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# ON DEGREES AND GENERA OF CURVES ON SMOOTH QUARTIC SURFACES IN $P^3$

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Our result is motivated by the results [GP] of Gruson and Peskin on characterization of the pair of degree $d$ and genus $g$ of a non-singular curve in $P^3$. In the last step, they construct the required curve $C$ on a singular quartic surface when $g \leqq (d-1)^2/8$. Here we consider curves on smooth quartic surfaces.

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THEOREM 1. *Let $k$ be an algebraically closed field of characteristic* 0 *and $d > 0$ and $g \geqq 0$ be integers. Then there is a non-singular curve $C$ of degree $d$ and genus $g$ on a non-singular quartic surface $X$ in $P^3$ if and only if* (1) $g = d^2/8 + 1$, *or* (2) $g < d^2/8$ and $(d, g) \neq (5, 3)$.

*Remark* 2. Under the notation of Theorem 1, $g = d^2/8 + 1$ if and only if $C$ is a complete intersection of $X$ and a hypersurface of degree $d/4$, which will be proved in the proof below.

*Proof of the only-if-part* ($\Rightarrow$) *of Theorem* 1. Let $H = \mathcal{O}_X(1)$. Since $(H \cdot H) > 0$, one has

$$(C \cdot H)^2 - (H \cdot H) \cdot (C \cdot C) = d^2 - 8(g-1) \geqq 0\,,$$

by Hodge index theorem, because $X$ is a $K3$ surface and $K_C = \mathcal{O}_C(C)$. One has $d^2 \equiv 0, 1, 4, 1 \pmod 8$ according as $d \equiv 0, 1, 2, 3 \pmod 4$. If $d^2 - 8(g-1) = 0$ then the classes of $\mathcal{O}_X(C)$ and $\mathcal{O}_X(H)$ are proportional. Since $X$ is a $K3$ surface and $(H \cdot H) = 4$, $\operatorname{Pic} X$ is torsion-free and $H$ is not divisible, whence $\mathcal{O}_X(C)$ is a multiple of $\mathcal{O}_X(H)$, which implies that $C$ is a complete intersection of $X$ and a hypersurface of degree $d/4$. It

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remains to show that $d^2 - 8(g-1) > 8$ when $d^2 - 8(g-1) > 0$, and we will treat three cases $d^2 - 8(g-1) = 8, 1, 4$.

*Case* (1) $d^2 - 8(g-1) = 8$: Let $d = 4d'$ ($d' \geqq 1$, $d' \in \boldsymbol{Z}$), then $2(g-1) = 2(2d'^2 - 1)$. Let $E = d'H - C$, then $(E \cdot H) = 0$ and $(E^2) = -2$. Since $X$ is a $K3$ surface, one has

$$h^0(E) + h^0(-E) \geqq \chi(\mathcal{O}(E)) = 2 + (E^2)/2 = 1 .$$

Thus $E$ or $-E$ gives a curve $E'$ such that $(E' \cdot H) = 0$, which contradicts the very ampleness of $H$.

*Case* (2) $d^2 - 8(g-1) = 1$: Let $d = 2d' - 1$ ($d' \geqq 1$, $d \in \boldsymbol{Z}$), then $2(g-1) = (d'^2 - d')$. Let $E = d'H - 2C$, then $(E \cdot H) = 2$, and $(E^2) = 0$.

Thus as in Case (1), $h^0(E) + h^0(-E) \geqq 2$. Since $(E \cdot H) = 2$ for ample $H$, one has $h^0(-E) = 0$ and $h^0(E) \geqq 2$. Thus $|E|$ has an effective member $E_0$. If $E' = (E_0)_{\text{red}}$ is irreducible, then $(E'^2) = 0$ and $(E' \cdot H) = 1, 2$ for very ample $H$. Thus $E' \cong \boldsymbol{P}^1$ and $(E'^2) = -2$, which contradicts $(E'^2) = 0$. Hence $(E_0)_{\text{red}}$ is reducible and by $(E_0 \cdot H) = 2$ for very ample $H$, one has $E_0 = E_1 + E_2$, where $E_1, E_2 \cong \boldsymbol{P}^1$, $\#(E_1 \cap E_2) \leqq 1$, and the intersection of $E_1$ and $E_2$ is transverse. Then $(E_0^2) = -4 + 2(E_1 \cdot E_2) \leqq -2$, which contradicts $(E_0^2) = 0$.

*Case* (3) $d^2 - 8(g-1) = 4$: Let $d = 4d' - 2$ ($d' \geqq 1$, $d' \in \boldsymbol{Z}$), then $2(g-1) = 4(d'^2 - d')$. If $E = d'H - C$, then $(E \cdot H) = 2$ and $(E^2) = 0$. Thus one gets a contradiction as in Case (2). If $d = 5$ and $g = 3$, then $d > 2g - 2$. Thus $h^0(\mathcal{O}_C(1)) = 3$, which implies that $C$ is a plane curve, but this contradicts the genus formula for plane curves. Thus "$\Rightarrow$" is proved.

PROPOSITION 3. *Let $d$ and $g$ be integers such that $0 \leqq g \leqq d - 3$. If* char $k \neq 2$, *then there exist a non-singular Kummer surface $X_0$ and effective divisors $H_0, C_0$ on $X_0$ such that*

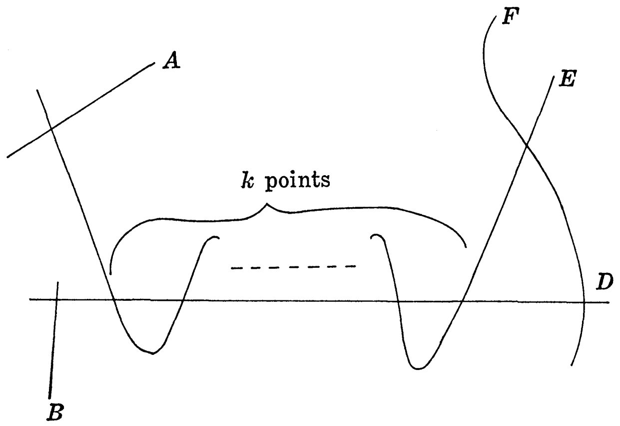
(1) $(H_0^2) = 4$, $(H_0 \cdot C_0) = d$, $(C_0^2) = 2g - 2$,

(2) *$H_0$ is numerically effective,*

(3) *$C_0$ is numerically effective if $g \geqq 2$,*

(4) *$\boldsymbol{Z}H_0 + \boldsymbol{Z}C_0$ is a direct summand of* Pic $X_0$.

*Proof.* Let $k = d - g - 3 \geqq 0$. Let $Y_1$ and $Y_2$ be elliptic curves with an isogeny $f: Y_1 \to Y_2$ of degree $2k + 1$. Let $P, Q \in Y_1$ be non-zero points such that $2P = 0$, $f(2Q) \neq 0$. Let $X_0$ be the non-singular Kummer surface




associated to $Y_1 \times Y_2$. Then $Y_1 \times 0$, $Q \times Y_2$, the graph of $f$, $P \times f(P)$, and $P \times 0$ give irreducible curves $D, F, E, A$, and $B$ in $X_0$ such that $D \cong E \cong A \cong B \cong \boldsymbol{P}^1$, and $F$ is an elliptic curve, with the configuration as in the picture with all the intersections transverse (cf. [MM] or [SI]). Let $H_0 = D + 3F$, and $C_0 = E + gF$. Then (1) is clear; (2) follows from $(H_0 \cdot D) = 1$ and $(H_0 \cdot F) = 1$; (3) follows from $(C_0 \cdot E) = g - 2$ and $(C_0 \cdot F) = 1$; and (4) follows from $(H_0 \cdot B) = 1$, $(H_0 \cdot A) = 0$, $(C_0 \cdot B) = 0$, and $(C_0 \cdot A) = 1$. q.e.d.

*Remark* 4. Let $k$ be the field of complex numbers. Then, in the local versal deformations space Def of $X_0$, the locus where $H_0$ and $C_0$ lift as line bundles is an 18-dimensional smooth subvariety Pol, and there is a dense subset Pol′ of Pol such that if $q \in$ Pol′, then the surface $X$ and line bundles $H$ and $C$ on $X$ lying over $q$ satisfy the conditions:

(1) $(H^2) = 4$, $(H \cdot C) = d$, $(C^2) = 2g - 2$,

(2) $H$ is numerically effective,

(3) $C$ is numerically effective if $g \geqq 2$, and

(4) $\operatorname{Pic} X = ZH + ZC$.

Indeed (1) is clear, whence $X$ is algebraic by [K, Theorem 8], and (4) follows from [K, Theorem 14]. As for (2) and (3), $2H_0$ and $2C_0$ (if $g \geqq 2$) are base point free by (1) of Theorem 5. The obstructions for lifting sections of $\mathcal{O}(2H_0)$ and $\mathcal{O}(2C_0)$ (if $g \geqq 2$) to Pol lie in $H^1(\mathcal{O}(2H_0))$ and $H^1(\mathcal{O}(2C_0))$ which are both 0 by Ramanujam's vanishing theorem.

We now quote results by Saint-Donat:

THEOREM 5 (Saint-Donat [SD] or cf. [MM]). *Let $X$ be a $K3$ surface defined over an algebraically closed field of characteristic $\neq 2$. Let $H$ be* *a numerically effective divisor on X. Then one has*

(1) *H is not base point free if and only if there exist irreducible curves* $E, \Gamma$, *and an integer* $k \geqq 2$ *such that* $H \sim kE + \Gamma$, $(E^2) = 0$, $(\Gamma^2) = -2$, $(E \cdot \Gamma) = 1$. *In this case, every member of* $|H|$ *is of the form* $E' + \Gamma$, *where* $E'$ *is a sum of* $k$ *effective divisors* $E_1, \cdots, E_k$ *such that* $E_i \sim E$ *for all* $i$.

(2) *Let* $(H^2) \geqq 4$. *Then H is very ample if and only if*

(i) *there is no irreducible curve E such that* $(E^2) = 0$, $(E \cdot H) = 1, 2$,

(ii) *there is no irreducible curve E such that* $(E^2) = 2$, $H \sim 2E$, *and*

(iii) *there is no irreducible curve E such that* $(E^2) = -2$, $(E \cdot H) = 0$.

PROPOSITION 6. *Let X, H, C be as in Remark* 4. *Then H is very ample and* $|C|$ *contains an irreducible smooth member.*

*Proof.* We will first check that $H$ satisfies the conditions (i)–(iii) in (2) of Theorem 5. We denote by disc $(A, B)$ the determinant of the intersection matrix of divisors $A$ and $B$. If there is a divisor $E$ such that $(E^2) = -2$, $(E \cdot H) = 0$, then disc $(E, H) = -8$ is divisible by disc $(H, C) = 8(g-1) - d^2$. However, by $g \leqq d - 3$, one has $d^2 \geqq (g+3)^2 > 8g$ and disc $(H, C) < -8$. This is a contradiction. Thus (iii) is checked, (i) is checked in the same way, and (ii) is obvious because $H$ is a part of the basis of Pic $X$. Hence $H$ is very ample. Assume that $g \geqq 2$. Then we use (1) of Theorem 5 to show that $C$ is base point free. If $C$ is not base point free, then there is a divisor $E$ such that $(E^2) = 0$, $(E \cdot C) = 1$. Then disc $(E, C) = -1$ is divisible by disc $(H, C)$, which is a contradiction, as we have seen above. Thus $C$ is base point free and $|C|$ has an irreducible smooth member because $(C^2) > 0$. Let $g = 1$. Then the equation

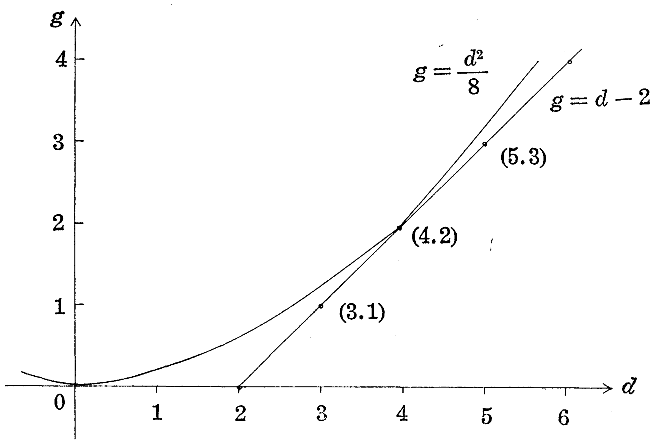
$$(xH + yC)^2 = 2x(2x + dy) = -2$$

does not have integral solutions $x, y$, because $d \geqq g + 3 = 4$.

Hence $X$ does not contain smooth rational curves by Remark 4, (4). By $(C^2) = 0$, $|C|$ or $|-C|$ contains an effective member. By $(C \cdot H) = d > 0$, $|C|$ contains an effective member $C_0$. Thus $C$ is numerically effective because otherwise $C_0$ contains an irreducible curve $Z \cong \boldsymbol{P}^1$, which is a contradiction. Hence by (1) of Theorem 5, $C$ is base point free, and $C$ is a multiple of an elliptic pencil. Since $C$ is a part of the basis of Pic $X$, $|C|$ is an elliptic pencil, and it contains a smooth elliptic curve. Let $g = 0$. Since $(C^2) = -2$ and $(C \cdot H) > 0$, one has $C \sim E + D$, where

$E \cong \boldsymbol{P}^1$ and $D$ is an effecitve divisor. Since disc $(H, C)$ divides disc $(H, E)$, one has $8 + (C \cdot H)^2 \leqq 8 + (E \cdot H)^2$. Thus $(D \cdot H) \leqq 0$, and $D = 0$. Hence $C = E$, and Proposition 6 is proved.

We can now finish the proof of the if-part ($\Leftarrow$) of Theorem 1. We use induction on $d$. We omit the proof for $(d, g) = (1, 0)$, $(2, 0)$, $(3, 1)$, since they are well known. We may assume that $g < d^2/8$, otherwise $C$ is given as a complete intersection. We may also assume that $g \geqq d - 2$ by Remark 4 and Proposition 6. Thus as shown by the picture, one sees




$d \geqq 6$. First, we assume that $(d, g) \neq (9, 10)$. Let $d' = d - 4$, and $g' = g - d + 2$. Then $d'^2 - 8g' = d^2 - 8g > 0$ and $(d', g') \neq (5, 3)$. Thus by the induction hypothesis, there exist a non-singular quartic $X'$ and a non-singular curve $C'$ on it of degree $d'$ and genus $g'$. Let $H'$ be an irreducible hyperplanesection of $X'$, and $C = C' + H'$. Since $d' = d - 4 \geqq 2$, one sees $(C \cdot C') = 2(g' - 1) + d' \geqq 0$, and $C$ is numerically effective. Since $(H'^2) = 4$, $C$ is base point free by (1) of Theorem 5. If we denote by the same $C$, a smooth member of $|C|$, then $C$ has degree $d$ and genus $g$. Thus $C$ and $X$ are the required pair for $d, g$. For $(d, g) = (9, 10)$, let $d' = 1$, $g' = 0$, and $C'$ a straight line on a smooth quartic surface $X'$. Let $H'$ be an irreducible hyperplanesection of $X$ and $C = C' + 2H'$. Then, one sees that $C, X$ are the required pair as in the above argument. This proves Theorem 1.

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