightarrow \mathscr{M} \models \theta(a_1, \cdots, a_n)\,.$$

Proof. This is well-known. See, e.g., Barwise [2: IV. 3.1]. □

Using this lemma, we obtain the following lemma.

LEMMA 8. *The set of all uniformly J-pointed perfect trees in $L_\alpha[K]$ is Σ_1 over $L_\alpha[K]$.*

Proof. Put $\sigma = |\alpha|_S$, (recall that $J = J^S_\alpha$). Let P be a perfect tree in $L_\alpha[K]$ and $\delta < \alpha$. Let $\beta(P, \delta, \sigma)$ denote the least admissible ordinal $\beta < \alpha$ such that $\max(\delta, \sigma, \omega) < \beta$ and $P \in L_\beta[K]$. The function β is Σ_1 over $L_\alpha[K]$. We can easily find a Π^1_1 formula θ in the language of set theory such that for any perfect tree $P \in L_\alpha[K]$:

$$P \text{ is uniformly } J\text{-pointed} \longleftrightarrow (\exists \delta < \alpha) L_{\beta(P,\delta,\sigma)}[K] \models \theta(P, \delta, \sigma)\,.$$

Thus the lemma follows from Lemma 7. □

§ 3. Forcing with uniform J_a^S-pointed perfect trees

Suppose $|a|_S > 0$ and put $J = J_a^S$. Let $\alpha > \omega$ be a countable $|a|_S$-recursively inaccessible ordinal and K a generic function with respect to the unbounded Levy forcing over L_α, which we fix throughout this section.

Let $\mathscr{L}(\alpha, \mathscr{T})$ be a ramified language containing names for all members of $L_\alpha[f]$. $\mathscr{L}(\alpha, \mathscr{T})$ includes: a numeral $\bar{n}$ for each $n \in \omega$, unranked variables $x, y, z, \cdots$; ranked variables $x^\beta, y^\beta, z^\beta, \cdots$ for each $\beta < \alpha$; and abstraction operator $\hat{}$. It is intended that $\mathscr{T}$ denotes $\{n \in \omega : f(n) = 1\}$, that x ranges over $L_\alpha[f]$, that x^β ranges over $L_\beta[f]$, and that $\hat{x}^\beta \phi(x^\beta)$ denotes the set:

$$\{x \in L_\beta[f] : L_\beta[f] \models \phi(x)\}.$$

$C(\beta)$ is the set of names for elements of $L_\beta[f]$ and $C = \bigcup_{\beta<\alpha} C(\beta)$.

Let $\mathscr{P}$ denote the set of all uniformly J-pointed perfect trees in $L_\alpha[K]$. $P, Q, R, \cdots$ denote the members of $\mathscr{P}$. For a ranked sentence ϕ of $\mathscr{L}(\alpha, \mathscr{T})$ and $P \in \mathscr{P}$, let $\rho(P, \phi)$ be the least admissible ordinal $\rho < \alpha$ such that $P \in L_\rho[K]$ and $\operatorname{rank}(\phi) < \rho$. The function ρ is Σ_1 over $L_\alpha[K]$. The forcing relation $P \Vdash \phi$, where ϕ is a sentence of $\mathscr{L}(\alpha, \mathscr{T})$, is defined inductively:

(1) ϕ is ranked. $P \Vdash \phi$ iff $(\forall f \in [P]) L_{\rho(P,\phi)}[f] \models \phi$;

(2) $\phi \vee \psi$ is not ranked. $P \Vdash \phi \vee \psi$ iff $P \Vdash \phi$ or $P \Vdash \psi$;

(3) $(\exists x^\beta)\phi(x^\beta)$ is not ranked. $P \Vdash (\exists x^\beta)\phi(x^\beta)$ if $P \Vdash \phi(c)$ for some $c \in C(\beta)$;

(4) $P \Vdash (\exists x)\phi(x)$ iff $P \Vdash \phi(c)$ for some $c \in C$;

(5) ϕ is not ranked. $P \Vdash \neg\phi$ iff $(\forall Q \subseteq P) \neg (Q \Vdash \phi)$.

Using Lemmas 7 and 8, it is easy to see that the forcing relation $P \Vdash \phi$, restricted Σ_1 sentences ϕ, is Σ_1 over $L_\alpha[K]$.

LEMMA 9. *For each P and ϕ, there exists a $Q \subseteq P$ such that $Q \Vdash \phi$ or $Q \Vdash \neg\phi$.*

Proof. In view of (5), we may assume that ϕ is ranked. By Lemma 6, we may also assume that $\phi \in L_\delta[P]$ for some P-admissible δ such that $\delta < \omega_1(J, P)$. Then, in $L_\delta[P]$, all sets are countable. Thus, in $L_\delta[P]$, we can enumerate all ranked sentences of rank $\leq \operatorname{rank}(\phi)$:

$$\phi = \phi_0, \phi_1, \cdots, \phi_n, \cdots \qquad (n \in \omega).$$

Let $\overset{P}{\Vdash}$ be the local Cohen forcing relation over $L_\delta[P]$. In $L_\delta[P]$, we can construct a family $\langle q_s : s \in \mathrm{Seq}(2)\rangle$ of elements of P such that:

(6) $q_s \overset{P}{\Vdash} \phi_n$ or $q_s \overset{P}{\Vdash} \neg\phi_n$, where $n = \ell h(s)$;

and

(7) $q_{s\frown\langle 0\rangle}$ and $q_{s\frown\langle 1\rangle}$ are incompatible extensions of q_s,

where Seq(2) is the set of all finite sequences of 0's and 1's. Let $Q = \{q \in P : (\exists s) q \subseteq q_s\}$. Then by (7) Q is a perfect subtree of P. By (6), it is easy to see that $Q \Vdash \phi$ or $Q \Vdash \neg\phi$. Since $Q \in L_\delta[P]$, $Q = \{\gamma\}^P$ for some $\gamma < \delta$. Therefore Q is uniformly J-pointed because P is. □

A real $f \in 2^\omega$ is said to be generic if for every dense subset $\mathscr{D}$ of $\mathscr{P}$ which is definable over $L_\alpha[K]$ there is a $P \in \mathscr{D}$ such that $f \in [P]$. For every $P \in \mathscr{P}$, there is a generic f such that $f \in [P]$. From Lemma 9, it follows that for every generic f and sentence ϕ:

$$L_\alpha[f] \models \phi \quad \text{iff} \quad (\exists P)(f \in [P] \text{ and } P \Vdash \phi)\,.$$

LEMMA 10. *If f is generic, then $L_\alpha[f]$ is admissible.*

Proof. We need to show that $L_\alpha[f]$ satisfies the Δ_0-Collection. Let $\phi(x, y)$ be a formula of $\mathscr{L}(\alpha, \mathscr{T})$ with no unranked quantifiers. We claim that if $P \Vdash (\forall n)(\exists y)\phi(n, y)$ then there exists a $Q \subseteq P$ and a $\beta < \alpha$ such that $Q \Vdash (\forall n)(\exists y^\beta)\phi(n, y^\beta)$. The proof of this claim is almost the same as that of [4: 3.12] with some notational changes. So, we omit the proof here. From the claim, it follows that $L_\alpha[f]$ satisfies the Δ_0-Collection. □

Proof of Theorem 1. Let $\alpha > \omega$ be a countable $|\alpha|_S$-recursively inaccessible ordinal and K be as before. Put $\sigma = |\alpha|_S$ and $J = J^S_\alpha$. In the case $\sigma = 0$, Theorem 1 is exactly Theorem 2, which has already been established by Sacks [4]. So we may assume $\sigma > 0$. Let $f_0 \in 2^\omega$ be a generic real over $L_\alpha[K]$ with respect to the forcing with uniform J-pointed perfect trees. By Lemma 6, for each $\xi < \alpha$, the set $\{P \in \mathscr{P} : \xi < \omega_1(J, P)\}$ is dense in $\mathscr{P}$. It is obviously definable over $L_\alpha[K]$. Therefore there is a $P \in \mathscr{P}$ such that $f_0 \in [P]$ and $\xi < \omega_1(J, P)$. Since P is J-pointed, we have:

$$\xi < \omega_1(J, P) \leqq \omega_1(J, f_0)\,.$$

Thus, we have $\alpha \leqq \omega_1(J, f_0)$. To see $\alpha = \omega_1(J, f_0)$, we must show that $\alpha \in RI(\sigma, f_0)$. At first we consider the case where $\sigma = \tau + 1$ for some τ. It is sufficient to prove that α is a limit of ordinals in $RI(\tau, f_0)$, since then by induction on τ we can show that $\alpha \in RI(\tau, f_0)$, (note that $\alpha \in RI(0, f_0)$ by Lemma 10). Suppose $\xi < \alpha$. We shall show that the following set $\mathscr{D}_\xi$ is dense in $\mathscr{P}$:

$$\mathscr{D}_\xi = \{P \in \mathscr{P}\colon (\exists\delta < \alpha)(\xi < \delta \text{ and } (\forall f \in [P])\delta \in RI(\tau, f))\}\,.$$

Assume this can be done. Using Lemma 7, it is easy to see that $\mathscr{D}_\xi$ is Σ_1 over $L_\alpha[K]$. Therefore, for every $\xi < \alpha$, there exists a $\delta < \alpha$ such that $\xi < \delta$ and $\delta \in RI(\tau, f_0)$

To show that $\mathscr{D}_\xi$ is dense in $\mathscr{P}$, take an arbitrary $P \in \mathscr{P}$. By Lemma 6, we may assume $\xi < \omega_1(J, P)$. Take a $\delta \in RI(\tau, P)$ so that $\xi < \delta < \omega_1(J, P)$. Such a δ exists because $\omega_1(J, P)$ is a limit of ordinals in $RI(\tau, P)$. Consider the local Cohen forcing relation $\Vdash^{P}$ over $L_\delta[P]$. Let δ^+ be the next P-admissible ordinal of δ, Then, $L_\delta[P]$ is countable in $L_{\delta^+}[P]$. So we can enumerate inside $L_{\delta^+}[P]$ all sentences of the appropriate forcing language:

$$\phi_0, \phi_1, \cdots, \phi_n, \cdots \qquad (n \in \omega)\,.$$

As in the proof of Lemma 9, we can construct a perfect subtree $Q \in L_{\delta^+}[P]$ of P such that:

$$(\forall f \in [Q])f \text{ is generic with respect to } \Vdash^{P}\,.$$

Q is uniformly J-pointed since $Q \in L_{\delta^+}[P]$, $\delta^+ < \omega_1(J, P)$ and P is uniformly J-pointed. To show that $\delta \in RI(\tau, f)$ for all $f \in [Q]$, take $f \in [Q]$. Let $\beta \leqq \delta$ be an arbitrary P-admissible ordinal $> \omega$, and $\Vdash^{P}_{\beta}$ be the local Cohen forcing relation over $L_\beta[P]$. It is easy to see that f is generic with respect to $\Vdash^{P}_{\beta}$, and so β is f-admissible. From this, by induction on τ, we see that $\delta \in RI(\tau, f)$.

Now we consider the case where σ is a limit ordinal. The proof is carried out in the same way. For any $\xi < \alpha$ and any $\tau < \sigma$, let $\mathscr{D}_{\xi\tau}$ be the set:

$$\{P \in \mathscr{P}\colon (\exists\delta < \alpha)(\xi < \delta \text{ and } (\forall f \in [P])\delta \in RI(\tau, f))\}\,.$$

Then $\mathscr{D}_{\xi\tau}$ is dense in $\mathscr{P}$ and definable over $L_\alpha[K]$. Therefore, we have that $\alpha = \omega_1(J, f_0)$ for any generic f_0 with respect to $\Vdash$. $\square$

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