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A REMARK ON SMITH'S RESULT ON A DIVISOR PROBLEM IN ARITHMETIC PROGRESSIONS

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§1. Introduction

Let $d_k(n)$ be the number of the factorizations of n into k positive numbers. It is known that the following asymptotic formula holds:

$$\sum_{\substack{n\leq x \\ n\equiv r (\mathrm{mod}\ q)}} d_k(n) = \varphi(q)^{-1}xP_k(\log(x)) + \Delta_k(q;r) ,$$

where r and q are co-prime integers with $0 < r < q$, P_k is a polynomial of degree $k-1$, $\varphi(q)$ is the Euler function, and $\Delta_k(q;r)$ is the error term. (See Lavrik [3]).

In 1982, R. A. Smith [5] proved that if $(r, q) = 1$, then for $x \geqq q^{\frac{1}{2}(k+1)}$,

$$\Delta_k(q;r) = F_k(0) + O(x^{(k-1)/(k+1)}(\log(2x))^{k-1}d_k(q)) , \tag{1.1}$$

where the function $F_k(s)$ is the meromorphic continuation of the Dirichlet series

$$\sum_{n\equiv r (\mathrm{mod}\ q)} d_k(n)n^{-s} .$$

The proof of Smith depends essentially on Deligne's famous work concerning Weil's conjecture [1].

A remaining problem is the estimation of the term $F_k(0)$. In the "Note added in proof" of [5], Smith announced the estimate $F_k(0) \ll q^{\frac{1}{2}k}(\log(q))^k$, so the explicit upper bound of $\Delta_k(q;r)$ obtained by Smith is as follows:

$$\Delta_k(q;r) = O(q^{\frac{1}{2}k}(\log(q))^k + x^{(k-1)/(k+1)}(\log(2x))^{k-1}d_k(q)) . \tag{1.2}$$

Furthermore, Smith conjectured that the upper bound of $F_k(0)$ can be improved to $q^{\frac{1}{2}(k-1)+\varepsilon}$ for any $\varepsilon > 0$. He said, "I will return to this problem at another time." But, unfortunately, he suddenly passed away in March 1983, at forty-six years old.

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In this note we shall prove this conjecture of Smith:

THEOREM. *If* $(q, r) = 1$, *then for any* $\varepsilon > 0$,

$$F_k(0) = O(q^{\frac{1}{2}(k-1)+\varepsilon}) ,$$

where the O-constant depends only on k and ε.

This result was already proved in 1982, and appeared in [4] in March 1983, without knowing the existence of Smith's paper [5].

Our method also depends on Deligne's work. We shall use Weinstein's version [7] of Deligne's result, which gives the following sharp estimate of the "hyper-Kloosterman sum".

LEMMA (Deligne-Weinstein). *If we put*

$$S(m_1, \cdots, m_k; q) = \sum_{\substack{1 \leqq \alpha_i \leqq q (1 \leqq i \leqq k) \\ \alpha_1 \cdots \alpha_k \equiv 1 (\bmod q)}} \exp(2\pi i q^{-1}(m_1\alpha_1 + \cdots + m_k\alpha_k)) ,$$

then,

$$S(m_1, \cdots, m_k; q) \ll k^{v(q)} q^{\frac{1}{2}(k-1)} (m_1, m_k, q)^{\frac{1}{2}} \cdots (m_{k-1}, m_k, q)^{\frac{1}{2}} ,$$

where $v(q)$ *is the number of distinct prime factors of* q, *and* (a, b, c) *is the greatest common divisor of* a, b *and* c.

In the next section, we shall prove the Theorem. In Section 3, we mention briefly further comments concerning the estimation of $\Delta_k (q; r)$.

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§ 2. Proof of the Theorem

The function $F(s) = F_k(s)$ is defined as the Dirichlet series

$$F(s) = \sum_{n \equiv r (\bmod q)} d_k(n) n^{-s}$$

for $s = \sigma + it$, $\sigma = \mathrm{Re}(s) > 1$. Then,

$$\begin{aligned} F(s) &= \sum_{\substack{1 \leqq \alpha_1, \cdots, \alpha_k \leqq q \\ \alpha_1 \cdots \alpha_k \equiv r}} \Big(\sum_{u_1 \equiv \alpha_1} u_1^{-s} \Big) \cdots \Big(\sum_{u_k \equiv \alpha_k} u_k^{-s} \Big) \\ &= q^{-ks} \sum_{\substack{1 \leqq \alpha_1, \cdots, \alpha_k \leqq q \\ \alpha_1 \cdots \alpha_k \equiv r}} \zeta(s, q^{-1}\alpha_1) \cdots \zeta(s, q^{-1}\alpha_k) , \end{aligned}$$

where,

$$\zeta(s, w) = \sum_{n=0}^{\infty} (n + w)^{-s} \qquad (0 < w \leqq 1)$$

is the Hurwitz zeta-function. The Hurwitz zeta-function can be analytically continued over whole plane, and holomorphic except at the pole of order one at $s = 1$. And, if $\operatorname{Re}(s) < 0$, then

$$\zeta(s, w) = -i\Gamma(1-s)(2\pi)^{s-1} \sum_{m=1}^{\infty} m^{s-1}\left\{e(mw)e\left(\frac{1}{4}s\right) - e(-mw)e\left(-\frac{1}{4}s\right)\right\},$$

where $\Gamma(s)$ is the Γ-function, and $e(x) = \exp(2\pi i x)$. (See Titchmarsh [6], Chap. II) Hence, the function $F(s)$ is also meromorphic over whole plane, holomorphic except at $s = 1$, and if $\operatorname{Re}(s) < 0$,

$$F(s) = q^{-ks} \sum_{\alpha_1, \cdots, \alpha_k} (-i\Gamma(1-s)(2\pi)^{s-1})^k \times \prod_{j=1}^{k} \sum_{m_j=1}^{\infty} m_j^{s-1}\left\{e(m_j\alpha_j/q)e\left(\frac{1}{4}s\right) - e(-m_j\alpha_j/q)e\left(-\frac{1}{4}s\right)\right\}.$$

Let $\varepsilon_j = \pm 1$ $(1 \leqq j \leqq k)$, and $E(\varepsilon_1, \cdots, \varepsilon_k)$ be the number of j such that $\varepsilon_j = -1$. Then,

$$F(s) = q^{-ks}(-i\Gamma(1-s)(2\pi)^{s-1})^k \sum_{\varepsilon_1, \cdots, \varepsilon_k = \pm 1} (-1)^{E(\varepsilon_1, \cdots, \varepsilon_k)} \times e\left(\frac{1}{4}(\varepsilon_1 + \cdots + \varepsilon_k)s\right) \sum_{u=1}^{\infty} u^{s-1} \sum_{m_1 \cdots m_k = u} \sum_{\substack{1 \leqq \alpha_1, \cdots, \alpha_k \leqq q \\ \alpha_1 \cdots \alpha_k \equiv r}} \times e(q^{-1}(\varepsilon_1 m_1 \alpha_1 + \cdots + \varepsilon_k m_k \alpha_k)).$$

Now we assume $(q, r) = 1$, and estimate the right-hand side by Deligne-Weinstein's lemma. Let α_j^* be the unique solution of the congruence $\alpha_j \alpha_j^* \equiv 1 \pmod{q}$, $1 \leq \alpha_j^* \leq q$. Then, $\alpha_k \equiv \alpha_1^* \cdots \alpha_{k-1}^* r$, so the last sum of the right-hand side is $S(\varepsilon_1 m_1, \cdots, \varepsilon_{k-1} m_{k-1}, \varepsilon_k m_k r; q)$. Using Weinstein's estimate and Stirling's formula, we have

$$F(s) \ll ((1 + |t|)^{\frac{1}{2} - \sigma_0} e^{-\frac{1}{2}\pi|t|} q^{-\sigma_0})^k e^{\frac{1}{2}\pi|t|k} \times \sum_{u=1}^{\infty} u^{\sigma_0 - 1} \sum_{m_1 \cdots m_k = u} k^{v(q)} q^{\frac{1}{2}(k-1)} (m_1, m_k r, q)^{\frac{1}{2}} \cdots (m_{k-1}, m_k r, q)^{\frac{1}{2}}, \tag{2.1}$$

for $s = \sigma_0 + it$, $\sigma_0 < 0$.

Let

$$Z(u) = \sum_{m_1 \cdots m_k = u} (m_1, m_k r, q)^{\frac{1}{2}} \cdots (m_{k-1}, m_k r, q)^{\frac{1}{2}}.$$

Then,

$$|Z(u)| \leqq \sum_{m_1 \cdots m_k = u} (m_1 \cdots m_{k-1}, q^{k-1})^{\frac{1}{2}} \ll u^{\varepsilon}(u, q^{k-1})^{\frac{1}{2}}.$$

So,

$$\sum_{u=1}^{U} |Z(u)| \ll U^{\varepsilon} \sum_{u=1}^{U} (u, q^{k-1})^{\frac{1}{2}} \ll U^{\varepsilon} \sum_{d \mid q^{k-1}} d^{\frac{1}{2}}[U/d] \ll U^{1+\varepsilon} q^{\varepsilon} .$$

Hence, by partial summation,

$$\sum_{u=1}^{\infty} u^{\sigma_0 - 1} Z(u) \ll q^{\varepsilon} .$$

Also, since $v(q) \ll \log(q)/\log\log(q)$, we have $k^{v(q)} \ll q^{\varepsilon}$. Substituting these estimates in (2.1), we have (for $\sigma_0 = \operatorname{Re}(s) < 0$)

$$F(s) \ll (1 + |t|)^{(\frac{1}{2} - \sigma_0)k} q^{\frac{1}{2}(k-1) - \sigma_0 k + \varepsilon} . \tag{2.2}$$

Now, let

$$f(s) = F(s) - d_k(a) a^{-s} ,$$

where $a \equiv r \pmod q$ and $0 < a < q$. Then, for $s = \sigma_0 + it$,

$$f(s) \ll (1 + |t|)^{(\frac{1}{2} - \sigma_0)k} a^{-\sigma_0} q^{\frac{1}{2}(k-1) - \sigma_0 k} (aq)^{\varepsilon} . \tag{2.3}$$

Next, it is easily shown that for $\operatorname{Re}(s) = \sigma_1 > 1$,

$$f(s) \ll q^{\varepsilon - \sigma_1} . \tag{2.4}$$

Now we introduce the function

$$\tilde{f}(s) = f(s) - M(s) ,$$

where $M(s)$ is the meromorphic part of $F(s)$ at $s = 1$. Smith [5] showed that $M(s)$ has the same meromorphic part as the function $\Phi_k(s; q)\zeta^k(s)$, where the definition of $\Phi_k(s; q)$ is as follows:

$$\Phi_k(s; q) = \varphi(q)^{-1} \sum_{d \mid q} d^{-s} \mu(d) ,$$

here $\mu(d)$ is the Möbius function.

Smith ([5], p. 263) proved that $\Phi_k(s; q)$ has the Taylor expansion

$$\Phi_k(s; q) = \sum_{n=0}^{\infty} C_1(n; q)(s - 1)^n$$

at $s = 1$, where the coefficients $C_1(n; q)$ satisfy the following estimates:

$$n! \cdot C_1(n; q) \ll q^{-1}(\log\log(3q))^n .$$

So we can easily show that $M(s)$ satisfies the estimate

$$M(s) \ll q^{-1}(\log\log(3q))^{k-1} \tag{2.5}$$

in the range $\{|s-1| \geqq \varepsilon/2\}$. Hence, $\tilde{f}(s)$ satisfies the same estimates as (2.3) and (2.4), if $\sigma_1 - 1$ is sufficiently small.

Applying the Phragmén-Lindelöf principle to the holomorphic function $\tilde{f}(s)$, we can deduce from (2.3) and (2.4) the following estimate of $\tilde{f}(s)$, for $\sigma_0 \leqq \sigma = \operatorname{Re}(s) \leqq \sigma_1$:

$$\begin{aligned} \tilde{f}(s) \leqq\ & (1+|t|)^{(\frac{1}{2}-\sigma_0)k(\sigma_1-\sigma)/(\sigma_1-\sigma_0)} a^{-\sigma_0(\sigma_1-\sigma)/(\sigma_1-\sigma_0)} \\ & \times q^{((\frac{1}{2}(k-1)-\sigma_0 k)(\sigma_1-\sigma)-\sigma_1(\sigma-\sigma_0))/(\sigma_1-\sigma_0)} (aq)^{\varepsilon} . \end{aligned} \tag{2.6}$$

If we substitute the values $\sigma_1 = 1+\frac{1}{2}\varepsilon$, $\sigma_0 = -\frac{1}{2}\varepsilon$ and $s=0$ in (2.6), we get

$$\tilde{f}(0) \ll q^{\frac{1}{2}(k-1)+\varepsilon} .$$

This result with (2.5) completes the proof of the theorem.

§ 3. Remarks on some estimations of $\varDelta_k$

Heath-Brown [2] handled the Dirichlet series $F_2(s)$, and got an estimate of $\varDelta_2(q;r)$. A generalization of Heath-Brown's argument leads to the following estimate ([4]): If $(q,r)=1$ and $x \geqq q^{\frac{1}{2}(k+1)}$, then for any $\varepsilon > 0$,

$$\varDelta_k(q;r) = O(x^{k/(k+2)+\varepsilon} q^{-1/(k+2)+\varepsilon}) . \tag{3.1}$$

This result improves Smith's estimate (1.2) in the range $q^{\frac{1}{2}(k+1)} \leqq x \leqq q^{(k^2+2k+2)/2k}$.

Now, using the estimate of our theorem, the result (1.1) leads to the estimate

$$\varDelta_k(q;r) = O(q^{\frac{1}{2}(k-1)+\varepsilon} + x^{(k-1)/(k+1)}(\log(2x))^{k-1} d_k(q)) \tag{3.2}$$

instead of (1.2), and this is sharper than (3.1) for any $x \geqq q^{\frac{1}{2}(k+1)}$. (We note here that better estimates are known for some special values of k such as $k=4$. For general k, however, results as sharp as the estimate (3.2) seem to be not known before.)

Our theorem is a direct consequence of Deligne-Weinstein's lemma, so it seems difficult to improve the result (3.2) by the method of this paper. On the other hand, there is a possibility to improve the estimate (3.1), if we can refine the generalization of Heath-Brown's argument [2]. Such a result will improve the estimate of $\varDelta_k(q;r)$ for some range of x.

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