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# A ZETA FUNCTION CONNECTED WITH THE EIGENVALUES OF THE LAPLACE-BELTRAMI OPERATOR ON THE FUNDAMENTAL DOMAIN OF THE MODULAR GROUP

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## §1. Introduction

Let $\lambda_0 = 0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots$ run over the eigenvalues of the discrete spectrum of the Laplace-Beltrami operator on $L^2(H/\Gamma)$, where $H$ is the upper half of the complex plane and we take $\Gamma = PSL(2, Z)$. It is well known that $\lambda_1 > \frac{1}{4}$. We put $\lambda_j = \frac{1}{4} + r_j^2$ for $j \geqq 0$. Let $\alpha$ be a positive number. Here we are concerned with the zeta function defined by

$$Z_\alpha(s) = \sum_{r_j>0} \frac{\sin(\alpha r_j)}{r_j^s} .$$

We shall prove the following theorem.

THEOREM. *For any positive $\alpha$, $Z_\alpha(s)$ is entire.*

This should be compared with Minakshisundaram and Pleijel [8], Guinand [4] and Delsarte [1] and also with the author's result which states that on the Riemann Hypothesis $\sum_{\gamma>0} (\sin(\alpha\gamma)/\gamma^s)$ is entire for any positive $\alpha$, where $\gamma$ runs over the imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$ (cf. [3]).

We recall first Selberg's trace formula. Let $h(r)$ satisfy the conditions;

1) $h(r) = h(-r)$
2) $h(r)$ is analytic in the strip $|\mathrm{Im}\, r| < \frac{1}{2} + \varepsilon$, $\varepsilon > 0$
3) $h(r) = O((1 + |r|^2)^{-1-\varepsilon})$ in this strip.

Then we have

$$\sum h(r_j) = \frac{1}{6}\int_{-\infty}^{\infty} r \,\mathrm{th}(\pi r) h(r) dr$$
$$+ \int_{-\infty}^{\infty} \left(\frac{1}{2} + \frac{2}{3\sqrt{3}}(e^{\pi r/3} + e^{-\pi r/3})\right)\frac{h(r)}{e^{\pi r} + e^{-\pi r}} dr$$

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$$+ 2\sum_{\{P_0\}}\sum_{k=1}^{\infty}\frac{\log N(P_0)}{N(P_0)^{k/2}-N(P_0)^{-k/2}}g(k\log N(P_0))$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\frac{\varphi'}{\varphi}\Big(\frac{1}{2}+ir\Big)dr-\frac{1}{\pi}\int_{-\infty}^{\infty}h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr$$
$$-2\log 2\,g(0)+\tfrac{1}{2}(1-\varphi(\tfrac{1}{2}))h(0)\,,$$

where the left hand side is over all the solutions $r_j$ of all the equations $\lambda_j=\frac{1}{4}+r_j^2$, $\{P_0\}$ runs over all primitive hyperbolic conjugacy classes in $\Gamma$, $N(P_0)$ is the square of the eigenvalue (greater than one) of a representative element $P_0$,

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iru}h(r)dr\,,\qquad \varphi(s)=\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}$$

(cf. Selberg [9] and Hejhal [6]). Let $Z(s)$ be the Selberg's zeta function defined by

$$Z(s)=\prod_{\{P_0\}}\prod_{k=0}^{\infty}(1-N(P_0)^{-k-s})\qquad \text{for } \operatorname{Re}s>1\,.$$

Then Selberg's trace formula describes the location of the poles and the zeros of $Z(s)$ and gives the functional equation of $Z(s)$. Using these, one can deduce the following formula for $T>0$,

$$N(T)\equiv|\{0\leqq r\leqq T\}|$$
$$=\frac{1}{4\pi}\int_0^T\Big(\frac{\varphi'}{\varphi}\Big(\frac{1}{2}+ir\Big)+\frac{\varphi'}{\varphi}\Big(\frac{1}{2}-ir\Big)\Big)dr$$
$$+S_Z(T)+\frac{1}{12}T^2-\frac{1}{\pi}T\log T+\frac{\log(e/2)}{\pi}T+w(T)\,,$$

where $r$ runs over $r_j$, $S_Z(T)=(1/\pi)\arg Z(\frac{1}{2}+iT)$ as usual and $w(T)$ satisfies $w'(T)\ll T^{-2}$ for $T>T_0$ (cf. Venkov [11]). Using this formula, one gets the following result which can be proved by the same method as the author's [2];

$$\sum_{0<r\leqq T}e^{i\alpha r}=\frac{1}{\pi}\frac{\Lambda(e^{\alpha/2})}{e^{\alpha/2}}T+\frac{e^{i\alpha T}}{6i\alpha}T$$
$$+\frac{1}{2\pi}e^{-\alpha/2}\Big(\sum_{\{P\},\,N(P)=e^{\alpha}}\tilde{\Lambda}(P)\Big)T+O(T/\log T)\,,$$

where $T>T_0$, $\Lambda(x)$ is the von Mangoldt function, $\{P\}$ runs over all hyperbolic conjugacy classes and we put

$$\tilde{\Lambda}(P)=\frac{\log N(P_0)}{1-N(P_0)^{-k}} \quad \text{and} \quad N(P)=N(P_0)^k \quad \text{if } P=P_0^k$$

with an integer $k \geqq 1$. By this we see that $Z_\alpha(s)$ is regular in $\operatorname{Re} s>1$. We shall prove its analytic continuation using the Selberg's trace formula.

Finally, we remark that $Z_\alpha(1)$ or $Z_\alpha(0)$ can be evaluated as a bi-product of the proof of the above theorem and they have some significant arithmetic meanings.(*)

## § 2. Proof of Theorem

We use Selberg's trace formula with

$$h(r)=e^{-(1/4+r^2)x}\sin(\alpha r)r\,.$$

Then

$$\begin{aligned} g(u)=&-\frac{(u-\alpha)}{8\sqrt{\pi}\,x^{3/2}}e^{-(u-\alpha)^2/4x}e^{-(1/4)x}\\ &+\frac{(u+\alpha)}{8\sqrt{\pi}\,x^{3/2}}e^{-(u+\alpha)^2/4x}e^{-(1/4)x}\end{aligned}$$

and

$$\begin{aligned} &2\sum_{r>0}e^{-(1/4+r^2)x}\sin(\alpha r)r\\ &=\frac{1}{2}(e^{\alpha/2}-e^{-\alpha/2})+\frac{1}{6}\int_{-\infty}^{\infty}r\,\mathrm{th}(\pi r)e^{-(1/4+r^2)x}\sin(\alpha r)r\,dr\\ &+\int_{-\infty}^{\infty}\left(\frac{1}{2}+\frac{2}{3\sqrt{3}}(e^{(1/3)r}+e^{-(1/3)r})\right)\frac{e^{-(1/4+r^2)x}\sin(\alpha r)r}{e^{\pi r}+e^{-\pi r}}dr\\ &+\frac{x^{-3/2}}{4\sqrt{\pi}}\sum_{\{P\}}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}(\alpha-\log N(P))e^{-(\alpha-\log N(P))^2/4x}e^{-(1/4)x}\\ &+\frac{x^{-3/2}}{4\sqrt{\pi}}\sum_{\{P\}}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}}(\alpha+\log N(P))e^{-(\alpha+\log N(P))^2/4x}e^{-(1/4)x}\\ &+\frac{\log\pi/2}{2\sqrt{\pi}}\alpha x^{-3/2}e^{-\alpha^2/4x}e^{-(1/4)x}\\ &-\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-(1/4+r^2)x}\sin(\alpha r)r\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)+\frac{\Gamma'}{\Gamma}(1+ir)\right)dr\\ &+\frac{1}{2\sqrt{\pi}}x^{-3/2}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n}(\alpha+2\log n)e^{-(2\log n+\alpha)^2/4x}e^{-(1/4)x}\\ &+\frac{1}{2\sqrt{\pi}}x^{-3/2}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n}(\alpha-2\log n)e^{-(2\log n-\alpha)^2/4x}e^{-(1/4)x}\,.\end{aligned}$$

(*) The results are announced in the author's "Zeros, Eigenvalues and Arithmetic" (Proc. of Japan Academy, 60 Ser. A (1984) p. 22–25).

Now we consider the integral

$$I \equiv \int_0^\infty x^{s-1} \sum_{r>0} e^{-r^2 x} \sin(\alpha r) r \, dx .$$

We remark that

$$\sum_{r>0} r \int_0^\infty x^{\sigma-1} e^{-r^2 x} dx = \sum_{r>0} \frac{1}{r^{2\sigma-1}} \int_0^\infty x^{\sigma-1} e^{-x} dx$$
$$= \Gamma(\sigma) \sum_{r>0} \frac{1}{r^{2\sigma-1}} \qquad \text{for } \sigma > 3/2 .$$

Hence for Re $s > 3/2$,

$$I = Z_\alpha(2s-1)\Gamma(s) .$$

Now

$$I = \left(\int_0^1 + \int_1^\infty\right) x^{s-1} \left(\sum_{r>0} e^{-r^2 x} \sin(\alpha r) r\right) dx$$
$$= I_1 + I_2 , \quad \text{say} .$$

$I_2$ is entire.

$$I_1 = \frac{1}{4}(e^{\alpha/2} - e^{-\alpha/2}) \int_0^1 x^{s-1} e^{(1/4)x} dx$$
$$+ \frac{1}{12} \int_0^1 x^{s-1} \int_{-\infty}^\infty r \operatorname{th}(\pi r) e^{-r^2 x} \sin(\alpha r) r \, dr \, dx$$
$$+ \int_0^1 x^{s-1} \int_{-\infty}^\infty \left(\frac{1}{4} + \frac{1}{3\sqrt{3}}(e^{\pi r/3} + e^{-\pi r/3})\right) \frac{e^{-r^2 x} \sin(\alpha r) r}{e^{\pi r} + e^{-\pi r}} dr \, dx$$
$$+ \frac{1}{8\sqrt{\pi}} \int_0^1 x^{s-5/2} \sum_{\{P\}} \frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}} (\alpha - \log N(P)) e^{-(\alpha - \log N(P))^2/4x} dx$$
$$+ \frac{1}{8\sqrt{\pi}} \int_0^1 x^{s-5/2} \sum_{\{P\}} \frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}} (\alpha + \log N(P)) e^{-(\alpha + \log N(P))^2/4x} dx$$
$$+ \frac{\log(\pi/2)}{4\sqrt{\pi}} \alpha \int_0^1 x^{s-5/2} e^{-\alpha^2/4x} dx$$
$$- \frac{1}{2\pi} \int_0^1 x^{s-1} \int_{-\infty}^\infty e^{-r^2 x} \sin(\alpha r) r \left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + ir\right) + \frac{\Gamma'}{\Gamma}(1 + ir)\right) dr \, dx$$
$$+ \frac{1}{4\sqrt{\pi}} \int_0^1 x^{s-5/2} \sum_{n=2}^\infty \frac{\Lambda(n)}{n} \big((\alpha - 2\log n) e^{-(\alpha - 2\log n)^2/4x}$$
$$+ (\alpha + 2\log n) e^{-(\alpha + 2\log n)^2/4x}\big) dx$$
$$= I_3 + I_4 + \cdots + I_9 + (I_{10} + I_{11}) , \qquad \text{say} .$$

We see at first that $I_3/\Gamma(s)$ and $I_5/\Gamma(s)$ are entire. Since

$$I_6 = \frac{1}{8\sqrt{\pi}} \int_1^\infty x^{-s+1/2} \sum_{\{P\}} \frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}} (\alpha - \log N(P)) e^{-(\log N(P)-\alpha)^2 x/4}\, dx$$

$I_6$ is entire. Similarly, $I_7$, $I_8$, $I_{10}$ and $I_{11}$ are entire.

We shall treat $I_9$ next.

$$I_9 = -\frac{1}{2\pi} \int_0^1 x^{s-1} \left(\int_0^1 + \int_1^\infty\right) e^{-r^2 x} \sin(\alpha r) r$$
$$\cdot \left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + ir\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - ir\right) + \frac{\Gamma'}{\Gamma}(1 + ir) + \frac{\Gamma'}{\Gamma}(1 - ir)\right) dr\, dx$$
$$= I_{12} + I_{13}, \qquad \text{say}.$$

$I_{12}/\Gamma(s)$ is entire.

By Stirling's formula, we see that

$$I_{13} = -\frac{1}{2\pi} \int_0^1 x^{s-1} \int_1^\infty e^{-r^2 x} \sin(\alpha r) r \left(4 \log r + \sum_{k=1}^\infty \frac{b_k}{r^{2k}}\right) dr\, dx$$
$$= I_{14}(s) + I_{15}(s), \qquad \text{say}.$$

We remark that $I_{15}(s)/\Gamma(s)$ is entire.

$$I_{14}(s) = -\frac{1}{\pi} \int_0^1 x^{s-2} (\nu_1(x) + \alpha\nu_2(x) - \alpha\nu_3(x))\, dx$$
$$= -\frac{1}{\pi} J_1(s) - \frac{\alpha}{\pi} J_2(s) + \frac{\alpha}{\pi} J_3(s), \qquad \text{say},$$

where we put

$$\nu_1(x) = \int_1^\infty e^{-r^2 x} \sin(\alpha r) r^{-1}\, dr,$$
$$\nu_2(x) = \int_0^\infty e^{-r^2 x} \cos(\alpha r) \log r\, dr$$

and

$$\nu_3(x) = \int_0^1 e^{-r^2 x} \cos(\alpha r) \log r\, dr.$$

$J_1(s)/\Gamma(s)$ is entire except a simple pole at $s = 1$. Since

$$J_3(s) = \frac{1}{s-1} \nu_3(1) - \frac{1}{s-1} \int_0^1 x^{s-1} \nu_3'(x)\, dx$$

for $\operatorname{Re} s > 1$, $J_3(s)/\Gamma(s)$ is entire except a simple pole at $s = 1$.

$$J_2(s) = -\frac{\sqrt{\pi}}{4}\int_0^1 x^{s-5/2}\log x\, e^{-\alpha^2/4x}\, dx$$
$$+\frac{\sqrt{\pi}}{4}\int_1^\infty x^{-s+1/2}\left(\int_0^1 e^{-y}e^{-(1/4)\alpha^2 x} - \frac{e^{-\alpha^2 x/4(1+y)}}{\sqrt{1+y}}\right)\frac{dy}{y}\,dx$$
$$+\frac{\sqrt{\pi}}{4}\int_1^\infty x^{-s+1/2}e^{-(1/4)\alpha^2 x}\,dx\int_1^\infty e^{-y}\frac{dy}{y}$$
$$-\frac{\sqrt{\pi}}{4}\int_1^\infty x^{-s+1/2}\int_1^\infty \frac{e^{-\alpha^2 x/4(1+y)}}{y\sqrt{1+y}}\,dy\,dx\,.$$

We see that the first three integrals are entire and that the last integral devided by $\Gamma(s)$ is entire except a simple pole at $s=1$. By the calculus of residues, we see that $I_{14}(s)/\Gamma(s)$ is entire, and hence $I_9/\Gamma(s)$ is entire.

We are left to treat $I_4$.

$$I_4 = \frac{1}{6}\int_0^1 x^{s-1}\left(\int_0^1 + \int_1^\infty\right) r^2\sin(\alpha r)e^{-r^2 x}\,dr\,dx$$
$$-\frac{1}{3}\int_0^1 x^{s-1}\int_0^\infty \frac{r^2\sin(\alpha r)}{e^{2\pi r}+1}e^{-r^2 x}\,dr\,dx$$
$$= (I_{16}+I_{17}) + I_{18}\,, \qquad \text{say}\,.$$

$I_{16}/\Gamma(s)$ and $I_{18}/\Gamma(s)$ are entire. To treat $I_{17}$ we put

$$\eta(x) = \int_1^\infty r^2\sin(\alpha r)e^{-r^2 x}\,dr\,,$$
$$\eta_1(x) = \int_1^\infty r^{-2}\sin(\alpha r)e^{-r^2 x}\,dr$$

and

$$\eta_2(x) = \int_1^\infty r^{-3}\cos(\alpha r)e^{-r^2 x}\,dr\,.$$

Then for $x>0$,

$$\eta(x) = \frac{\sin\alpha}{2}\frac{e^{-x}}{x} + \frac{\alpha\cos\alpha}{4}\frac{e^{-x}}{x^2} + \frac{e^{-x}}{4x^2}\sin\alpha$$
$$-\frac{1}{4x^2}\eta_1(x) + \frac{e^{-x}}{8x^3}\alpha\cos\alpha - \frac{\eta_2(x)}{4x^3}\alpha$$
$$-\frac{\alpha^2}{8}\sin\alpha\frac{e^{-x}}{x^3} - \frac{\alpha^3\cos\alpha}{16}\frac{e^{-x}}{x^4}$$
$$+\frac{\alpha^4}{16}\frac{\eta_1(x)}{x^4} + \frac{\alpha^3}{8}\frac{\eta_2(x)}{x^4}\,.$$

We put $F_1(s-3) = \int_0^1 x^{s-3}\eta_1(x)\,dx$ and

$$F_2(s-4) = \int_0^1 x^{s-4}\eta_2(x)\,dx\,.$$

Then $F_1(s-3)$ is regular in $\operatorname{Re} s > 2$ and $F_2(s-4)$ is regular in $\operatorname{Re} s > 3$. We remark that in $\operatorname{Re} s > 3$,

$$F_2(s-4) = \frac{\eta_2(1)}{s-2} + \frac{\cos\alpha}{2}\frac{1}{s-2}(\Gamma(s-3) - E(s-3)) - \frac{\alpha}{2(s-2)}F_1(s-4)$$

and

$$\begin{aligned} F_1(s-3) = {} & \frac{\eta_1(1)}{s-2} + \frac{\sin\alpha}{2(s-2)}(\Gamma(s-2) - E(s-2)) \\ & + \frac{\alpha\cos\alpha}{4(s-2)}(\Gamma(s-3) - E(s-3)) - \frac{1}{2(s-2)}F_1(s-3) \\ & - \frac{\alpha^2}{4(s-2)}F_1(s-4) - \frac{\alpha}{2(s-2)}F_2(s-4)\,, \end{aligned}$$

where $E(s-2) \equiv \int_1^\infty x^{s-3}e^{-x}\,dx$ is entire.

Hence in $\operatorname{Re} s > 3$,

$$\begin{aligned} \frac{\alpha^2}{4}\frac{s-3}{s-2}F_1(s-4) = {} & -\left(s - \frac{3}{2}\right)F_1(s-3) + \eta_1(1) \\ & + \frac{\sin\alpha}{2}(\Gamma(s-2) - E(s-2)) + \frac{\alpha\cos\alpha}{4}(\Gamma(s-3) - E(s-3)) \\ & - \frac{\alpha}{2(s-2)}\eta_2(1) - \frac{\alpha\cos\alpha}{4(s-2)}(\Gamma(s-3) - E(s-3))\,. \end{aligned}$$

From these relations we see that $F_1(s)$ and $F_2(s)$ can be continued analytically to the whole complex plane except simple poles at $s = -1, -2, \cdots$.

Now

$$\begin{aligned} 6I_{17} & = \int_0^1 x^{s-1}\eta(x)\,dx \\ & = \frac{\sin\alpha}{2}(\Gamma(s-1) - E(s-1)) + \frac{\alpha\cos\alpha}{4}(\Gamma(s-2) - E(s-2)) \\ & \quad + \frac{\sin\alpha}{4}(\Gamma(s-2) - E(s-2)) - \frac{1}{4}F_1(s-3) \end{aligned}$$

$$+\frac{\alpha\cos\alpha}{8}(\Gamma(s-3)-E(s-3))-\frac{1}{4}\alpha F_2(s-4)$$
$$-\frac{\alpha^2\sin\alpha}{8}(\Gamma(s-3)-E(s-3))-\frac{\alpha^3\cos\alpha}{16}(\Gamma(s-4)-E(s-4))$$
$$+\frac{\alpha^4}{16}F_1(s-5)+F_2(s-5)\,.$$

Hence we see that $I_{17}/\Gamma(s)$ is entire except, at most, simple poles at $s=4, 3, 2$ and 1. However as we see immediately by calculating the residues that $I_{17}/\Gamma(s)$ is entire.

Thus we have proved that $Z_\alpha(2s-1)$ is entire.

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