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TENSOR PRODUCTS OF POSITIVE DEFINITE QUADRATIC FORMS, VII

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In this paper we generalize results of the third paper of this series. As a corollary we can show the following: Let L_i $(1 \leq i \leq n)$ be a positive definite quadratic form which is equivalent to one of Cartan matrices of Lie algebras of type A_n $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8 and assume that $\otimes_{i=1}^n L_i \cong \otimes_{i=1}^m M_i$ where M_i $(1 \leq i \leq m)$ is positive definite quadratic forms and satisfies that $\operatorname{rk} M_i \geq 2$ and $M_i \cong K \otimes L$ implies $\operatorname{rk} K$ or $\operatorname{rk} L = 1$. Then we have $n = m$ and L_i is equivalent to a constant multiple of $M_{s(i)}$ for some permutation s. Therefore we get the uniqueness of decompositions with respect to tensor products in this case.

We explain notations and terminology.

By a positive lattice we mean a lattice on a positive definite quadratic space over the rational number field. Let L be a positive lattice. We put

$$m(L) = \min_{\substack{x \in L \\ x \neq 0}} Q(x),$$

where $Q(\ )$ is a quadratic form associated with L. Put $\mathfrak{M}(L) = \{x \in L \mid Q(x) = m(L)\}$ and denote by $\tilde{L}$ a submodule of L spanned by $\mathfrak{M}(L)$. If $\mathfrak{M}(L \otimes M) \subset \{x \otimes y \mid x \in L, y \in M\}$ holds for every positive lattice M, then L is called of E-type and then $\mathfrak{M}(L \otimes M) = \mathfrak{M}(L) \otimes \mathfrak{M}(M)$, $m(L \otimes M) = m(L)m(M)$ hold. Unless L is isometric to the tensor product of positive lattices M, N with $\operatorname{rk} M > 1$, $\operatorname{rk} N > 1$, L is called indecomposable with respect to tensor products.

Let A be a finite set and $[\ ,\ ]$ a mapping from $A \times A$ to $\{t \mid 0 \leq t \leq 1\}$ satisfying

(i) $[a, a'] = 1$ if and only if $a = a'$, and

(ii) $[a, a'] = [a', a]$ for $a, a' \in A$.

Then we call $(A, [\ ,\ ])$ or simply A a weighted graph.

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Let A be a weighted graph. A is called connected unless there exist subsets A_1, A_2 of A such that $A = A_1 \cup A_2$, $A_1 \cap A_2 = \phi$ and $[a_1, a_2] = 0$ for any $a_i \in A_i$.

Let A, B be weighted graphs. For $(a, b), (a', b') \in A \times B$ we define $[(a, b), (a', b')]$ by $[a, a']\cdot[b, b']$. Then $A \times B$ becomes a weighted graph. If there exists a bijection σ from A on B such that $[a, a'] = [\sigma(a), \sigma(a')]$ $(a, a' \in A)$, then we say that A, B are isometric and write $\sigma : A \cong B$. Unless a weighted graph A is isometric to $B \times C(|B| > 1, |C| > 1)$, we say that A is indecomposable.

Let L be a positive lattice with a bilinear form $B(\ ,\ )$ $(B(x, x) = Q(x))$. Put $G(L) = \mathfrak{M}(L)/\pm$ and for $a, b \in G(L)$, put $[a, b] = |B(a, b)|/m(L)$. Then $G(L)$ becomes a weighted graph.

Let L, M be positive lattices. Then it is obvious that the isometry $\sigma : L \cong M$ induces the isometry $\tilde{\sigma} : G(L) \cong G(M)$, and that $G(L \otimes M) \cong G(L) \times G(M)$ if either L or M is of E-type.

LEMMA 1. *Let A, A', B, C be weighted graphs and assume that $A = \{e_i\}_{i=1}^n$ and $\sigma : A \times B \cong A' \times C$. Take any element $b \in B$ and fix it. Define $f_i \in A'$, $c_i \in C$, $g_{ij} \in A$, $b_{ij} \in B$ by*

$$\sigma(e_i, b) = (f_i, c_i) \quad \textit{and} \quad \sigma(g_{ij}, b_{ij}) = (f_i, c_j)\,.$$

Then we have $[e_i, e_j] = 0$ if $b_{ij} \neq b$.

LEMMA 2. *Let A, A', B, C be weighted graphs and assume that $A = \{e_i\}_{i=1}^n$ is connected and $\sigma : A \times B \cong A' \times C$. Take any element $b \in B$ and put $\sigma(e_i, b) = (f_i, c_i)$. Then we have*

$$A \cong \{\sigma(e_i, b) | 1 \leq i \leq n\} = \{f_i | 1 \leq i \leq n\} \times \{c_i | 1 \leq i \leq n\}\,.$$

Lemmas 1, 2 are proved in Section 1 of [4] when $A = A'$. Moreover we did not use the condition $A = A'$. Hence the proof in case of $A = A'$ is valid with trivial changes like that f_i is regarded as an element not of A but of A'.

LEMMA 3. *Let A, B, C be connected weighted graphs and let σ be an isometry from $A \times B$ on $A \times C$. If there exist $b_0 \in B$, $c_0 \in C$ such that $\sigma(x, b_0) = (f(x), c_0)$ for every $x \in A$, then f is an isometry from A on A and there is an isometry g from B on C with $\sigma(x, y) = (f(x), g(y))$ $(x \in A, y \in B)$.*

This is proved in [3].

THEOREM. *Let A_i $(1 \leq i \leq n)$, B_i $(1 \leq i \leq m)$ be connected weighted graphs and suppose that $|A_i| > 1$, $|B_j| > 1$ and A_i, B_j are indecomposable $(1 \leq i \leq n, 1 \leq j \leq m)$. Assume*

$$\sigma : \prod_{i=1}^{n} A_i \cong \prod_{i=1}^{m} B_i .$$

Then we have $n = m$ and there exist a permutation s and isometries $\sigma_i : A_i \cong B_{s(i)}$ and σ is equal to the product of σ_i.

Proof. Without loss of generality we may assume that $|A_1| \geq |A_i|, |B_j|$ $(1 \leq i \leq n, 1 \leq j \leq m)$. Take any element $e_i \in A_i$ $(i \geq 2)$ and put $e = e_2 \times \cdots \times e_n \in \prod_{i=2}^{n} A_i$. By interchanges of B_i we may assume that the projection of $\sigma(A_1 \times e)$ on B_1 includes at least two distinct elements. Applying Lemma 2 to $A = A_1$, $B = \prod_{i=2}^{n} A_i$, $A' = B_1$, $C = \prod_{i=2}^{m} B_i$, we get

$$\sigma(A_1 \times e) \subset B_1 \times c \qquad \text{for some } c \in C ,$$

since A_1 is indecomposable. By the assumption on A_1 we have $\sigma(A_1 \times e) = B_1 \times c$. Hence by virtue of Lemma 3 there exist isometries $f : A_1 \cong B_1$, $g : \prod_{i=2}^{n} A_i \cong \prod_{i=2}^{m} B_i$ such that $\sigma(x \times y) = (f(x), g(y))$ for $x \in A_1$, $y \in \prod_{i=2}^{n} A_i$. Therefore our theorem is inductively proved.

THEOREM. *Let L_i $(1 \leq i \leq n)$ be a positive lattice of E-type and assume that*

(i) $[L_i : \tilde{L}_i] < \infty$,

(ii) *$\tilde{L}_i$ is indecomposable,*

(iii) *L_i is indecomposable with respect to tensor products, and*

(iv) $\operatorname{rk} L_i > 1$.

Suppose that $\sigma : \otimes_{i=1}^{n} L_i \cong \otimes_{i=1}^{m} M_i$ where M_i $(1 \leq i \leq m)$ is a positive lattice satisfying the above conditions (iii), (iv) *for M_i instead of L_i. Then we have $n = m$ and, interchanging M_i if necessary, $\sigma = \otimes \sigma_i$ where σ_i is an isometry from L_i on $M_i^{a_i}$ (scaling of M_i by a positive constant a_i).*

Proof. Put $L = \otimes_{i=1}^{n} L_i$. Since L_i $(1 \leq i \leq n)$ is of E-type, L is also of E-type and $\tilde{L} = \otimes_{i=1}^{n} \tilde{L}_i$ and then $[L : \tilde{L}] < \infty$. Since $\tilde{L}_i$ is indecomposable and $[L_i : \tilde{L}_i] < \infty$, L_i is also indecomposable. Hence L and $\tilde{L}$ are indecomposable [2]. Then by virtue of Theorem in [4] M_i is of E-type and satisfies the conditions (i) and (ii) for M_i instead of L_i. Therefore without loss of generality we may assume that $m(L_i) = m(M_j) = 1$ $(1 \leq i \leq n, 1 \leq j \leq m)$ and interchanging L_i if necessary, L_1 satisfies

(1) $\operatorname{rk} L_1 \geq \operatorname{rk} L_i$, $\operatorname{rk} M_j$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$,

(2) if $\operatorname{rk} L_1 = \operatorname{rk} M_j$, then $dL_1 \leq dM_j$.

Since $\tilde{L}_i$, $\tilde{M}_j$ are indecomposable, associated graphs $G(L_i)$, $G(M_j)$ are connected. σ induces an isometry $\tilde{\sigma} : \prod G(L_i) = \prod G(M_i)$. Fix any element $e_i \in \mathfrak{M}(L_i)$ $(i \geq 2)$ and e_i is regarded as an element of $G(L_i)$. Then we have

$$\tilde{\sigma}(G(L_1) \times e_2 \times \cdots \times e_n) = \prod G_i,\ G_i \subset G(M_i)\,.$$

Denoting by M_i^0 a submodule of M_i spanned by elements of $\mathfrak{M}(M_i)$ which are projected in G_i, we get

$$\sigma(\tilde{L}_1 \otimes e_2 \otimes \cdots \otimes e_n) = M_1^0 \otimes \cdots \otimes M_m^0\,.$$

Put $\overline{M}_i = M_i \cap \boldsymbol{Q}M_i^0$, then $\overline{M}_i$ is a direct summand of M_i and $[\overline{M}_i : M_i^0] < \infty$. Comparing direct summands, we have

$$\sigma(L_1 \otimes e_2 \otimes \cdots \otimes e_n) = \overline{M}_1 \otimes \cdots \otimes \overline{M}_m\,.$$

By the assumption (iii) we may assume $\operatorname{rk} \overline{M}_i = 1$ $(i \geq 2)$, interchanging M_i if necessary. Then $\overline{M}_i$ $(i \geq 2)$ is spanned by an element f_i in $\mathfrak{M}(M_i)$. Since we assumed $m(L_i) = m(M_j) = 1$, there is an isometry σ_1 such that $\sigma(x \otimes e_2 \otimes \cdots \otimes e_n) = \sigma_1(x) \otimes f_2 \otimes \cdots \otimes f_m$. By virtue of Lemma 3 the isometry σ_1 is independent of e_i $(i \geq 2)$, since the sign can be absorbed in f_i. The assumption on L_1 implies $\operatorname{rk} M_1 \leq \operatorname{rk} L_1 = \operatorname{rk} \sigma_1(L_1) \leq \operatorname{rk} M_1$ and then $dL_1 \leq dM_1 \leq d\sigma(L_1) = dL_1$ and then $\sigma_1(L_1) = M_1$. Moreover Lemma 3 implies that there is an isometry $\tilde{\sigma}_2 : \prod_{i \geq 2} G(L_i) \cong \prod_{i \geq 2} G(M_i)$ such that $\tilde{\sigma} = \tilde{\sigma}_1 \times \tilde{\sigma}_2$ on $\prod_{i=1}^{n} G(L_i)$. Therefore for any fixed $e_1 \in \mathfrak{M}(L_1)$ we have

$$\sigma(e_1 \otimes e) = \sigma_1(e_1) \otimes \sigma_2(e) \qquad \text{for } e \in \mathfrak{M}(\bigotimes_{i \geq 2} L_i)\,,$$

where $\sigma_2(e) \in \mathfrak{M}(\otimes_{i \geq 2} M_i)$.

Since σ is an isometry, σ_2 is an isometry from $\otimes_{i \geq 2} \tilde{L}_i$ on $\otimes_{i \geq 2} \tilde{M}_i$. Moreover $[\otimes_{i \geq 2} L_i : \otimes_{i \geq 2} \tilde{L}_i] < \infty$ and $e_1 \otimes (\otimes_{i \geq 2} L_i)$, $\sigma_1(e_1) \otimes (\otimes_{i \geq 2} M_i)$ are direct summands. Hence $\sigma_2(\otimes_{i \geq 2} L_i) = \otimes_{i \geq 2} M_i$ follows. For $e_1' \in \mathfrak{M}(L_1)$ we have, similarly, $\sigma(e_1' \otimes e) = \sigma_1(e_1') \otimes \sigma_2'(e)$ for $e \in \mathfrak{M}(\otimes_{i \geq 2} L_i)$ where σ_2' is an isometry from $\otimes_{i \geq 2} L_i$ on $\otimes_{i \geq 2} M_i$. Since $\tilde{\sigma} = \tilde{\sigma}_1 \times \tilde{\sigma}_2$, $\sigma_2(e) = \pm \sigma_2'(e)$ holds for $e \in \mathfrak{M}(\otimes_{i \geq 2} L_i)$. If $B(e_1, e_1') \neq 0$, then $B(e_1, e_1') = B(e_1 \otimes e, e_1' \otimes e) = B(\sigma_1(e_1) \otimes \sigma_2(e), \sigma_1(e_1') \otimes \sigma_2'(e)) = B(e_1, e_1')B(\sigma_2(e), \sigma_2'(e))$ implies $B(\sigma_2(e), \sigma_2'(e)) = 1$ and then $\sigma_2(e) = \sigma_2'(e)$ since $m(M_i) = 1$ and $\sigma_2(e), \sigma_2'(e) \in \mathfrak{M}(\otimes_{i \geq 2} M_i)$. Thus $\sigma_2 = \sigma_2'$ holds on $\mathfrak{M}(\otimes_{i \geq 2} L_i)$ and then $\sigma_2 = \sigma_2'$ on $\otimes_{i \geq 2} L_i$. Since $G(L_1)$ is connected,

σ_2 is independent of the choice of e_1. Thus we have proved $\sigma = \sigma_1 \otimes \sigma_2$ on $\otimes_{i\geq 1} L_i$ where $\sigma_1 : L_1 \cong M_1$, $\sigma_2 : \otimes_{i\geq 2} L_i = \otimes_{i\geq 2} M_i$. Theorem is inductively proved.

COROLLARY. *Let L_i be those in Theorem. Then the orthogonal group of $\otimes L_i$ is generated by the orthogonal group of L_i $(1 \leq i \leq n)$ and interchanges of L_i and L_j if L_i, L_j are isometric.*

This follows directly from Theorem.

EXAMPLES. Let L be a positive lattice which is associated to the Cartan matrix of one of Lie algebras of type A_n $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8. Then L is of E-type [1] and the conditions (i), (ii), (iv) are obviously satisfied. The condition (iii) is checked as follows: Suppose $L \cong M \otimes N$, $\mathrm{rk}\, M$, $\mathrm{rk}\, N > 1$. Then by virtue of Theorem in [4], M, N are of E-type and $M = \tilde{M}$, $N = \tilde{N}$ since $L = \tilde{L}$. Without loss of generality we may assume $m(M) = 1$, $m(N) = 2$. For $e \in \mathfrak{M}(M)$, $f_1, f_2 \in \mathfrak{M}(N)$, $Z \ni B(e \otimes f_1, e \otimes f_2) = B(f_1, f_2)$ follows. If $B(f_1, f_2)$ is even for every $f_1, f_2 \in \mathfrak{M}(N)$, then N is decomposable since the scale of N is $2Z$. This is a contradiction. Hence for some $f_1, f_2 \in \mathfrak{M}(N)$, $B(f_1, f_2) = 1$ holds. Then we have $Z \ni B(e_1 \otimes f_1, e_2 \otimes f_2) = B(e_1, e_2)$ for $e_i \in \mathfrak{M}(M)$ and then the scale of M is Z. Therefore M is decomposable. This is also a contradiction.

Other examples are found in [4].

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