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ON THE NUMBER OF DIFFEOMORPHISM CLASSES IN A CERTAIN CLASS OF RIEMANNIAN MANIFOLDS

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§ 0. Introduction

The study of finiteness for Riemannian manifolds, which has been done originally by J. Cheeger [5] and A. Weinstein [13], is to investigate what bounds on the sizes of geometrical quantities imply finiteness of topological types, —e.g. homotopy types, homeomorphism or diffeomorphism classes— of manifolds admitting metrics which satisfy the bounds. For a Riemannian manifold M we denote by R_M and K_M respectively the curvature tensor and the sectional curvature, by $\mathrm{Vol}(M)$ the volume, and by $\mathrm{diam}(M)$ the diameter.

CHEEGER'S FINITENESS THEOREM I [5]. *For given n, Λ, $V > 0$ there exist only finitely many pairwise non-diffeomorphic (non-homeomorphic) closed $n(\neq 4)$-manifolds (4-manifolds) which admit metrics such that* $|K_M| \leq \Lambda^2$, $\mathrm{diam}(M) \leq 1$, $\mathrm{Vol}(M) \geq V$.

He proved directly finiteness up to homeomorphism for all dimension, and then for $n \neq 4$ used the results of Kirby and Siebenmann which show that finiteness up to homeomorphism implies finiteness up to diffeomorphism. For $n = 4$, he put a stronger bound on $\|\nabla R\|$, where ∇R denotes the covariant derivative of curvature tensor R. For given n, Λ, Λ_1, $V > 0$, we denote by $\mathfrak{M}^n(\Lambda, \Lambda_1, V)$ a class of closed n-dimensional Riemannian manifolds M which satisfy the following bounds;

$$|K_M| \leq \Lambda^2, \quad \|\nabla R_M\| \leq \Lambda_1, \quad \mathrm{diam}(M) \leq 1, \quad \mathrm{Vol}(M) \geq V,$$

and set $\mathfrak{M}(\Lambda, \Lambda_1, V) = \bigcup_n \mathfrak{M}^n(\Lambda, \Lambda_1, V)$.

CHEEGER'S FINITENESS THEOREM II [5]. *For given n, Λ, Λ_1, $V > 0$, the number $\sharp_{\mathrm{diff}}\,\mathfrak{M}^n(\Lambda, \Lambda_1, V)$ of diffeomorphism classes in $\mathfrak{M}^n(\Lambda, \Lambda_1, V)$ is finite.*

In the proof of the Cheeger finiteness theorem and our results as

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well, an estimate of the injectivity radius $i(M)$ of the exponential map on M plays an important role. But since in his proof Ascoli's theorem is used essentially, it seems to us that it is impossible to bound the number $\#_{\mathrm{diff}}\mathfrak{M}^n(\Lambda, \Lambda_1, V)$ explicitly from above by using the proof as in [5]. The main purpose of this paper is to show the existence of an upper bound for $\#_{\mathrm{diff}}\mathfrak{M}(\Lambda, \Lambda_1, V)$ and express upper bounds for $\#_{\mathrm{diff}}\mathfrak{M}^n(\Lambda, \Lambda_1, V)$ and $\#_{\mathrm{diff}}\mathfrak{M}(\Lambda, \Lambda_1, V)$ explicitly in terms of a priori given constants. For a Riemannian manifold we denote by d the distance function induced from the Riemannian metric.

We obtain the following theorems.

THEOREM 1. *For given $n, \Lambda, \Lambda_1, R>0$ there exist $\varepsilon_1 = \varepsilon_1(n) > 0$, $r_1 = r_1(n, \Lambda, \Lambda_1, R) > 0$ such that if complete n-dimensional manifolds M and $\overline{M}$ satisfy the following conditions, then M is diffeomorphic to $\overline{M}$;*

1) $|K_M|, |K_{\overline{M}}| \leq \Lambda^2$, $\|\nabla R_M\|, \|\nabla R_{\overline{M}}\| \leq \Lambda_1$, $i(M), i(\overline{M}) \geq R$,

2) *for some r, $r \leq r_1$, and ε, $\varepsilon \leq \varepsilon_1$, there exist $2^{-(n+8)}r$-dense and $2^{-(n+9)}r$-discrete subsets $\{p_i\} \subset M$, $\{q_i\} \subset \overline{M}$ such that the correspondence $p_i \to q_i$ is bijective and $(1+\varepsilon)^{-1} \leq d(q_i, q_j)/d(p_i, p_j) \leq 1+\varepsilon$ for all p_i, p_j with $d(p_i, p_j) \leq 20r$. ε_1 and r_1 can be written explicitly; e.g.*

$$\varepsilon_1 = 10^{-20}(n+1)^{-8}(n!)^{-2}2^{-(2n^2+41n)},$$
$$r_1 = \min\{R/140, \varepsilon_1/20\Lambda, \sqrt[3]{10^{-3}n^{-5}2^{-((n^2+17n)/2)}\Lambda_1^{-1}}, (10(2n^2\Lambda^2+1))^{-1}\}.$$

For a metric space X a subset A is *δ-dense* iff for any $x \in X$, $d(x, A) < \delta$. A subset A is *δ-discrete* iff any two points of A have the distance at least δ.

Let ω_n denote the volume of the standard unit n-sphere. If we set $R = \min\{\pi/\Lambda, (n-1)V/(2\omega_{n-2}e^{(n-1)\Lambda})\}$, then R gives a lower bound of the injectivity radii $i(M)$ for all M in $\mathfrak{M}^n(\Lambda, \Lambda_1, V)$, and every M in $\mathfrak{M}(\Lambda, \Lambda_1, V)$ has the dimension at most n_0, where $n_0 = 2\max\{[\log(k^{k+2}/k!\,V)], k\} + 3$, $k = [\pi e^{2\Lambda+1}] + 1$, (§ 1. Lemma). Let $\varepsilon_1 = \varepsilon_1(n)$, $r_1 = r_1(n, \Lambda, \Lambda_1, R)$ be as in Theorem 1.

THEOREM 2.

$$\#_{\mathrm{diff}}\mathfrak{M}^n(\Lambda, \Lambda_1, V) \leq (2^{2n+17}/\varepsilon_1 r_1^2)^{\binom{N_0}{2}+1}N_0,$$
$$\#_{\mathrm{diff}}\mathfrak{M}(\Lambda, \Lambda_1, V) \leq \sum_{n=0}^{n_0}(2^{2n+17}/\varepsilon_1 r_1^2)^{\binom{N_0}{2}+1}N_0,$$

where, $N_0 = [e^{\Lambda(n-1)}/(\Lambda 2^{-(n+9)}r_1)^n]$.

Here we descrive another application of Theorem 1. For a bi-Lipschitz map $f\colon X\to Y$ between two metric spaces X and Y, set

$$l(f):=\inf\{L;\ L^{-1}\leq d(f(x),f(y))/d(x,y)\leq L \text{ for all } x,y\in X\}.$$

Definition. Define $\rho(X, Y)$ by

$$\begin{cases}\inf\{\log l(f);\ f\colon X\to Y \text{ is bi-Lipschitz map}\}\\ \infty \quad \text{if any bi-Lipschitz map does not exist.}\end{cases}$$

It is clear that ρ is symmetric and satisfies the triangle inequality. In the case X and Y are compact, Ascoil's theorem implies

$$\rho(X, Y)=0 \quad \text{iff } X \text{ is isometric to } Y.$$

For a positive integer n we denote by $\mathfrak{A}^n$ a class of complete n-dimensional Riemannian manifolds M with

$$|K_M|<\infty,\quad \|\nabla R_M\|<\infty,\quad i(M)>0\,.$$

Of course $\mathfrak{A}^n$ contains all compact Riemannian manifolds of dimension n. Conversely, according to [7] every noncompact n-manifold admits a metric which belongs to the class $\mathfrak{A}^n$. A theorem of Shikata [12] states that there exists an $\varepsilon(n)>0$ depending only on n such that if compact n-dimensional Riemannian manifolds M and N satisfy $\rho(M, N)<\varepsilon(n)$, then they are diffeomorphic. We do not know whether ρ is distance on $\mathfrak{A}^n$, but can extend the Shikata theorem to the class $\mathfrak{A}^n$. Let $\varepsilon_1=\varepsilon_1(n)$ be as in Theorem 1 again.

Corollary 3. *If M and $N\in\mathfrak{A}^n$ satisfy $\rho(M, N)<\log(1+\varepsilon_1)$, then they are diffeomorphic.*

Recently M. Gromov [8], [9] states without giving detail of the proof that a similar result to Theorem 1 holds without the assumption for $\|\nabla R\|$. But our Theorem 1 is still valid for noncompact manifolds. However the assumption for $\|\nabla R\|$ is essential in the proof of our Theorem 1. Our proof is of course different from Gromov's one. The main tool of our proof is a technique of center of mass which is developed in [2].

The remainder of the paper is organized as follows: Assuming Theorem 1, the proofs of Theorem 2 and Corollary 3 are given in Section 1. Theorem 1 is proved in Section 2 - Section 4.

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§ 1. Proofs of Theorem 2 and Corollary 3

For a $\delta > 0$, a system of points $\{x_i\}$ in a metric space X is called a *δ-maximal system* of X if $\{x_i\}$ is maximal with respect to the property that the distance between any two of them is greater than or equal to δ. $\{x_i\}$ is a δ-maximal iff it is a δ-dense and δ-discrete subset. We show that there exists a δ-maximal system of every Riemannian manifold M. Take a sequence X_i of compact subsets of M such that $\bigcup_i X_i = M$, $\mathring{X}_{i+1} \supset X_i$, where $\mathring{A}$ denotes the interior of a set A. We denote by $i(X_k)$ the infimum of the injectivity radius of the exponential map at points of X_k, and set $r_k := \frac{1}{2}\min\{\delta, i(X_k)\}$. Take a δ-maximal system $\{p_i^1\}_{1\le i\le N_1}$ of X_1. Notice that since the balls $B(p_i^1, r_1)$ have compact closure, they are contained in some X_{k_1}, and together with the fact that $B(p_i^1, r_1)$ are disjoint, this implies

$$N_1 \le \mathrm{Vol}(X_{k_1})/\min \mathrm{Vol}(B(p_i^1, r_1)).$$

By induction, it is possible to take a δ-maximal system $\{p_i^k\}_{1\le i\le N_k}$ of X_k such that $p_i^k = p_i^j$ for every $j < k$ and every i, $1 \le i \le N_j$. Then the system $\bigcup_{k=1}^{\infty}\{p_i\}_{N_{k-1}+1\le i\le N_k}$ is a δ-maximal system of M, where $N_0 := 0$.

Proof of Corollary 3 *assuming Theorem* 1. By the assumption there exists a bi-Lipschitz map $f\colon M \to N$ such that $l(f) < 1 + \varepsilon_1(n)$. We may assume

$$|K_M|,\ |K_N| \le \Lambda^2,\ \|\nabla R_M\|,\ \|\nabla R_N\| \le \Lambda_1, \quad i(M),\ i(N) \ge R,$$

for some Λ, Λ_1, $R > 0$. Let $r_1 = r_1(n, \Lambda, \Lambda_1, R)$ be as in Theorem 1, and take a $(1+\varepsilon_1)2^{-(n+9)}r_1$-maximal system $\{p_i\}$ of M. Since f is bi-Lipschitz, it is surjective. Therefore it is easy to show that $\{f(p_i)\}$ is $2^{-(n+8)}r_1$-dense and $2^{-(n+9)}r_1$-discrete. Q.E.D.

To prove Theorem 2 we recall an injectivity radius estimate. From now on, for given n and $\delta > 0$, let $v(\delta)$ (resp. $\tilde{v}(\delta)$) denote the volume of a δ-ball in the n-dimensional hyperbolic space with constant curvature $-\Lambda^2$ (resp. n-sphere with constant curvature Λ^2). The following lemma is a dimension independent version of [5], [10] and [11].

LEMMA. *For given Λ, $V > 0$, there exist $n_0 = n_0(\Lambda, V)$ and $R_0 = R_0(\Lambda, V) > 0$ such that if M is an n-dimensional compact Riemannian manifold such that* $|K_M| \le \Lambda^2$, $\mathrm{diam}(M) \le 1$, $\mathrm{Vol}(M) \ge V$, *then*

(1) $n = \dim M \leq n_0$,

(2) $i(M) \geq \min\{\pi/\Lambda, (n-1)V/(2\omega_{n-2}e^{(n-1)\Lambda})\} \geq R_0$,

where n_0 *and* R_0 *can be written explicitly as*

$$n_0 = 2\max\{[\log(k^{k+2}/k!\,V)], k\} + 3, \quad k = [\pi e^{2\Lambda+1}] + 1,$$
$$R_0 = \min_{2\leq n\leq n_0}\{\pi/\Lambda, (n-1)V/(2\omega_{n-2}e^{(n-1)\Lambda})\}.$$

Proof. For (1), the Rauch comparison theorem yields

$$V \leq \mathrm{Vol}(M) \leq v(1) \leq \omega_{n-1}e^{(n-1)\Lambda},$$

where

$$\omega_{n-1} = \begin{cases} 2\pi^m/(m-1)! & (n = 2m) \\ 2(2\pi)^m/(2m-1)(2m-3)\cdots 3\cdot 1 & (n = 2m+1). \end{cases}$$

Notice that

$$\lim_{n\to\infty} \omega_{n-1}e^{(n-1)\Lambda} = 0.$$

It is an easy calculation to estimate such an n_0 that $\omega_{n-1}e^{(n-1)\Lambda} < V$ for all $n > n_0$. For (2), it suffices to bound the lengths of closed geodesics from below. Suppose that there is a closed geodesic with length l. The Rauch comparison theorem implies that $\mathrm{Vol}(M)$ is not greater than the volume of the tublar neighborhood of radius 1 of a geodesic segment with length l in the n-dimensional hyperbolic space with constant curvature $-\Lambda^2$. Therefore we get

$$\begin{aligned} \mathrm{Vol}(M) &\leq l\cdot\omega_{n-2}\int_0^1 (\sinh \Lambda t/\Lambda)^{n-2}\cosh \Lambda t\,dt \\ &= l\cdot\omega_{n-2}(\sinh \Lambda)^{n-1}/(n-1)\Lambda^{n-1} \\ &\leq l\cdot\omega_{n-2}e^{(n-1)\Lambda}/(n-1). \end{aligned}$$

Hence $l \geq (n-1)V/(\omega_{n-2}e^{(n-1)\Lambda})$, and this yields (2). Q.E.D.

Proof of Theorem 2 *assuming Theorem* 1. For each $M_\alpha \in \mathfrak{M}^n(\Lambda, \Lambda_1, V)$, take a $2^{-(n+8)}r_1$-maximal system $\{p_i^\alpha\}_i$ of M_α. Note that since $\mathrm{diam}(M_\alpha) \leq 1$,

$$\#\{p_i^\alpha\}_i \leq v(1)/\tilde{v}(2^{-(n+9)}r_1) \leq [e^{(n-1)\Lambda}/(\Lambda 2^{-(n+9)}r_1)^n] = N_0.$$

Set $m := \#_{\mathrm{diff}}\mathfrak{M}^n(\Lambda, \Lambda_1, V)$, $L := 1/(2^{-(n+8)}r_1)$ and $\varepsilon_1' := \varepsilon_1/(2(1+\varepsilon_1)L)$. Suppose that

$$m > (2^{2n+17}/\varepsilon_1 r_1^2)^{\binom{N_0}{2}+1}N_0 > ([L/2\varepsilon_1'] + 1)^{\binom{N_0}{2}+1}N_0.$$

Then $\mathfrak{M}^n(\Lambda, \Lambda_1, V)$ contains at least $[m/N_0]$ pairwise non-diffeomorphic manifolds $\{M_\alpha\}_{\alpha\in A}$ with the $2^{-(n+8)}r_1$-maximal systems whose numbers are all the same, say N_1, $N_1 \le N_0$. We concider the set

$$\Sigma := \left\{(i_k, j_k);\ 1 \le k \le \binom{N_1}{2} := N_1'\right\}$$

of all the distinct pairs of the indices of the systems $\{p_i^\alpha\}_i$ for $\{M_\alpha\}_{\alpha\in A}$. For each M_α and M_β ($\alpha, \beta \in A$), and for each $(i_k, j_k) \in \Sigma$, we set $l(\alpha, \beta; k) = d(p_{i_k}^\beta, p_{j_k}^\beta)/d(p_{i_k}^\alpha, p_{j_k}^\alpha)$. Notice that $L^{-1} \le l(\alpha, \beta; k) \le L$. We fix some $\alpha \in A$. For $(i_1, j_1) \in \Sigma$ there is a $t_1 \in [L^{-1}, L]$ such that if

$$A_1 := \{\beta \in A;\ l(\alpha, \beta; 1) \in [t_1 - \varepsilon_1', t_1 + \varepsilon_1']\}$$

then $\# A_1 \ge [m/N_0]([L/2\varepsilon_1'] + 1)^{-1}$. By induction, for $(i_k, j_k) \in \Sigma$ there is a $t_k \in [L^{-1}, L]$ such that if

$$A_k := \{\beta \in A_{k-1};\ l(\alpha, \beta; k) \in [t_k - \varepsilon_1', t_k + \varepsilon_1']\}$$

then $\# A_k \ge [m/N_0]([L/2\varepsilon_1'] + 1)^{-k}$. By the assumption on m, it is possible to take distinct pair β and β' in $A_{N_1'}$. Then $|l(\alpha, \beta; k) - l(\alpha, \beta'; k)| \le 2\varepsilon_1'$ for all k, $1 \le k \le N_1'$, and this implies $(1 + \varepsilon_1)^{-1} \le l(\beta, \beta'; k) \le 1 + \varepsilon_1$. This is a contradiction since by Theorem 1 M_β is diffeomorphic to $M_{\beta'}$. The estimate for $\#_{\mathrm{diff}}\, \mathfrak{M}(\Lambda, \Lambda_1, V)$ is an immediate consequence of the previous lemma (1) and the estimate for $\#_{\mathrm{diff}}\, \mathfrak{M}^n(\Lambda, \Lambda_1, V)$. Q.E.D.

§ 2. Construction of local diffeomorphisms

The rest of this paper is devoted to the proof of Theorem 1. For given n, Λ, $R>0$, set $R_0 := \frac{1}{2}\min\{R, \pi/\Lambda\}$ and let r and ε be adjustable parameters with $0 < r \le R_0/70$, $0 < \varepsilon \le 2^{-(n+14)}$. From now on we denote by M and $\overline{M}$ complete n-dimensional Riemannian manifolds which satisfy the conditions in Theorem 1 for r and ε. In the final part of the proof, we will set $r \le r_1$, and $\varepsilon \le \varepsilon_1$. We use the bound for $\|\nabla R\|$ actually only in Section 4. Let $\{p_i\} \subset M$ and $\{q_i\} \subset \overline{M}$ be $2^{-(n+8)}r$-dense and $2^{-(n+9)}r$-discrete subsets as in Theorem 1. For given $p \in M$ and $\delta > 0$, we denote by M_p the tangent space of M at p, and by $B(p, \delta)$ the δ-ball with center p. Note that all δ-balls with $\delta \le R_0$ in M and $\overline{M}$ are convex and that by the Rauch comparison theorem, for any $v, w \in M_p$ with $\|v\|, \|w\| \le t$, $t \le R_0$

$$\sin \Lambda t/\Lambda t \le d(\exp_p v, \exp_p w)/\|v - w\| \le \sinh \Lambda t/\Lambda t\,.$$

The purpose of this section is to prove the following lemma.

LEMMA 2.1. *For each i there exists a linear isometry I_i from M_{p_i} to $\overline{M}_{q_i}$ such that if $F_i := \exp_{q_i} \circ I_i \circ \exp_{p_i}^{-1}: B(p_i, R_0) \to B(q_i, R_0)$, then $d(F_i(p_j), q_j) \leq \delta_1 r$ for every p_j with $d(p_i, p_j) \leq 10r$, where*

$$\delta_1 = 2(n+1)(6^{n+2} n! 2^{(n/2)+7})^{1/2}(40\Lambda r + 2\varepsilon)^{1/4}.$$

Proof. Set $\tilde{p}_j := \exp_{p_i}^{-1}(p_j)$ and $\tilde{q}_j := \exp_{q_i}^{-1}(q_j)$. Then $\{\tilde{p}_j\}$ and $\{\tilde{q}_j\}$ are $2^{-(n+7)}r$-dense and $2^{-(n+10)}r$-discrete subsets of the $10r$-ball around 0 and satisfy $(1+\varepsilon)^{-1}e^{-20\Lambda r} \leq \|\tilde{q}_j - \tilde{q}_k\| / \|\tilde{p}_j - \tilde{p}_k\| \leq (1+\varepsilon)e^{20\Lambda r}$ for all $j, k, j \neq k$. Hence Lemma 2.1 is a direct consequence of the following.

LEMMA 2.1′. *Let $\{x_i\}$ be a $2^{-(n+7)}r$-dense and $2^{-(n+10)}r$-discrete subset of $B(0, r) \subset \boldsymbol{R}^n$ with $x_1 = 0$. If a system $\{y_i\}$ of points in $B(0, r)$ with $y_1 = 0$ satisfies $(1+\varepsilon)^{-1} \leq \|y_i - y_j\| / \|x_i - x_j\| \leq 1 + \varepsilon$ for every $i \neq j$. Then there exists a linear isometry I of R^n such that*

$$\|I(x_i) - y_i\| \leq (n+1)(6^{n+2} \cdot n! \cdot 2^{(n/2)+7} \cdot \varepsilon^{1/2})^{1/2} r$$

for every i.

For the proof of the Lemma 2.1′, it is convenient to introduce the following notion, a normal system, and to investigate some properties of a normal system. This is done in Lemma 2.3 – Lemma 2.5.

DEFINITION 2.2. For $0 \leq \eta < 1$ and $r > 0$, we say that a system of n points $\{p_i\}_{1 \leq i \leq n}$ of $\boldsymbol{R}^n$ is *(r, η)-normal* if $(1-\eta)r \leq \|p_i\| \leq r$, $|\langle p_i, p_j \rangle| \leq \eta r^2$ *for* every $i \neq j$.

LEMMA 2.3. *For every $L \geq n+1$,* let $\{p_i\}_{1 \leq i \leq n}$ *be an $(r, 2^{-L})$-normal system for $\boldsymbol{R}^n$. If we set $p_i' := p_i - \langle p_i, u_1 \rangle u_1 - \cdots - \langle p_i, u_{i-1} \rangle u_{i-1}$, $u_i := p_i' / \|p_i'\|$ inductively, then*

(1) $\|p_i'\| \geq (1 - 2^{-(L-i)})^{1/2} r \geq (1 - 2^{-(L-i)}) r$,

(2) $|\langle p_k, u_i \rangle| \leq 2^{-(L-i)} r$

for every i, k with $k > i$.

Proof. For $i = 1$, (1) and (2) are trivial. Assume (1), (2) for j, $1 \leq j \leq i$. Then we get

$$\begin{aligned} \|p_{i+1}'\|^2 &= \|p_{i+1}\|^2 - \langle p_{i+1}, u_1 \rangle^2 - \cdots - \langle p_{i+1}, u_i \rangle^2 \\ &\geq ((1 - 2^{-L})^2 - 2^{-2(L-1)} - \cdots - 2^{-2(L-i)}) r^2 \\ &\geq (1 - 2^{-(L-i-1)}) r^2 \geq (1 - 2^{-(L-i-1)})^2 r^2, \end{aligned}$$

and for $k > i+1$,

$$\begin{aligned}|\langle p_k, u_{i+1}\rangle| &\le \|p'_{i+1}\|^{-1}(|\langle p_k, p_{i+1}\rangle| + |\langle p_{i+1}, u_1\rangle||\langle p_k, u_1\rangle| + \cdots \\ &\quad + |\langle p_{i+1}, u_i\rangle||\langle p_k, u_i\rangle|) \\ &\le 2(2^{-L} + 2^{-2(L-1)} + \cdots + 2^{-2(L-i)})r \le 2^{-L+i+1}r.\end{aligned}$$

Thus for $L \ge n+1$, the Gram-Schmidt orthonormalization procedure yields the orthonormal basis $\{u_i\}$ for $\boldsymbol{R}^n$ via an $(r, 2^{-L})$-normal system $\{p_i\}$.

LEMMA 2.4. *If $\{p_i\}_{1\le i\le n}$ is an $(r, 2^{-L})$-normal system for $\boldsymbol{R}^n$, and if for some $\delta > 0$, x and y in $\boldsymbol{R}^n$ satisfy*

$$\|x\|, \|y\| \le r, \quad |\|x\| - \|y\|| \le \delta, \quad |\|x - p_i\| - \|y - p_i\|| \le \delta$$

for all i, $1 \le i \le n$, then $\|x - y\| \le 3(n + 2^{-L+n+4})\delta$.

Proof. Notice that

$$|\langle p_i, x-y\rangle| = 2^{-1}|\|x\|^2 - \|y\|^2 + \|p_i - y\|^2 - \|p_i - x\|^2| \le 3\delta r.$$

By induction, we show that

$$(*) \qquad |\langle u_i, x-y\rangle| \le 3(1 + 2^{-L+i+1})^2\delta.$$

This is trivial for $i = 1$. Assume $(*)$ for j, $1 \le j \le i$. Then we have with Lemma 2.3

$$\begin{aligned}|\langle u_{i+1}, x-y\rangle| &\le \|p'_{i+1}\|^{-1}(|\langle p_{i+1}, x-y\rangle| + |\langle p_{i+1}, u_1\rangle||\langle u_1, x-y\rangle| + \cdots \\ &\quad + |\langle p_{i+1}, u_i\rangle||\langle u_i, x-y\rangle|) \\ &\le 3(1 - 2^{-L+i+1})^{-1}(1 + 2^{-(L-1)}(1 + 2^{-L+2})^2 + \cdots \\ &\quad + 2^{-(L-i)}(1 + 2^{-L+i+1})^2)\delta \\ &\le 3(1 + 2^{-L+i+2})(1 + 2^{-L+2} + \cdots + 2^{-L+i+1})\delta \\ &\le 3(1 + 2^{-L+i+2})^2\delta.\end{aligned}$$

Hence we conclude that

$$\|x - y\| \le \sum_1^n |\langle u_i, x-y\rangle| \le \sum_1^n 3(1 + 2^{-L+i+1})^2\delta \le 3(n + 2^{-L+n+4})\delta.$$

Q.E.D.

LEMMA 2.5. *For k, $1 \le k \le n$, and $L \ge k+2$, let $\{e_i\}_{1\le i\le k} \subset \boldsymbol{R}^n$ be a $(1, 2^{-L})$-normal system for the linear subspace spanned by $\{e_i\}$ with $\|e_i\| = 1$ for all i. If two unit vectors x and y which belong to* Span $\{e_i\}_{1\le i\le k}$ *satisfy the following inequalities;*

$$|\measuredangle(e_i, x) - \measuredangle(e_i, y)| \le \alpha \quad (1 \le i \le k-1), \ \langle x, e_k\rangle \ge 3/4, \ \langle y, e_k\rangle \ge 3/4,$$

then $\measuredangle(x,y) \leq 6((k-1)+2^{-L+k+3})\alpha$, *where* $\measuredangle(x,y)$ *denotes the angle between* x *and* y.

Proof. Notice that $|\langle e_i, x\rangle - \langle e_i, y\rangle| \leq \alpha$ $(1 \leq i \leq k-1)$, and

$$2^{-1}\measuredangle(x,y) \leq \sin\measuredangle(x,y) \leq \|x-y\|.$$

Hence it suffices to estimate $\|x-y\|$ from above. Let $\{u_i\}$ be an orthonormal basis for Span $\{e_i\}$ obtained by the Gram Schmidt process from $\{e_i\}$. From Lemma 2.4 (*), we get $|\langle u_i, x-y\rangle| \leq (1+2^{\;L+i+1})^2\alpha$ $(1 \leq i \leq k-1)$. By Lemma 2.3,

$$\begin{aligned}\langle u_k, x\rangle &\geq \|e_k'\|^{-1}(\langle e_k, x\rangle - |\langle e_k, u_1\rangle||\langle u_1, x\rangle| - \cdots - |\langle e_k, u_{k-1}\rangle||\langle u_{k-1}, x\rangle|)\\ &\geq \langle e_k, x\rangle - 2^{-L+1} - \cdots - 2^{-L+k-1} \geq 3/4 - 2^{-L+k} \geq 1/2.\end{aligned}$$

Hence the inequality;

$$|\langle u_k, x\rangle^2 - \langle u_k, y\rangle^2| = \left|\sum_1^{k-1}(\langle u_i, x\rangle^2 - \langle u_i, y\rangle^2)\right| \leq 2\sum_1^{k-1}\langle u_i, x-y\rangle|$$

implies

$$|\langle u_k, x-y\rangle| \leq 2\sum_1^{k-1}|\langle u_i, x-y\rangle|,$$

and this yields that

$$\begin{aligned}\|x-y\| &\leq \sum_1^{k}|\langle u_i, x-y\rangle| \leq 3\sum_1^{k-1}(1+2^{\;L+i+1})^2\alpha\\ &\leq 3((k-1)+2^{-L+k+3})\alpha.\end{aligned}$$

Q.E.D.

From now we return to the situation in Lemma 2.1′. Let $\{x_i\}$ be a $2^{-(n+7)}r$-dense and $2^{-(n+10)}r$-discrete subset of $B(0,r)$ and let $\{y_i\}$ be a system of points in $B(0,r)$ with $y_1 = 0$ such that

$$(1+\varepsilon)^{-1} \leq \|y_i - y_j\|/\|x_i - x_j\| \leq 1+\varepsilon \quad \text{for every } i \neq j.$$

LEMMA 2.6. $|\measuredangle(x_i,x_j) - \measuredangle(y_i,y_j)| \leq 2^{(n/2)+8}\varepsilon^{1/2}$ *for every* $i \neq j$.

Proof. Set $\alpha_{i,j} := \measuredangle(x_i,x_j)$ and $\beta_{i,j} := \measuredangle(y_i,y_j)$. First we show that $|\cos\alpha_{i,j} - \cos\beta_{i,j}| \leq 2^{(n+13)}\varepsilon$. Set $\kappa = 1+\varepsilon$, then we get

$$\begin{aligned}\cos\alpha_{i,j} &= (\|x_i\|^2 + \|x_j\|^2 - \|x_i - x_j\|^2)/2\|x_i\|\|x_j\|\\ &\leq (\kappa^2(\|y_i\|^2 + \|y_j\|^2) - \kappa^{-2}\|y_i - y_j\|^2)/2\|x_i\|\|x_j\|\\ &= (\kappa^2(2\|y_i\|\|y_j\|\cos\beta_{i,j} + \|y_i - y_j\|^2) - \kappa^{-2}\|y_i - y_j\|^2)/2\|x_i\|\|x_j\|\\ &= \kappa^2\cos\beta_{i,j}\cdot\|y_i\|\|y_j\|/\|x_i\|\|x_j\| + (\kappa^2 - \kappa^{-2})\|y_i - y_j\|^2/2\|x_i\|\|x_j\|\\ &\leq \kappa^4\cos\beta_{i,j} + (\kappa^2 - \kappa^{-2})(2^{(n+10)}\kappa + \kappa^2),\end{aligned}$$

$$\cos\alpha_{i,j} - \cos\beta_{i,j} \leq (\kappa^4 - 1)\cos\beta_{i,j} + (\kappa^2 - \kappa^{-2})(2^{(n+10)}\kappa + \kappa^2)$$
$$\leq 2^{(n+13)}\varepsilon:$$

Hence we can get that $|\cos\alpha_{i,j} - \cos\beta_{i,j}| \leq 2^{(n+13)}\varepsilon$, and this yields

$$2(\sin(|\alpha_{i,j} - \beta_{i,j}|/2))^2 \leq 2^{(n+13)}\varepsilon,$$
$$|\alpha_{i,j} - \beta_{i,j}| \leq 2\sin^{-1}((2^{n+12}\varepsilon)^{1/2})$$
$$\leq 2^{(n/2)+8}\varepsilon^{1/2} \quad (\varepsilon \leq 2^{-(n+14)}).$$ Q.E.D.

LEMMA 2.7. *There exist* $\{x_{m_j}\}_{1\leq j\leq n} \subset \{x_i\}$ *and* $\{y_{m_j}\}_{1\leq j\leq n} \subset \{y_i\}$ *such that they are* $(r, 2^{-(n+4)})$*-normal systems for* $\boldsymbol{R}^n$.

Proof. Take an orthogonal basis $\{w_j\}$ for $\boldsymbol{R}^n$ such that $\|w_j\| = (1 - 2^{-(n+6)})r$, and by deneseness, take $\{x_{m_j}\}_{1\leq j\leq n} \subset \{x_i\}$ such that $\|x_{m_j} - w_j\| \leq 2^{-(n+7)}r$. An easy calculation shows that $\{X_{m_j}\}_{1\leq j\leq n}$ and the corresponding $\{y_{m_j}\}_{1\leq j\leq n}$ have the required properties. Q.E.D.

Proof of Lemma 2.1′. Let $\{u_i\}$ and $\{v_i\}$ be the orthonormal bases for $\boldsymbol{R}^n$ obtained by applying the Gram-Schmidt process to $\{x_{m_i}\}$ and $\{y_{m_i}\}$ respectively. A required linear isometry I of $\boldsymbol{R}^n$ is defined by $I(u_i) := v_i$. If we set $X_k = I(x_{m_k})/\|I(x_{m_k})\|$ and $Y_k = y_{m_k}/\|y_{m_k}\|$, then we have with Lemma 2.3 (1)

$$\langle v_k, X_k\rangle, \langle v_k, Y_k\rangle \geq 1 - 2^{-(n+4-k)}.$$

This yields

$$\langle X_k, Y_k\rangle \geq \cos(\sphericalangle(X_k, v_k) + \sphericalangle(v_k, Y_k))$$
$$\geq 2\cos^2\theta - 1 \qquad (\cos\theta := 1 - 2^{-(n+4-k)})$$
$$\geq 1 - 2^{-(n+2-k)} \geq 3/4.$$

ASSERTION 1. $\sphericalangle(I(x_{m_k}), y_{m_k}) \leq (6k-5)6^{k-2}(k-1)!\,\varepsilon'$. $\varepsilon' := 2^{(n/2)+8}\varepsilon^{1/2}$.

Proof. From the triangle inequality and Lemma 2.6, we have

$$\sphericalangle(y_{m_i}, I(x_{m_k})) \leq \sphericalangle(I(x_{m_k}), I(x_{m_i})) + \sphericalangle(I(x_{m_i}), y_{m_i})$$
$$\leq \sphericalangle(y_{m_k}, y_{m_i}) + \sphericalangle(I(x_{m_i}), y_{m_i}) + \varepsilon',$$

and similarly,

$$\sphericalangle(y_{m_i}, I(x_{m_k})) \geq \sphericalangle(y_{m_k}, y_{m_i}) - \sphericalangle(I(x_{m_i}), y_{m_i}) - \varepsilon',$$

hence,

$$|\sphericalangle(y_{m_i}, I(x_{m_k})) - \sphericalangle(y_{m_i}, y_{m_k})| \leq \sphericalangle(I(x_{m_i}), y_{m_i}) + \varepsilon'.$$

Clearly, $\sphericalangle(I(x_{m_1}), y_{m_1}) = 0$. Assume the assertion for i, $1 \leq i \leq k-1$, then we get for every i $(1 \leq i \leq k-1)$

$$\begin{aligned}|\sphericalangle(y_{m_i}, I(x_{m_k})) - \sphericalangle(y_{m_i}, y_{m_k})| &\leq (6i-5)6^{i-2}(i-1)!\varepsilon' + \varepsilon' \\ &\leq ((6k-11)6^{k-3}(k-2)! + 1)\varepsilon'.\end{aligned}$$

Notice that $\{y_{m_i}/\|y_{m_i}\|\}_{1 \leq i \leq k}$ is a $(1, 2^{-(n+3)})$-normal system for its spanning subspace. Hence applying Lemma 2.5 to $\{y_{m_i}/\|y_{m_i}\|\}_{1 \leq i \leq k}$, X_k and Y_k in place of $\{e_i\}_{1 \leq i \leq k}$, x and y, we conclude

$$\sphericalangle(I(x_{m_k}), y_{m_k}) \leq (6k-5)6^{k-2}(k-1)!\varepsilon'. \qquad \text{Q.E.D.}$$

ASSERTION 2. $|\|I(x_i) - y_{m_k}\| - \|y_i - y_{m_k}\|| \leq (2k!6^k\varepsilon')^{1/2}r$ for every i and every k, $1 \leq k \leq n$.

This and Lemma 2.4 complete the proof of Lemma 2.1′.

Proof of Assertion 2. Assertion 1 and the triangle inequality imply that

$$\begin{aligned}\sphericalangle(I(x_i), y_{m_k}) &\leq \sphericalangle(I(x_i), I(x_{m_k})) + \sphericalangle(I(x_{m_k}), y_{m_k}) \\ &\leq \sphericalangle(y_i, y_{m_k}) + ((6k-5)6^{k-2}(k-1)! + 1)\varepsilon',\end{aligned}$$

and similarly,

$$\sphericalangle(I(x_i), y_{m_k}) \geq \sphericalangle(y_i, y_{m_k}) - ((6k-5)6^{k-2}(k-1)! + 1)\varepsilon',$$

hence,

$$|\sphericalangle(I(x_i), y_{m_k}) - \sphericalangle(y_i, y_{m_k})| \leq ((6k-5)6^{k-2}(k-1)! + 1)\varepsilon'.$$

Therefore we have

$$\begin{aligned}&|\|I(x_i) - y_{m_k}\|^2 - \|y_i - y_{m_k}\|^2| \\ &\leq |\|I(x_i)\|^2 - \|y_i\|^2| + 2\|y_{m_k}\| \, |\|y_i\| \cos \sphericalangle(y_i, y_{m_k}) \\ &\qquad - \|I(x_i)\| \cos \sphericalangle(I(x_i), y_{m_k})|,\end{aligned}$$

where $|\|I(x_i)\|^2 - \|y_i\|^2| \leq 2\varepsilon r^2$ and

$$\begin{aligned}&|\|y_i\| \cos \sphericalangle(y_i, y_{m_k}) - \|I(x_i)\| \cos \sphericalangle(I(x_i), y_{m_k})| \\ &\leq r(|\sphericalangle(y_i, y_{m_k}) - \sphericalangle(I(x_i), y_{m_k}) + \varepsilon) \\ &\leq ((6k-5)6^{k-2}(k-1)! + 2)\varepsilon' r.\end{aligned}$$

Hence the inequality

$$|\|I(x_i) - y_{m_k}\| - \|y_i - y_{m_k}\|| \leq |\|I(x_i) - y_{m_k}\|^2 - \|y_i - y_{m_k}\|^2|^{1/2}$$

implies the required estimate. Q.E.D.

§ 3. Reduction and C^0-estimates

In this section we average the local diffeomorphisms F_i, constructed in the previous section, with a center of mass technique to obtain a smooth map $F: M \to \overline{M}$ and control the C^0 error between F and F_i. Let ψ be a smooth function such that

$$\psi|[0, 4] = 1, \quad \psi|[5, \infty) = 0, \quad 0 \geq \psi' \geq -2.$$

For every $x \in M$, define the weights $\phi_i(x)$ of $F_i(x)$ by

$$\phi_i(x) := \psi(d(x, p_i)/r)/\sum_j \psi(d(x, p_j)/r).$$

Notice that all p_j with $d(x, p_j) \leq 5r$ are finite and the corresponding $F_j(x)$ are contained in some convex ball B. It is easy from convexity argument to see that for a fixed $x \in M$, the function $C_x: \overline{M} \to \boldsymbol{R}$ defined by $C_x(y) = \frac{1}{2} \sum_i \phi_i(x) d^2(y, F_i(x))$ is C^∞ strongly convex on B, and has a unique minimum point on $\overline{M}$. Setting

$$F(x) := \text{the unique minimum point of } C_x$$

we define a map $F: M \to \overline{M}$. We show that F is smooth. Define a map V from a sufficiently small neighborhood of the graph of F in $M \times \overline{M}$ to the tangent bundle $T\overline{M}$ by

$$V(x, y) := -\sum_i \phi_i(x) \exp_y^{-1}(F_i(x)).$$

Since $V(x, y) = (\operatorname{grad} C_x)(y)$, we have $V(x, F(x)) = 0$. Let $K: TT\overline{M} \to T\overline{M}$ be the connection map, and define a map $D_2 V_{(x,y)}: \overline{M}_y \to \overline{M}_y$ by $D_2 V_{(x,y)}(\dot{y}(0)) = \nabla_{\dot{y}(0)} V(x, y(t))$, where we consider $V(x, y(t))$ as a vector field along a smooth curve $y(t)$ with $\dot{y}(0) = y$. Notice that

$$K(d/dt\ V(x, y(t))|_{t=0}) = D_2 V_{(x,y)}(\dot{y}(0)),$$

and $D_2 V_{(x,y)}$ is a linear map. From the standard Jacobi fields estimates (See (4.3) in the proof of Lemma 4.2),

$$\|D_2 V_{(x,y)}(\dot{y}(0)) - \dot{y}(0)\| \leq (30\Lambda r)^2 \|\dot{y}(0)\| < \|\dot{y}(0)\|.$$

This yields that $D_2 V_{(x,y)}$ is a linear isomorphism, and hence for $y = F(x)$, the space spanned by $\{d/dt\ V(x, y(t))|_{t=0}\}$ and the (horizontal) tangent space of the zero section of $T\overline{M}$ at $(F(x), 0)$ span $(T\overline{M})_{(F(x),0)}$. Therefore the implicit function theorem implies the smoothness of F.

From now on we fix $x_0 \in M$ and set $y_0 := F(x_0)$.

LEMMA 3.1. *dF_{x_0} has maximal rank iff*

$$(*)\qquad \sum_i d/dt\ \psi(d(x(t), p_i)/r)|_{t=0}\cdot \exp_{y_0}^{-1}(F_i(x_0)) + \sum_i \psi(d(x_0, p_i)/r)\cdot d(\exp_{y_0}^{-1})(dF_i(\dot{x}(0))) \neq 0$$

for every smooth curve $x(t)$ with $x(0) = x_0$ and $\dot{x}(0) \neq 0$.

Proof. Differentiating the curve $V(x(t), F(x(t)))$ in the zero section of $T\overline{M}$, we have

$$(3.2)\qquad d/dt\ V(x(t), y_0)|_{t=0} + D_2 V_{(x_0, y_0)}(dF(\dot{x}(0))) = 0\,.$$

Hence dF_{x_0} has maximal rank iff $d/dt\ V(x(t), y_0)|_{t=0} \neq 0$. Since $V(x_0, y_0) = 0$,

$$(3.3)\qquad \begin{aligned} &d/dt\ V(x(t), y_0)|_{t=0} \\ &\quad = -\sum_i d/dt\ \psi_i(d(x(t), p_i)/r)|_{t=0}\cdot \exp_{y_0}^{-1}(F_i(x_0))/\sum_j \psi_j(d(x_0, p_j)/r) \\ &\qquad -\sum_i \phi_i(x_0)\cdot d(\exp_{y_0}^{-1})(dF_i(\dot{x}(0)))\,. \end{aligned}$$

This completes the proof. Q.E.D.

We will show in Section 4 that in the above (*), the norm of the first term is smaller than that of the second if r and ε are taken sufficiently small. To do this we must first estimate the numbers of the sum in each term.

LEMMA 3.4. *If $N_1 := \#\{i;\ \psi(d(x_0, p_i)/r) = 1\}$ and $N_2 := \#\{i\colon \psi(d(x_0, p_i)/r) \neq 0\}$, then $N_2/N_1 \leq 6^n$.*

Proof. Since $\{p_i\}$ is $2^{-(n+8)}r$-dense, the union of $B(p_i, 2^{-(n+8)}r)$ with $d(x_0, p_i) \leq 4r$ covers the $3.9r$-ball around x_0, and since $\{p_i\}$ is $2^{-(n+9)}r$-discrete, the family of $B(p_i, 2^{-(n+10)}r)$ with $d(x_0, p_i) \leq 5r$ are disjoint and contained in the $5.1r$-ball around x_0. It follows from the Rauch comparison theorem that

$$N_1 \geq \tilde{v}(3.9r)/v(2^{-(n+8)}r), \quad N_2 \leq v(5.1r)/\tilde{v}(2^{-(n+10)}r)\,.$$

Hence we can get an explicit bound for N_2/N_1. Q.E.D.

Now we fix i and k such that $d(x_0, p_i)$, $d(x_0, p_k) \leq 5r$, and estimate $d(F_i(x_0), F_k(x_0))$.

LEMMA 3.5. *$|d(q_j, F_k(x_0)) - d(q_j, F_i(x_0))| \leq \delta_2 r$ for every j with $d(p_i, p_j)$, $d(p_k, p_j) \leq 10r$, where $\delta_2 = 2(\delta_1 + 600\,\Lambda r)$.*

Proof. Notice that

$$e^{-20\Lambda r} \leq d(F_k(x_0), F_k(p_j))/d(x_0, p_j) \leq e^{20\Lambda r}.$$

By Lemma 2.1,

$$|d(q_j, F_k(x_0)) - d(F_k(p_j), F_k(x_0))| \leq \delta_1 r.$$

Hence the triangle inequality implies

$$(3.6) \qquad |d(p_j, x_0) - d(q_j, F_k(x_0))| \leq (\delta_1 r + 40\Lambda r \cdot d(p_j, x_0)) \leq \delta_2 r/2.$$

From the same estimate for i, we have the required bound. Q.E.D.

Here we assume the following bound on ε and r in order to bound $\delta_2 \leq 1/2$;

$$(**) \qquad \varepsilon,\ 20\Lambda r \leq 2^{-18}(n+1)^{-4}(6^{n+2}n!\,2^{(n/2)+7})^{-2}.$$

This bound assures that $d(F_i(x_0), F_k(x_0)) \leq 2r/3$.

Lemma 3.7. *$d(F_k(x_0), F_i(x_0)) \leq \delta_3 r$, where $\delta_3 = 8(n+1)\delta_2$.*

Proof. Take a $q_{m_0} \in \{q_i\}$ such that $d(q_{m_0}, F_k(x_0)) \leq 2^{-(n+8)}r$, and let x_k and x_i denote the images of $F_k(x_0)$ and $F_i(x_0)$ by $\exp^{-1}_{q_{m_0}}$. Then from the above bound (**) we have that $\|x_k\|$, $\|x_i\| \leq r$. By Lemma 2.7, we can choose $\{q_{m_j}\}_{1\leq j\leq n}$ out of $\{q_i\}$ such that if $\tilde{q}_{m_j}$ denotes the image of q_{m_j} by $\exp^{-1}_{q_{m_0}}$, then $\{\tilde{q}_{m_j}\}_{1\leq j\leq n}$ 1 is an $(r, 2^{-(n+4)})$ normal system for $\overline{M}_{q_{m_0}}$. Notice that $\{p_{m_j}\}_{1\leq j\leq n}$ corresponding to $\{q_{m_j}\}_{1\leq j\leq n}$ are contained in $B(p_k, 10r) \cap B(p_i, 10r)$. From Lemma 3.5 we have

$$|\|\tilde{q}_{m_j} - x_k\| - \|\tilde{q}_{m_j} - x_i\|| \leq 2\delta_2 r, \quad 0 \leq j \leq n,$$

and together with Lemma 2.4 this yields

$$d(F_k(x_0), F_i(x_0)) \leq 8(n+1)\delta_2 r. \qquad \text{Q.E.D.}$$

From the definition of F it is clear that $d(F(x_0), F_i(x_0)) \leq \delta_3 r$ for every i with $d(x_0, p_i) \leq 5r$. Hence we have with Lemma 3.4

$$(3.8) \qquad \begin{aligned} &\|\sum_i d/dt\, \psi(d(x(t), p_i)/r)|_{t=0} \cdot \exp^{-1}_{y_0}(F_i(x_0))\| \\ &\leq N_2(2/r)\delta_3 r \|\dot{x}(0)\| \leq 2\cdot 6^n \delta_3 N_1 \|\dot{x}(0)\|. \end{aligned}$$

§ 4. C^1-estimates

To estimate the second term in Lemma 3.1 (*) from below, we must control the error between $d(\exp^{-1}_{y_0})(dF_i(\dot{x}(0)))$ and $d(\exp^{-1}_{y_0})(dF_k(\dot{x}(0)))$. To do this it is essential to estimate $\|dF_k(\dot{x}(0)) - PdF_i(\dot{x}(0))\|$ from above, where P denotes the parallel translation along the minimizing geodesic from $F_i(x_0)$ to $F_k(x_0)$. This is done in Lemma 4.5.

LEMMA 4.1. *For each* $x \in \overline{M}$, *let* $\{q_{\alpha_j}\} := \{q_i\} \cap B(x, r)$ *and* $N' := \#\{q_{\alpha_j}\}$. *The map* $\Phi\colon B(x, r/2) \to \boldsymbol{R}^{N'}$ *defined by* $\Phi^j(y) = d^2(q_{\alpha_j}, y)$ *satisfies the following;*

(1) Φ *is an embedding, and* $\|d\Phi(v)\| \geq r\|v\|$ *for every tangent vector* v *on* $B(x, r/2)$,

(2) $N' \leq 2^{n(n+11)}$.

Proof. The convexity of each component Φ^i of Φ implies the injectivity of Φ. For a given tangent vector v on $B(x, r/2)$, let γ be a geodesic with $\dot{\gamma}(0) = v/\|v\|$. Take a q_{α_j} such that $d(q_{\alpha_j}, \gamma(r/2)) \leq 2^{-(n+8)}r$. Comparing the triangle with vertices $(\gamma(0), \gamma(r/2), q_{\alpha_j})$ to a triangle with the same edge length in the sphere with constant curvature Λ^2, we have that cos $\measuredangle(\dot{\gamma}(0), \dot{\sigma}(0)) \geq 1/2$, where σ denote a unique minimizing geodesic from $\gamma(0)$ to q_{α_j}. This yields that

$$\|d\Phi(v)\| \geq |d\Phi^j(v)| \geq r\|v\| .$$

The same proof as in Lemma 3.4 implies (2). Q.E.D.

We fix i and k with $d(p_i, x_0)$, $d(p_k, x_0) \leq 5r$ and take an embedding $\Phi\colon B(F_k(x_0), r/2) \to \boldsymbol{R}^{N'}$ defined in the previous lemma for $F_k(x_0)$, where we set $\{q_{\alpha_j}\} := \{q_i\} \cap B(F_k(x_0), r)$. For a unit tangent vector v at x_0, let γ, σ_k and σ_i be geodesics such that $\dot{\gamma}(0) = v$, $\dot{\sigma}_k(0) = dF_k(v)$ and $\dot{\sigma}_i(0) = dF_i(v)$. For every q_{α_j}, we set

$$f_j(t) = d^2(p_{\alpha_j}, \gamma(t)) , \quad g_{m,j}(t) = \Phi^j(F_m \cdot \gamma(t)) ,$$
$$h_{m,j}(t) = \Phi^j(\sigma_m(t)) , \quad m = k, i .$$

LEMMA 4.2. *On* $[0, r/2]$,

(1) $2(1 - \Lambda^2 f_j) \leq f_j'' \leq 2(1 + \Lambda^2 f_j)$,

$2(1 - \Lambda^2 h_{m,j})e^{-20\Lambda r} \leq h_{m,j}'' \leq 2(1 + \Lambda^2 h_{m,j})e^{20\Lambda r}$,

(2) $|g_{m,j}'' - h_{m,j}''| \leq \Omega_1 r$

where $\Omega_1 = 82 + 10n^3\Omega r$,

$$\Omega = 60n(n-1)(10\Lambda_1 r^2 + 4\Lambda^2 r + 400n^{3/2}\Lambda(\Lambda r)^3)e^{10(2n^2\Lambda^2+1)r} .$$

(2) *is the only place where we need the assumption for* $\|\nabla R\|$.

Proof. We consider geodesic veriations

$$\alpha(t, s) = \exp_{q_{\alpha_j}} s(\exp_{q_{\alpha_j}}^{-1} F_m(\gamma(t))),$$
$$\beta(t, s) = \exp_{q_{\alpha_j}} s(\exp_{q_{\alpha_j}}^{-1} \sigma_m(t)).$$

Then for a fixed t, we have Jacobi fields

$$J_0(s) = \frac{\partial \alpha}{\partial t}(t, s) \quad \text{and} \quad J(s) = \frac{\partial \beta}{\partial t}(t, s),$$

and the second variation formula yields

$$g''_{m,j}(t) = 2(\langle \nabla_{J_0} J_0, T_0 \rangle + \langle J_0, \nabla_{T_0} J_0 \rangle)(1), \quad h''_{m,j}(t) = 2\langle J, \nabla_T J \rangle(1),$$

where T_0 and T denote the vector fields $\partial\alpha/\partial s$ and $\partial\beta/\partial s$. We assert that

$$(*) \qquad (1 - \Lambda^2\|T\|^2)\|J(1)\|^2 \leq \langle J, \nabla_T J \rangle(1) \leq (1 + \Lambda^2\|T\|^2)\|J(1)\|^2,$$

which implies (1). Let τ be a geodesic with $\|\dot{\tau}\| = \|T\|$ in the n-sphere S with constant curvature Λ^2 and I a linear isometry from $\overline{M}_{q_{\alpha_j}}$ to $S_{\tau(0)}$, and W the vector field along τ defined by using the parallel translations along $\beta(t, \ )$ and τ and I. Then a standard comparison argument implies

$$\langle J, J' \rangle(1) = I_0(J, J') \geq I_0(W, W) \geq I_0(V, V) = \langle V, V' \rangle(1),$$

where I_0 denote the index form and V the Jacobi field along τ with $V(0) = 0$ and $V(1) = W(1)$. It is easy to check that

$$\|V(s)\|^2 = s^2\|J^T(1)\|^2 + \frac{\sin^2 \Lambda\|T\|s}{\sin^2 \Lambda\|T\|}(\|J(1)\|^2 - \|J^T(1)\|^2),$$
$$\langle V, V' \rangle(1) = \|J^T(1)\|^2 + \Lambda\|T\| \cot \Lambda\|T\| \cdot (\|J(1)\|^2 - \|J^T(1)\|^2),$$

where J^T denote the tangential component of J. Hence we have that $\langle J, J' \rangle(1) \geq (1 - \Lambda^2\|T\|^2)\|J(1)\|^2$. Let P be a parallel vector field along $\beta(t, \ )$, then we get

$$|\langle J(s) - sJ'(s), P \rangle'| = |s\langle R(T, J)T, P \rangle| \leq 2\Lambda^2\|T\|^2\|J\|s.$$

The integration implies

$$(4.3) \qquad \|J(1) - J'(1)\| \leq \Lambda^2\|T\|^2\|J(1)\|.$$

It follows

$$|\langle J, J' \rangle(1)| \leq \|J(1)\|\|J'(1)\| \leq (1 + \Lambda^2\|T\|^2)\|J(1)\|^2.$$

For (2), we get with $(*)$

$$\begin{aligned} |g''_{m,j}(t) - h''_{m,j}(t)| &\le 2|\langle J_0, J_0'\rangle(1) - \langle J, J'\rangle(1)| + 2|\langle \nabla_{J_0}J_0, T_0\rangle(1)| \\ &\le e^{20\Lambda r}(2 + 8\Lambda^2 r^2) - e^{-20\Lambda r}(2 - 8\Lambda^2 r^2) + 2\|\nabla_{J_0(1)}J_0\| \cdot 2.3r \\ &\le 82\Lambda r + 4.6r\|\nabla_{J_0(1)}J_0\| . \end{aligned}$$

Let $\{e_i\}$ be an orthonormal basis for M_{p_m} and $\{x_i\}$, $\{y_i\}$ the normal coordinate systems on $B(p_m, 10r)$, $B(q_m, 10r)$ based on $\{e_i\}$, $\{I_m(e_i)\}$ respectively. Let $\Gamma^k_{i,j}$ and $\bar{\Gamma}^k_{i,j}$ be the Cristoffel symbols with respect to $\{x_i\}$ and $\{y_i\}$ and let $c: = F_m \circ \gamma$. Note that

$$\dot{c}: = \sum_i \dot{c}_i \frac{\partial}{\partial y_i}, \quad \ddot{c}_k + \sum_{i,j} \Gamma^k_{i,j}(\gamma(t))\dot{c}_i\dot{c}_j = 0 ,$$

$$\begin{aligned} \nabla_{\dot{c}}\dot{c} &= \sum_k (\ddot{c}_k + \sum_{ij} \bar{\Gamma}^k_{i,j}(c(t))\dot{c}_i\dot{c}_j)\frac{\partial}{\partial y_k} \\ &= \sum_{k,i,j} (-\Gamma^k_{i,j}(\gamma(t)) + \bar{\Gamma}^k_{i,j}(c(t)))\dot{c}_i\dot{c}_j \frac{\partial}{\partial y_k} . \end{aligned}$$

By the Rauch comparison theorem, we get

$$|\dot{c}_i| \le e^{10\Lambda r}\|\dot{c}\| \le e^{30\Lambda r} , \qquad \left\|\frac{\partial}{\partial y_k}\right\| \le e^{10\Lambda r} ,$$

$$|\Gamma^k_{i,j}| \le e^{10\Lambda r}\left\|\nabla_{\partial/\partial x_i}\frac{\partial}{\partial x_j}\right\|, \quad |\bar{\Gamma}^k_{i,j}| \le e^{10\Lambda r}\left\|\nabla_{\partial/\partial y_i}\frac{\partial}{\partial y_j}\right\| ,$$

and from a Cheeger's result (See [4], Lemma 4.3), we can estimate with (**) in Section 3

$$\left\|\nabla_{\partial/\partial x_i}\frac{\partial}{\partial x_j}\right\|, \quad \left\|\nabla_{\partial/\partial y_i}\frac{\partial}{\partial y_j}\right\| \le \Omega .$$

Therefore we conclude that $\|\nabla_{\dot{c}}\dot{c}\| \le 2n^3 e^{80\Lambda r}\Omega$, and this yields (2). Q.E.D.

The following lemma is used in the proof of Lemma 4.5.

LEMMA 4.4. *Let φ; $[0, t] \to \boldsymbol{R}$ be a C^2-function such that $\varphi(0) = 0$ and $|\varphi(s)| \le \alpha$, $|\varphi''(s)| \le \kappa$ on $[0, t]$. Then $|\varphi'(0)| \le \alpha/t + \kappa t/2$.*

LEMMA 4.5. *$\|PdF_i(v) - dF_k(v)\| \le 2^{n(n+11)/2}(11\delta_3 + \Omega_1 r/2)$, where P denotes the parallel translation along the minimizing geodesic from $F_i(x_0)$ to $F_k(x_0)$.*

Proof. Let τ be a geodesic with $\dot{\tau}(0) = PdF_i(v)$ and let $u_j(t): = \Phi^j(\tau(t))$. We apply the previous lemma to $h_{k,j} - u_j$. On $[0, r/2]$ we have with (3.6) and Lemma 4.2 (2)

$$\begin{aligned}|h_{k,j}-h_{i,j}| &\le |h_{k,j}-g_{k,j}|+|g_{k,j}-f_j|+|f_j-g_{i,j}|+|g_{i,j}-h_{i,j}|\\ &\le 4\delta_2 r^2+\Omega_1 r^3/4.\end{aligned}$$

and the Rauch comparison theorem implies

$$|h_{i,j}-u_j| \le d(\sigma_i(0), \tau(0)) \cosh \Lambda r\cdot 4r \le 5\delta_3 r^2,$$

hence

$$|h_{k,j}-u_j| \le (4\delta_2+5\delta_3+\Omega_1 r/4)r^2.$$

Together with Lemma 4.2 (1), Lemma 4.4 applied to $\varphi = h_{k,j}-u_j$ yields

$$\begin{aligned}|d\Phi^j(\dot\sigma_k(0)-\dot\tau(0))| &\le 2(4\delta_2+5\delta_3+\Omega_1 r/4)r+82\Lambda r^2/4\\ &\le (11\delta_3+\Omega_1 r/2)r.\end{aligned}$$

By Lemma 4.1, we conclude

$$\|PdF_i(v)-dF_k(v)\| \le 2^{n(n+11)/2}(11\delta_3+\Omega_1 r/2). \qquad \text{Q.E.D.}$$

Let P_k, P_i denote the parallel translation along the minimizing geodesics from y_0 to $F_k(x_0)$, $F_i(x_0)$, and for simplicity, set

$$v_m := dF_m(v), \quad \tilde v_m := d(\exp_{y_0}^{-1})(dF_m(v)), \quad m = i, k.$$

LEMMA 4.6. $\|\tilde v_k - \tilde v_i\| \le \delta_4$, *where* $\delta_4 = 2^{n(n+11)/2}(12\delta_3+\Omega_1 r/2)$.

Proof. From standard estimate of the Jacobi equation and an easy comparison argument, we get

$$\|P_k\tilde v_k - v_k\|, \quad \|P_i^{-1}v_i-\tilde v_i\|, \quad \|Pv_i - P_kP_i^{-1}v_i\| \le \Lambda^2 r^2.$$

Together with Lemma 4.5, this yields

$$\begin{aligned}\|\tilde v_k-\tilde v_i\| &= \|P_k\tilde v_k - P_k\tilde v_i\|\\ &\le \|P_k\tilde v_k - v_k\| + \|v_k - Pv_i\| + \|Pv_i - P_kP_i^{-1}v_i\|\\ &\quad + \|P_kP_i^{-1}v_i - P_k\tilde v_i\|\\ &\le 2^{n(n+11)/2}(12\delta_3+\Omega_1 r/2). \qquad \text{Q.E.D.}\end{aligned}$$

Proof of Theorem 1. By Lemma 4.6, we have

$$\|\sum_i \psi(d(x_0,p_i)/r)\tilde v_i - \sum_i \psi(d(x_0,p_i)/r)\tilde v_k\| \le \delta_4 N_2,$$

hence with Lemma 3.4

$$\|\sum_i \psi(d(x_0,p_i)/r)\tilde v_i\| \ge (0.9-6^n\delta_4)N_1.$$

If we set $\varepsilon \le \varepsilon_1$, $r \le r_1$, then we get with (3.8)

$$\|\sum_i \psi(d(x_0, p_i)/r)\tilde{v}_i\| > \|\sum_i d/dt\ \psi(d(\gamma(t), p_i)/r)|_{t=0} \cdot \exp_{y_0}^{-1}(F_i(x_0))\| + 0.1N_1 .$$

By Lemma 3.1, F is an immersion. Furthermore the above inequality and (3.3) imply

$$\|d/dt\ V(\gamma(t), y_0)|_{t=0}\| > 0.1\, N_1/N_2 .$$

On the other hand, a standard Jacobi fields estimate (4.3) yields

$$\|\nabla_{dF(v)} V(x_0, F(\gamma(t)))\| \leq 4N_2\|dF(v)\| .$$

Hence we have with (3.2) and Lemma 3.4

$$\|dF(v)\| \geq N_1/40N_2^2 \geq \tilde{v}(2^{-(n+10)}r)/40 \cdot 6^n v(5.1r) > 0 .$$

This conclude that F must be surjective, and hence injective since it is a homotopy equivalence by its construction. Q.E.D.

Added in proof. Recently we have received a preprint, S. Peters "Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds", where the finiteness of diffeomorphism classes of Cheeger type is proved for all dimensions without the assumption for $\|\nabla R\|$ by using a similar method to our Theorem 1.

REFERENCES

[1] R. Bishop and R. Crittenden, Geometry of manifolds, Academic Press, New-York, 1964.

[2] P. Buser and H. Karcher, Gromov's almost flat manifolds, Astérisque, 1981.

[3] J. Cheeger, Comparison and finiteness theorems for Riemannian manifolds, Ph. D. Thesis, Princeton Univ., 1967.

[4] ——, Pinching theorems for a certain class of Riemannian manifolds, Amer. J. Math., **91** (1969), 807–834.

[5] ——, Finiteness theorems for Riemannian manifolds, Amer. J. Math., **92** (1970), 61–74.

[6] J. Cheeger and D. Ebin, Comparison theorems in Riemannian geometry, North-Holland, 1975.

[7] R. Greene, Complete metrics of bounded curvature on noncompact manifolds, Arch. Math., **31** (1978), 89–95.

[8] M. Gromov, Almost flat manifolds, J. Differential Geom., **13** (1978), 231–241.

[9] ——, Structures métriques pour les variétés riemanniennes, rédigé par J. Lafontaine et P. Pansu, Cedic-Fernand Nathan, Paris, 1981.

[10] E. Heintz and H. Karcher, A general comparison theorem with applications to volume estimates for submanifolds, Ann. Sci. Ecole Norm. Sup., **11** (1978), 451–470.

[11] M. Maeda, Volume estimate of submanifolds in compact Riemannian manifolds, J. Math. Soc. Japan, **30** (1978), 533–551.

[12] Y. Shikata, On a distance function on the set of differentiable structures, Osaka J. Math., **3** (1966), 65–79.

[13] A. Weinstein, On the homotopy type of positively-pinched manifolds, Arch. Math., **18** (1967), 523–524.

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