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A DECOMPOSITION THEOREM ON DIFFERENTIAL POLYNOMIALS OF THETA FUNCTIONS

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Let $\tau = (\tau_{ij})$ be a symmetric complex $g \times g$ matrix with the positive definite imaginary part. A theta function of level n means an entire function $f(z)$ in g complex variables $z = (z_1, \cdots, z_g)$ satisfying the difference relations:

$$f(z + \hat{b} + b\tau) = \exp(-\pi n\sqrt{-1}(b\tau{}^t b + 2z{}^t b))f(z)\,, \qquad ((\hat{b}, b) \in Z^g \times Z^g)\,.$$

Denoting by $\Theta_0^{(n)}$ the vector space of theta functions of level n, we get the graded algebra of theta functions;

$$\Theta_0 = \sum_{n\geq 1} \Theta_0^{(n)}\,.$$

Theta series

$$\vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z) = \sum_{\ell \in Z^g} \exp\left(\pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau{}^t\left(\ell + \frac{a}{n}\right) + 2z{}^t\left(\ell + \frac{a}{n}\right)\right)\right),$$
$$(a \in Z^g/nZ^g)$$

form a canonical basis of $\Theta_0^{(n)}$, and thus

$$\dim \Theta_0^{(n)} = n^g\,.$$

In the present article we shall prove the following decomposition theorem:

The algebra of differential polynomials of theta functions has a canonical linear basis

$$\left\{\left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z)\,|\,j \in Z^g_{\geq 0},\ a \in Z^g/nZ,\ n \geq 1\right\},$$

i.e. any differential polynomial is uniquely expressed as a linear combination of $(\partial/\partial z)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z)$, $(j \in Z^g_{\geq 0},\ a \in Z^g/nZ^g,\ n \geq 1)$ with constant

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coefficients depending on τ. More precisely we have the explicit expressions of the components of the decomposition.

The key is a very similar idea as making transvectants in the classical invariant theory, however the Lie algebra is Heisenberg Lie algebra instead of $s\ell_2$. The algebra Θ_0 of theta functions is embedded in a graded algebra Θ of auxiliary theta functions in $2g$ complex variables $(u, z) = (u_1, \cdots, u_g, z_1, \cdots, z_g)$ with the following properties,

1° A realization $\langle \mathscr{E}, \mathscr{D}_1, \cdots, \mathscr{D}_g, \Delta_1, \cdots, \Delta_g \rangle$ of Heisenberg Lie algebra acts on Θ as derivations,

2° Θ_0 is the subalgebra consisting of all the elements φ such that $\mathscr{D}_i\varphi = 0$ $(1 \leq i \leq g)$,

3° $\left\{\Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z) \,\middle|\, j \in Z^g_{\geq 0},\ a \in Z^g/nZ^g,\ n \geq 1\right\}$ is a canonical linear basis of Θ,

4° The mapping

$$\Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z) \longrightarrow \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z)\,, \qquad (j \in Z^g_{\geq 0},\ a \in Z^g/nZ^g,\ n \geq 1)$$

induces an algebra isomorphism of Θ onto the algebra of differential polynomials of theta functions.

We shall also characterize differential polynomials of theta functions which are theta functions.

The associative law for the structure constants of

$$C\left[\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau|z), \cdots\right]$$

with respect to the basis must be very important relations between

$$\left\{\left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}\left(\tau\middle|\frac{\hat{a}}{n}\right) \middle| j \in Z^g_{\geq 0};\ a, \hat{a} \in Z^g/nZ^g;\ n \geq 1\right\}.$$

Notations.

$Z_{\geq 0} = \{\text{non-negative integers}\}$, $Z^g_{\geq 0} = \{j = (j_1, \cdots, j_g) | j_i \in Z_{\geq 0}\}$,
$j \pm \varepsilon_i = (j_1, \cdots, j_{i-1}, j_i \pm 1, j_{i+1}, \cdots, j_g)$, $j! = j_1! \cdots j_g!$,
$\binom{j}{p} = \binom{j_1}{p_1} \cdots \binom{j_g}{p_g}$, $\binom{j}{k^{(1)}, \cdots, k^{(r)}} = \binom{j_1}{k_1^{(1)}, \cdots, k_1^{(r)}} \cdots \binom{j_g}{k_g^{(1)}, \cdots, k_g^{(r)}}$,
$|j| = j_1 + \cdots + j_g$, $u = (u_1, \cdots, u_g)$, $z = (z_1, \cdots, z_g)$, $u^j = u_1^{j_1} \cdots u_g^{j_g}$,
$z^j = z_1^{j_1}, \cdots, z_g^{j_g}$,
$\left(\frac{\partial}{\partial u}\right)^j = \left(\frac{\partial}{\partial u_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial u_g}\right)^{j_g}$, $\left(\frac{\partial}{\partial z}\right)^j = \left(\frac{\partial}{\partial z_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial z_g}\right)^{j_g}$,

$$\left(2\pi n\sqrt{-1}u + \frac{\partial}{\partial u}\right)^j = \left(2\pi n\sqrt{-1}u_1 + \frac{\partial}{\partial z_1}\right)^{j_1} \cdots \left(2\pi n\sqrt{-1}u_g + \frac{\partial}{\partial z_g}\right)^{j_g}.$$

§1. Auxiliary theta functions

1.1. An auxiliary theta function of level n means a function $\varphi(u, z)$ in $2g$ complex variables $(u, z) = (u_1, \cdots, u_g, z_1, \cdots, z_g)$ such that

1° $\varphi(u, z)$ is a polynomial in $u = (u_1, \cdots, u_g)$ whose coefficients are entire functions in $z = (z_1, \cdots, z_g)$,

2° $\varphi(u + b, z + \hat{b} + b\tau) = \exp(-\pi n\sqrt{-1}(b\tau^t b + 2z^t b))\varphi(u, z)$, $((\hat{b}, b) \in Z^g \times Z^g)$.

Denoting by $\Theta^{(n)}$ the vector space of auxiliary theta functions of level n, we obtain a graded algebra

$$\Theta = \sum_{n\geq 1} \Theta^{(n)}$$

of auxiliary theta functions, which contains the graded algebra Θ_0 of theta functions as the subalgebra of polynomials of degree zero in u. Auxiliary theta series are also defined as follows,

$$\begin{aligned} &\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z) \\ (1.1)\qquad &= (2\pi n\sqrt{-1})^{|j|} \sum_{\ell \in Z^g} \left(u + \ell + \frac{a}{n}\right)^j \\ &\cdot \exp \pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau^t\left(\ell + \frac{a}{n}\right) + 2z^t\left(\ell + \frac{a}{n}\right)\right) \\ &(j \in Z^g_{\geq 0},\ a \in Z^g/nZ^g,\ n \geq 1). \end{aligned}$$

LEMMA 1.1.

$$(1.2)\qquad \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z) = \left(2\pi n\sqrt{-1}u + \frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z),$$

$$\begin{aligned} &\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u + b, z + \hat{b} + b\tau) \\ (1.3)\qquad &= \exp(-\pi n\sqrt{-1}(b\tau^t b + 2z^t b))\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z) \\ &(\hat{b}, b) \in Z^g \times Z^g). \end{aligned}$$

Proof. For a, b, $\hat{b}$ in Z^g we have

$$\left(2\pi n\sqrt{-1}\,u + \frac{\partial}{\partial z}\right)^{j} \exp\left(\pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau^{t}\left(\ell + \frac{a}{n}\right) + 2z^{t}\left(\ell + \frac{a}{n}\right)\right)\right)$$

$$= (2\pi n\sqrt{-1})^{|j|}\left(u + \ell + \frac{a}{n}\right)^{j}$$

$$\exp\left(\pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau^{t}\left(\ell + \frac{a}{n}\right) + 2z\left(\ell + \frac{a}{n}\right)\right)\right),$$

$$\left(u + \ell + b + \frac{a}{n}\right)^{j}$$

$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(\ell + b + \frac{a}{n}\right)\tau^{t}\left(\ell + b + \frac{a}{b}\right) + 2z^{t}\left(\ell + b + \frac{a}{n}\right)\right)\right)$$

$$= \exp\left(\pi n\sqrt{-1}(b\tau^{t}b + 2z^{t}b)\left(u + \ell + b + \frac{a}{n}\right)^{j}\right.$$

$$\cdot \exp\left(\pi n\sqrt{-1}\left(\left(\ell + \frac{a}{n}\right)\tau^{t}\left(\ell + \frac{a}{n}\right)\tau^{t}\left(\ell + \frac{a}{n}\right)\right.\right.$$

$$\left.\left. + 2(z + \hat{b} + b\tau)\left(\ell + \frac{a}{n}\right)\right)\right).$$

Hence, making the sum with respect to $\ell \in Z^{g}$, we obtain (1.2), (1.3).

THEOREM 1.1. *$\left\{\vartheta_{j}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z) \mid j \in Z^{g}_{\geq 0},\ a \in Z^{g}/nZ^{g}\right\}$ is a basis of the space Θ^{n} of auxiliary theta functions of level n.*

Proof. By virtue of Lemma 1.1 $\vartheta_{j}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z)$ $(j \in Z^{g}_{\geq 0},\ a \in Z^{g}/nZ^{g})$ belong to $\Theta^{(n)}$, and obviously they are linearly independent. Let $\varphi(u, z) = \sum_{j} u^{j} f_{j}(z)$ be an element of $\Theta^{(n)}$, and let $u^{k} f_{k}(z)$ be one of terms with maximal degree k in u. Then, comparing the coefficients of u^{k} in the both sides of

$$\sum_{j} (u + b)^{j} f_{j}(z + \hat{b} + b\tau) = \exp(-\pi n\sqrt{-1}(b\tau^{t}b + 2z^{t}b)) \sum_{j} u^{j} f_{j}(z),$$

we have

$$f_{k}(z + \hat{b} + b\tau) = \exp(-\pi n\sqrt{-1}(b\tau^{t}b + 2z^{t}b)) f_{k}(z).$$

This means that there exists a system $(\alpha_{a})_{a \in Z^{g}/nZ^{g}}$ of constants such that

$$f_{k}(z) = \sum_{a} \alpha_{a} \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z),$$

and thus

$$\varphi(u, z) - \sum_a \alpha_a \vartheta_k^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z)$$

is an element in $\Theta^{(n)}$ without u^k-term and all the new terms are of lower degree than k in u. Proceeding this process successively, we can express $\varphi(u, z)$ as a linear sum of $\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z)$ $(j \in Z^g_{\geq 0},\ a \in Z^g/nZ^g)$.

1.2. Denoting the projection operators by

$$\sigma^{(n)}\colon \Theta \longrightarrow \Theta^{(n)}\ , \qquad (n \geq 1)$$

we define differential operators

$$\begin{aligned} \mathscr{E} &= \sum_{n\geq 1} n\sigma^{(n)}\ , \\ \mathscr{D}_i &= \sum_{n\geq 1} \frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial u_i} \circ \sigma^{(n)}\ , \\ \Delta_i &= \sum_{n\geq 1} \left(2\pi n\sqrt{-1}\, u_i + \frac{\partial}{\partial z_i}\right) \circ \sigma^{(n)}\ , \\ \mathscr{D}^j &= \mathscr{D}_1^{j_1} \cdots \mathscr{D}_g^{j_g}\ , \qquad \Delta_1^{j_1} \cdots \Delta_g^{j_g}\ . \end{aligned}$$

PROPOSITION 1.1.

$$\mathscr{D}_i \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z) = n j_i \vartheta_{j-\varepsilon_i}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z)\ , \tag{1.4}$$

$$\Delta_i \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z) = \vartheta_{j+\varepsilon_i}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z)\ , \tag{1.5}$$

$$\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z) = \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z)\ , \tag{1.6}$$

$$\frac{1}{p!}\mathscr{D}^p \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z) = \binom{j}{p} n^{|p|} \vartheta_{j-p}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z)\ , \tag{1.7}$$

$$\frac{1}{j!}\mathscr{D}^j \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z) = n^{|j|} \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z) \tag{1.8}$$

$$(j, p \in Z^g_{\geq 0},\ j \geq p,\ a \in Z^g/nZ^g,\ n \geq 1).$$

Proof. From the expression

$$\vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z) = \left(2\pi n\sqrt{-1}\, u + \frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z)$$

it follows (1.4), (1.5), (1.6). Applying (1.4) and (1.5) successively, we have (1.7), (1.8).

PROPOSITION 1.2. *$\mathscr{E}, \mathscr{D}_1, \cdots, \mathscr{D}_g, \Delta_1, \cdots, \Delta_g$ are derivations of Θ such that*

$$(1.9)\qquad [\mathscr{E}, \mathscr{D}_i] = [\mathscr{E}, \Delta_i] = [\mathscr{D}_i, \mathscr{D}_j] = [\Delta_i, \Delta_j] = 0\,,$$
$$[\mathscr{D}_i, \Delta_{i'}] = \begin{cases} \mathscr{E} & (i = i') \\ 0 & (i \neq i') \end{cases} \qquad (1 \leq i,\ i',\ j \leq g)\,.$$

Proof. By virtue of Proposition 1.2 $\mathscr{E}, \mathscr{D}_1, \cdots, \mathscr{D}_g, \Delta_1, \cdots, \Delta_g$, map Θ into itself. Since $\Theta = \sum_{n\geq 1} \Theta^{(n)}$ is a graded algebra, $\mathscr{E}, \mathscr{D}_1, \cdots, \mathscr{D}_g, \Delta_1, \cdots, \Delta_g$ are derivations of Θ. By simple calculation we have (1.9).

Proposition 1.2 states $\langle \mathscr{E}, \mathscr{D}_1, \cdots, \mathscr{D}_g, \Delta_1, \cdots, \Delta_g \rangle$ is a realization of Heisenberg Lie algebra acting on Θ as derivations.

PROPOSITION 1.3. *The graded algebra of theta functions is the subalgebra consisting of all the elements φ such that $\mathscr{D}_i\varphi = 0$ $(1 \leq i \leq g)$.*

Proof. Each ϕ in Θ_0 contains no u_i and

$$\mathscr{D}_i = \sum_{n\geq 1} \frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial u_i} \circ \sigma^{(n)} \qquad (1 \leq i \leq g)\,,$$

hence we have $\mathscr{D}_i\varphi = 0$ $(1 \leq i \leq g)$. Conversely, assume

$$\mathscr{D}_i\Big(\sum \alpha_{j,a/n,n} \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z)\Big) = 0 \qquad (1 \leq i \leq g)\,.$$

Then it follows

$$\sum nj_i \alpha_{j,a/n,n} \vartheta_{j-\varepsilon_i}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z) = 0 \qquad (1 \leq i \leq g)\,.$$

This means $\alpha_{j,a/n,n} = 0$ for $j \neq 0$.

§ 2. Projection operators

2.1. In order to express the projection operators

$$\sigma_j^{(n)}\colon \Theta \longrightarrow \Delta^j\Theta_0^{(n)} \qquad (j \in Z_{\geq 0}^g,\ n \geq 1)\,,$$

we need a lemma.

LEMMA 2.1.

$$(2.1)\quad \Big(\sum_{p\leq k} \frac{(-1)^{|p|}}{p!} n^{-|p|} \Delta^p \mathscr{D}^p\Big) \vartheta_k^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,u, z) = \begin{cases} \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z) & (k = 0) \\ 0 & (k \neq 0) \end{cases},$$

$$\left(\Delta^j\left(\sum_p \frac{(-1)^{|p|}}{p!} n^{-|p|}\Delta^p \mathscr{D}^p\right)\frac{1}{j!} n^{-|j|}\mathscr{D}^j\right)\vartheta_k^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z)$$

$$(2.2) \qquad = \begin{cases} \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z) & (k = j) \\ 0 & (k \neq j) \end{cases},$$

$$(j, k \in Z^g_{\geq 0},\ a \in Z^g/nZ^g,\ n \geq 1)\,.$$

Proof. From (1.4), (1.5), (1.6), (1.7) it follows

$$\left(\sum_p \frac{(-1)^{|p|}}{p!} n^{-|p|}\Delta^p \mathscr{D}^p\right)\vartheta_k^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z)$$

$$= \sum_{p \leq k} (-1)^{|p|}\binom{k}{p}\Delta^p \vartheta_{k-p}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z)$$

$$= \left(\sum_{p \leq k} (-1)^{|p|}\binom{k}{p}\right)\cdot \vartheta_k^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z)$$

$$= \begin{cases} \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z) & (k = 0) \\ 0 & (k \neq 0) \end{cases},$$

$$\left(\Delta^j\left(\sum_p \frac{(-1)^{|p|}}{p!} n^{-|p|}\Delta^p \mathscr{D}^p\right)\frac{1}{j!} n^{-|j|}\mathscr{D}^j\right)\vartheta_k^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z)$$

$$= \Delta^j\left(\sum_p \frac{(-1)^{|p|}}{p!} n^{-|p|}\Delta^p \mathscr{D}^p\right)\binom{k}{j}\vartheta_{k-j}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z)$$

$$= \binom{k}{j}\Delta^j\left(\sum_p \frac{(-1)^{|p|}}{p!} n^{-|p|}\Delta^p \mathscr{D}^p\right)\vartheta_{k-j}^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z)$$

$$= \begin{cases} \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z) = \vartheta_j^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid u, z) & (j = k) \\ 0 & (j \neq k) \end{cases}$$

THEOREM 2.1. *Θ has the direct sum decomposition*

$$(2.3) \qquad \Theta = \sum_{i \in Z^g_{\geq 0}} \Delta^j \Theta_0 = \sum_{n \geq 1} \sum_{j \in Z^g_{\geq 0}} \Delta^j \Theta_0^{(n)}$$

such that Δ^j induces a vector space isomorphism of $\Theta_0^{(n)}$ onto $\Delta^j\Theta_0^{(n)}$. The projection operators

$$\sigma_j^{(n)}:\ \Theta \longrightarrow \Delta^j \Theta_0^{(n)}$$

are given by

$$(2.4) \qquad \sigma_j^{(n)} = \Delta^j\left(\sum_p \frac{(-1)^{|p|}}{p!} n^{-|p|}\Delta^p \mathscr{D}^p\right)\frac{1}{j!} n^{-|j|}\mathscr{D}^j \circ \sigma^{(n)}$$

$$(j \in Z^g_{\geq 0},\ n \geq 1)\,.$$

Proof. The first part of the assertion is a direct consequence of the fact: $\left\{\vartheta_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,u, z)\,|\,j\in Z^g_{\geq 0},\ a\in Z^g/nZ^g,\ n\geq 1\right\}$, $\left\{\vartheta_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,u, z)\,|\,a\in Z^g/nZ^g\right\}$ and $\left\{\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,z)\,|\,a\in Z^g/nZ^g\right\}$ are the basis of Θ, $\Delta^j\Theta_0^{(n)}$ and $\Theta_0^{(n)}$, respectively. The expression (2.4) is a direct consequence of (2.2).

COROLLARY. *The inverse mapping of* Δ^j: $\Theta_0^{(n)}\to\Delta^j\Theta_0^{(n)}$ *is given by*

$$\left(\sum_p\frac{(-1)^{|p|}}{p!}n^{-|p|}\Delta^p\mathscr{D}^p\right)\frac{1}{j!}n^{-|j|}\mathscr{D}^j\qquad(j\in Z^g_{\geq 0},\ n\geq 1)\,. \tag{2.5}$$

Proof. Since the mapping

$$\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,z)\longrightarrow\Delta^j\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,z)=\vartheta_j^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,u, z)$$

is a bijection, (2.4) implies (2.5).

§ 3. Decomposition theorem on differential polynomials of theta functions

3.1. First let us prove the algebra isomorphic theorem:

THEOREM 3.1. *The replacement*

$$\Delta^j\varphi(z)\longrightarrow\left(\frac{\partial}{\partial z}\right)^j\varphi(z)\qquad(j\in Z^g_{\geq 0},\ \varphi\in\Theta_0)$$

induces a Θ_0*-algebra isomorphism of* Θ *onto the algebra*

$$C\left[\cdots,\left(\frac{\partial}{\partial z}\right)^j\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,z),\cdots\right]$$

of differential polynomials of theta functions, namely

1° $G\left(\cdots,\Delta^j\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,z),\cdots\right)=0$,
if and only if $G\left(\cdots,\left(\frac{\partial}{\partial z}\right)^j\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,z),\cdots\right)=0$,

2° $G\left(\cdots,\Delta^j\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,z),\cdots\right)=G\left(\cdots,\left(\frac{\partial}{\partial z}\right)^j\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,z),\cdots\right)$
if and only if $G\left(\cdots,\left(\frac{\partial}{\partial z}\right)^j\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,z),\cdots\right)\in\Theta_0$.

Proof. It is enough to assume $G\left(\cdots,\Delta^j\vartheta^{(n)}\begin{bmatrix}a/n\\0\end{bmatrix}(\tau\,|\,u, z),\cdots\right)$ belongs to $\Theta^{(m)}$ with some m. If $G\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z), \cdots\right) = 0$, then putting $u = 0$, we obtain $G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z), \cdots\right) = 0$. By virtue of the direct decomposition theorem we may put

$$G\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z), \cdots\right) = \sum_h \Delta^h \phi_h(z)$$

with $\phi_h \in \Theta_0^{(m)}$. If we assume $G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z), \cdots\right) = 0$, then we have

$$\begin{aligned} \sum_h \left(\frac{\partial}{\partial z}\right)^h \phi_h(z) &= G\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z), \cdots\right)_{|u=0} \\ &= G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z), \cdots\right) = 0\,. \end{aligned}$$

Therefore it is enough to show $\phi_h(z) = 0$ under the condition

$$\sum_h \left(\frac{\partial}{\partial z}\right)^h \phi_h(z) = 0 \quad \text{and} \quad \phi_h(z) \in \Theta_0^{(m)}\,.$$

For each $b \in Z^g$ it follows

$$\begin{aligned} &\phi_h(z + b\tau) = \exp(-\pi m\sqrt{-1}(b\tau{}^t b + 2z{}^t b))\phi_h(z)\,, \\ &\sum_h \left(\frac{\partial}{\partial z}\right)^h \phi_h(z + b\tau) = \sum_h \left(\frac{\partial}{\partial z}\right)^h (\exp(-\pi m\sqrt{-1}(b\tau{}^t b + 2z{}^t b))\phi_h(z)) \\ &\qquad = \exp(-\pi m\sqrt{-1}(b\tau{}^t b + 2z{}^t b)) \sum_h \sum_p \binom{h}{p} \\ &\qquad\quad \cdot (-2\pi m\sqrt{-\,b})^p \left(\frac{\partial}{\partial z}\right)^{h-p} \phi^h(z) \qquad (b \in Z^g)\,, \end{aligned}$$

and thus

$$(*) \qquad \sum_h \sum_p \binom{h}{p}(-2\pi m\sqrt{-1}b)^p \left(\frac{\partial}{\partial z}\right)^{h-p} \phi_h(z) = 0 \qquad (b \in Z^g)\,.$$

Let h_0 be one of maximal h in the above sum. Then, the coefficients of b^{h_0} in the polynomial relation (*) in b is given by $(-2\pi m\sqrt{-1})^{|h_0|}\phi_{h_0}(z)$, hence we may conclude $\phi_{h_0}(z) = 0$. Proceeding this process successively we have $\phi_h(z) = 0$, i.e. $G\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z), \cdots\right) = 0$. Since $G\Big(\cdots,$ $\Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau\,|\,z), \cdots\Big)$ belongs to $\Theta^{(m)}$, assuming

$$G\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) = G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right),$$

we have

$$\begin{aligned} &G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right)_{\mid z \to z+\hat{b}+b\tau} \\ &\quad = G\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right)_{\mid (u,z) \to (u+b, z+\hat{b}+b\tau)} \\ &\quad = \exp(-\pi m\sqrt{-1}(b\tau^t b + 2z^t b))\, G\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) \\ &\quad = \exp(-\pi m\sqrt{-1}(b\tau^t b + 2z^t b))\, G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) \end{aligned}$$

i.e.

$$G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) \in \Theta_0^{(m)} .$$

Conversely, if

$$G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) \in \Theta_0^{(m)} ,$$

then applying 1° for

$$\begin{aligned} &F\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) \\ &\quad = G\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) - G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) \end{aligned}$$

we obtain

$$F\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) = 0 ,$$

i.e.

$$G\left(\cdots, \Delta^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right) = G\left(\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ n \end{bmatrix}(\tau \mid z), \cdots\right).$$

Combining Theorem 2.1 and Theorem 3.1 we obtain the decomposition theorem.

THEOREM 3.2. *The algebra* $C\left[\cdots, \left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z), \cdots\right]$ *of differential polynomials of theta functions has a canonical linear basis*

$$\left\{\left(\frac{\partial}{\partial z}\right)^j \vartheta^{(n)}\begin{bmatrix} a/n \\ 0 \end{bmatrix}(\tau \mid z) \mid j \in Z^g_{\geq 0},\ a \in Z^g/nZ^g,\ n \geq 1\right\}, \tag{3.1}$$

namely differential polynomials of theta functions are uniquely expressed as linear combinations of (3.1) *with constant coefficients depending on* τ.

3.2. In order to express the decomposition of differential polynomials of theta functions explicitly, we introduce differential polynomials in $Y_1, \cdots, Y_r$

$$\begin{aligned} &F^{(n_1,\cdots,n_r)}_{j^{(1)},\cdots,j^{(r)};h}(Y_1, \cdots, Y_r \mid z) \\ &= \frac{1}{h!(n_1 + \cdots + n_r)^{|h|}} \sum_p \frac{(-1)^{|p|}}{p!}\left(\frac{1}{n_1 + \cdots + n_r}\frac{\partial}{\partial z}\right)^p \\ &\cdot \left\{\sum_{\substack{k^{(1)}+\cdots+k^{(r)}=p+h \\ k^{(\alpha)}\leq j^{(\alpha)}}} \binom{p+h}{k^{(1)}, \cdots, k^{(r)}} \frac{1}{(j^{(1)} - k^{(1)})!} \cdots \frac{1}{(j^{(1)} - k^{(r)})!}\right. \\ &\left.\cdot \left(\frac{1}{n_1}\frac{\partial}{\partial z}\right)^{j^{(1)}-k^{(1)}} Y_1 \cdots \left(\frac{1}{n_r}\frac{\partial}{\partial z}\right)^{j^{(r)}-k^{(r)}} Y_r\right\} \\ &\qquad (j^{(1)}, \cdots, j^{(r)},\ h \in Z^g_{\geq 0},\ n_1, \cdots, n_r \geq 1)\,. \end{aligned} \tag{3.2}$$

THEOREM 3.3. *For theta functions* $\varphi_\alpha(z) \in \Theta^{(n_\alpha)}_0$ $(1 \leq \alpha \leq r)$ $F^{(n_1,\cdots,n_r)}_{j^{(1)},\cdots,j^{(r)};h} \times (\varphi_1, \cdots, \varphi_r \mid z)$, $(j^{(1)}, \cdots, j^{(r)}, h \in Z^g_{\geq 0})$ *are theta functions of level* $n_1 + \cdots + n_r$ *such that*

$$\begin{aligned} &\frac{1}{j^{(1)}! \cdots j^{(r)}!}\left(\frac{1}{n_1}\frac{\partial}{\partial z}\right)^{j^{(1)}} \varphi_1(z) \cdots \left(\frac{1}{n_r}\frac{\partial}{\partial z}\right)^{j^{(r)}} \varphi_r(z) \\ &= \sum_{h \leq j^{(1)}+\cdots+j^{(r)}} \left(\frac{\partial}{\partial z}\right)^h F^{(n_1,\cdots,n_r)}_{j^{(1)},\cdots,j^{(r)};h}(\varphi_1, \cdots, \varphi_r \mid z) \\ &= \sum_c \lambda_{(j^{(1)},\cdots,j^{(r)};h),c/(n_1+\cdots+n_r)}(\varphi_1, \cdots, \varphi_r)\left(\frac{\partial}{\partial z}\right)^h \vartheta^{(n_1+\cdots+n_r)} \\ &\cdot \begin{bmatrix} c/(n_1 + \cdots + n_r) \\ 0 \end{bmatrix}(\tau \mid z)\,, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &\lambda_{(j^{(1)},\cdots,j^{(r)};h),c/(n_1+\cdots+n_r)}(\varphi_1, \cdots, \varphi_r) \\ &= \frac{1}{(n_1 + \cdots + n_r)^g} \sum_{\hat{c} \in Z^g/(n_1+\cdots+n_r)Z^g} \exp \frac{2\pi\sqrt{-1}\hat{c}^t c}{n_1 + \cdots + n_r} \\ &\cdot \vartheta^{(n_1+\cdots+n_r)}\begin{bmatrix} c/(n_1 + \cdots + n_r) \\ 0 \end{bmatrix}(\tau \mid 0)^{-1} F^{(n_1,\cdots,n_r)}_{j^{(1)},\cdots(r);h} \\ &\cdot \left(\varphi_1, \cdots, \varphi_r \,\middle|\, \frac{\hat{c}}{n_1 + \cdots + n_r}\right). \end{aligned} \tag{3.4}$$

Proof. Putting

$$\frac{1}{j^{(1)}!\cdots j^{(r)}!}\left(\frac{1}{n_1}\Delta\right)^{j^{(1)}}\varphi_1(z)\cdots\left(\frac{1}{n_r}\Delta\right)^{j^{(r)}}\varphi_r(z)$$
$$=\sum \Delta^h\psi_h(z)$$

with $\psi_h(z)\in\Theta_0^{(n_1\cdots+n_r)}$, by virtue of Corollary of Theorem 2.1 (1.6) and (1.7) we have

$$\begin{aligned}\psi_h(z)&=\frac{1}{h!\,(n_1+\cdots+n_r)^{|h|}}\sum_p\frac{(-1)^{|p|}}{p!}\left(\frac{1}{n_1+\cdots+n_r}\Delta\right)^p\mathscr{D}^{p+h}\\&\quad\cdot\frac{1}{j^{(1)}!\cdots j^{(r)}!}\left(\frac{1}{n_1}\Delta\right)^{j^{(1)}}\varphi_1(z)\cdots\left(\frac{1}{n_r}\Delta\right)^{j^{(r)}}\varphi_r(z)\\&=\frac{1}{h!(n_1+\cdots+n_r)^{|h|}}\sum_p\frac{(-1)^{|p|}}{p!}\left(\frac{1}{n_1+\cdots+n_r}\Delta\right)^p\\&\left\{\sum_{k^{(1)}+\cdots+k^{(r)}=p+h}\begin{bmatrix}p+h\\k^{(1)},\cdots,k^{(r)}\end{bmatrix}\frac{n_1^{|k^{(1)}|}\cdots n_r^{|k^{(r)}|}}{j^{(1)}!\cdots j^{(r)}!\,n_1^{|j^{(1)}|}\cdots n_r^{|j^{(r)}|}}\right.\\&\qquad\left.\cdot n^{-|k^{(1)}|}\mathscr{D}^{k^{(1)}}\Delta^{j^{(1)}}\varphi_1(z)\cdots n^{-|k^{(r)}|}\mathscr{D}^{k^{(r)}}\Delta^{j^{(r)}}\varphi_r(z)\right\}\\&=\frac{1}{h!\,(n_1+\cdots+n_r)^{|h|}}\sum_p\frac{(-1)^{|p|}}{p!}\left(\frac{1}{n_1+\cdots+n_r}\Delta\right)^p\\&\left\{\sum_{\substack{k^{(1)}+\cdots+k^{(r)}=p+h\\k^{(\alpha)}\le j^{(\alpha)}}}\begin{bmatrix}p+h\\k^{(1)},\cdots,k^{(r)}\end{bmatrix}\frac{1}{(j^{(1)}-k^{(1)})!\cdots(j^{(r)}-k^{(r)})!}\right.\\&\qquad\left.\cdot\left(\frac{1}{n_1}\Delta\right)^{j^{(1)}-k^{(1)}}_{\varphi_1(z)}\cdots\left(\frac{1}{n_r}\Delta\right)^{j^{(r)}-k^{(r)}}_{\varphi_r(z)}\right\}\\&=F^{(n_1,\cdots,n_r)}_{j^{(1)},\cdots,j^{(r)};\,h}(\varphi_1,\cdots,\varphi_r\,|\,z)\,.\end{aligned}$$

Hence, replacing Δ_i by $\partial/\partial z_i$ $(1\le i\le g)$, we prove the first assertion of Theorem 3.3. Putting

$$\begin{aligned}&F^{(n_1,\cdots n_r)}_{j^{(1)},\cdots,j^{(r)};\,h}(\varphi_1,\cdots,\varphi_r\,|\,z)\\&=\sum_{c\in Z^g/(n_1+\cdots+n_r)Z^g}\lambda_{h,c}\vartheta^{(n_1+\cdots+n_r)}\begin{bmatrix}c/(n_1+\cdots+n_r)\\0\end{bmatrix}(\tau\,|\,z)\,,\end{aligned}$$

we have

$$\begin{aligned}&F^{(n_1,\cdots,n_r)}_{j^{(1)},\cdots,j^{(r)};\,h}\left(\varphi_1,\cdots,\varphi_r\,\middle|\,\frac{\hat{c}}{n_1+\cdots+n_r}\right)\\&=\sum_c\lambda_{h,c}\vartheta^{(n_1+\cdots+n_r)}\begin{bmatrix}c/(n_1+\cdots+n_r)\\0\end{bmatrix}\left(\tau\,\middle|\,\frac{\hat{c}}{n_1+\cdots+n_r}\right)\\&=\sum_c\lambda_{h,c}\exp\left(\frac{2\pi\sqrt{-1}\hat{c}^tc}{n_1+\cdots+n_r}\right)\vartheta^{(n_1+\cdots+n_r)}\begin{bmatrix}c/(n_1+\cdots+n_r)\\0\end{bmatrix}(\tau\,|\,0)\\&\qquad(c\in Z^g/(n_1+\cdots+n_r)Z^g)\,.\end{aligned}$$

Hence, by virtue of the orthogonal relation for characters

$$\sum_c \exp\left(\frac{2\pi\sqrt{-1}\hat{c}^t c}{n_1+\cdots+n_r}\right) = \begin{cases}(n_1+\cdots+n_r)^g & \hat{c}\equiv 0 \mod (n_1+\cdots+n_r)\\ 0 & \hat{c}\not\equiv 0 \mod (n_1+\cdots+n_r)\,,\end{cases}$$

it follows

$$\lambda_{h,c} = \frac{1}{(n_1+\cdots+n_r)^g}\sum_{\hat{c}}\exp\left(\frac{-2\pi\sqrt{-1}\hat{c}^t c}{n_1+\cdots+n_r}\right)\vartheta^{(n_1+\cdots+n_r)}$$
$$\cdot\begin{bmatrix}c/(n_1+\cdots+n_r)\\ 0\end{bmatrix}(\tau\,|\,0)^{-1}F^{(n_1;\cdots;n_r)}_{j^{(1)},\ldots,j^{(r)};\,h}\left(\varphi_1,\cdots,\varphi_r\,\middle|\,\frac{\hat{c}}{n_1+\cdots+n_r}\right).$$

Specializing

$$(\varphi_1(z),\varphi_2(z)) \quad\text{to}\quad \left(\vartheta^{(n_1)}\begin{bmatrix}a_1/n_1\\ 0\end{bmatrix}(\tau\,|\,z),\ \vartheta^{(n_r)}\begin{bmatrix}a_r/n_r\\ 0\end{bmatrix}(\tau\,|\,z)\right),$$

we obtain the explicit expression of structure constants of

$$C\left[\cdots,\left(\frac{\partial}{\partial z}\right)^j\vartheta^{(n)}\begin{bmatrix}a/n\\ 0\end{bmatrix}(\tau\,|\,z),\cdots\right]$$

with respect to the basis

$$\left\{\left(\frac{\partial}{\partial z}\right)^j\vartheta^{(n)}\begin{bmatrix}a/n\\ 0\end{bmatrix}(\tau\,|\,z)\right\}.$$

Theorem 3.4. *The structure constants of*

$$C\left[\cdots,\left(\frac{\partial}{\partial z}\right)^j\vartheta^{(n)}\begin{bmatrix}a/n\\ 0\end{bmatrix}(\tau\,|\,z),\cdots\right]$$

are given by

$$\left(\frac{\partial}{\partial z}\right)^{j^{(1)}}\vartheta^{(n_1)}\begin{bmatrix}a_1/n_1\\ 0\end{bmatrix}(\tau\,|\,z)\left(\frac{\partial}{\partial z}\right)^{j^{(2)}}\vartheta^{(n_2)}\begin{bmatrix}a_2/n_2\\ 0\end{bmatrix}(\tau\,|\,z)$$
$$=\sum_h\sum_c \gamma^{(h,c/(n_1+n_2),n_1+n_2)}_{(j^{(1)},a_1/n_2,n_1),(j^{(2)},a_2/n_2,n_2)}(\tau)\left(\frac{\partial}{\partial z}\right)^h\vartheta^{(n_1+n_2)}\begin{bmatrix}c/(n_1+n_2)\\ 0\end{bmatrix}(\tau\,|\,z)\,,$$

$$\gamma^{(h,c/(n_1+n_2),n_1+n_2)}_{(j^{(1)},a_1n_1,n_1)(j^{(2)},a_2/n_2,n_2)}(\tau)$$
$$(3.5)\qquad = \frac{j^{(1)}!\,j^{(2)}!\,n_1^{|j^{(1)}|}n_2^{|j^{(2)}|}}{h!\,(n_1+n_2)^{g+|h|}}\sum_{\hat{c}\in Z^g/(n_1+n_2)Z^g}\exp\left(\frac{-2\pi\sqrt{-1}\hat{c}^t c}{n_1+n_2}\right)\vartheta^{(n_1+n_2)}$$
$$\cdot\begin{bmatrix}c/(n_1+n_2)\\ 0\end{bmatrix}(\tau\,|\,0)^{-1}\left[\sum_p\frac{(-1)^{|p|}}{p!}\left(\frac{1}{n_1+n_2}\frac{\partial}{\partial z}\right)^p\sum_{\substack{k^{(1)}+k^{(2)}=p+h\\ k^{(1)}\le j^{(1)},k^{(2)}\le j^{(2)}}}\right.$$
$$\cdot\begin{bmatrix}p+h\\ k^{(1)},k^{(2)}\end{bmatrix}\frac{1}{(j^{(1)}-k^{(1)})!\,(j^{(2)}-k^{(2)})!}\left(\frac{1}{n_1}\frac{\partial}{\partial z}\right)^{j^{(1)}-k^{(1)}}\vartheta^{(n_1)}\begin{bmatrix}a_1/n_1\\ 0\end{bmatrix}$$

$$\cdot\,(\tau\,|\,z)\left(\frac{1}{n_2}\frac{\partial}{\partial z}\right)^{j^{(2)}-k^{(2)}}\vartheta^{(n_2)}\begin{bmatrix}a_2/n_2\\0\end{bmatrix}(\tau\,|\,z)\Bigg]_{z=\hat{c}(n_1+n_2)}.$$

For theta functions $\varphi_\alpha(z)$ $(1 \leq \alpha \leq z)$, if a differential polynomial $G(\cdots, (\partial/\partial z)^j\varphi_\alpha(z), \cdots)$ is a theta function, then by virtue of Theorems 2.1 and 3.1 $G(\cdots, (\partial/\partial z)^j\varphi_\alpha(z), \cdots)$ is itself the Θ_0-component of the decomposition. Hence, Theorem 3.4 implies the following characterization of differential polynomials of $\varphi_\alpha(z)$ $(1 \leq \alpha \leq r)$ which are also theta functions.

THEOREM 3.5. *For theta functions* $\varphi_\alpha(z) \in \Theta_0^{(n_\alpha)}$ *the space*

$$C\left[\cdots, \left(\frac{\partial}{\partial z}\right)^j \varphi_\alpha(z), \cdots\right] \cap \Theta_0^{(m)}$$

is linearly spanned by

$$F^{(n_1,\cdots,n_1,\cdots,n_r\cdots n_r)}_{(j^{(1,1)},\cdots,j^{(1,e_1)},\cdots,j^{(r,1)},\cdots,j^{(r,e_r)};\,0}(\underbrace{\varphi_1, \cdots, \varphi_1}_{e_1}, \cdots, \underbrace{\varphi_r, \cdots, \varphi_r}_{e_r}\,|\,z)$$

$$(\sum_\alpha e_\alpha n_\alpha = m;\ j^{(1,1)}, \cdots, j^{(1,e_1)}, \cdots, j^{(r,1)}, \cdots, {}^{(r,e_r)} \in Z^g_{\geq 0})\,.$$

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