K. Miyake
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A GENERALIZATION OF HILBERT'S THEOREM 94

KATSUYA MIYAKE

§1. Introduction

Let k be an algebraic number field of finite degree. We denote the absolute class field of k by $\tilde{k}$, and the absolute ideal class group of k by $C\ell(k)$.

For an unramified abelian extension K/k, let $P_k(K)$ be the subgroup of $C\ell(k)$ consisting of the all classes the ideals of which become principal in K, and $S_k(K)$ be the subfield of $\tilde{k}$ corresponding to $P_k(K)$ by class field theory. The collection

$$\{S_k(K) \mid K \text{ is an intermediate field of } \tilde{k}/k.\}$$

stands for the solution for the problem on capitulation of ideals of k. Its members seem rather special among intermediate fields of $\tilde{k}/k$, but little is known about their number theoretical characterization.

Our concern in this paper is the degree $[\tilde{k}: S_k(K)]$ which is equal to the order $|P_k(K)|$. The following theorems are classical:

HILBERT'S THEOREM 94. *If K/k is an unramified cyclic extension, then $[K: k]$ divides $|P_k(K)|$.*

THE PRINCIPAL IDEAL THEOREM. *$P_k(\tilde{k}) = C\ell(k)$, $S_k(\tilde{k}) = k$, and $|P_k(\tilde{k})| = [\tilde{k}: k]$.*

This theorem has been generalized as follows (cf. [3, Theorems 5 and 7]):

THEOREM. *Let $\tilde{\tilde{k}}$ be the second class field of k, that is, the absolute class field of $\tilde{k}$. Let φ be an endomorphism of $\mathrm{Gal}(\tilde{\tilde{k}}/k)$, and $K(\varphi)$ be the subfield of $\tilde{\tilde{k}}$ corresponding to the subgroup*

$$\langle g^{-1}\cdot\varphi(g) \mid g \in \mathrm{Gal}(\tilde{\tilde{k}}/k)\rangle \cdot \mathrm{Gal}(\tilde{\tilde{k}}/\tilde{k})\,.$$

Then the degree $[K(\varphi): k]$ divides $|P_k(K(\varphi))|$.

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Though we have not yet obtained the generality including all of these theorems, we give a generalization of the first one in this paper. Let us denote the maximal unramified central extension of K/k by $C(K/k)$. Then its genus field coincides with $\tilde{k}$. Therefore the degree $[C(K/k)\colon S_k(K)]$ is a multiple of $|P_k(K)| = [\tilde{k}\colon S_k(K)]$. We show

THEOREM 1. *The degree* $[K\colon k]$ *divides* $[C(K/k)\colon S_k(K)]$.

COROLLARY. *If* $C(K/k)$ *coincides with the genus field* $\tilde{k}$, *then* $[K\colon k]$ *divides* $[\tilde{k}\colon S_k(K)] = |P_k(K)|$.

It is well known that every central extension of a cyclic extension coincides with its genus field. Therefore the corollary contains Hilbert's Theorem 94 as a special case.

We shall prove a stronger result. For an intermediate field F of K/k, define the subfield $S_F(K)$ of $\tilde{F}$ as above for the unramified abelian extension K/F. Then it is not hard to see that $S_F(K)$ contains $S_k(K)$.

THEOREM 2. *Let* F *be a cyclic extension of* k *of the maximal degree contained in* K. *Then* $[K\colon k]$ *divides*

$$[C(K/k) \cap S_F(K)\cdot K\colon S_k(K)]\ .$$

A number theoretical description of the quotient will be given. (See Theorem 3 in § 2).

As for the proofs, our basis is Artin [1], by which we reduce the things to group theoretic investigation of the transfers of the metabelian group $\mathrm{Gal}\,(\tilde{K}/k)$. The results are then also translated into theorems on the structure of the idele groups in Section 4 by the same way as in [3].

§ 2. The main theorem and its consequences

Let K/k be an unramified abelian extension of algebraic number fields. In addition to the notation given in the preceding section, let $\lambda_{K/k}\colon C\ell(k) \to C\ell(K)$ be the homomorphism induced by lifting ideals of k to the ones of K naturally. Then $P_k(K)$ is the kernel of $\lambda_{K/k}$. We denote the homomorphism of $C\ell(K)$ to $C\ell(k)$ induced from the norm map of K over k by $N_{K/k}\colon C\ell(K) \to C\ell(k)$.

Let F be an abelian extension of k contained in K. The field $S_F(K)$ is the subfield of $\tilde{F}$ corresponding to $P_F(K) = \mathrm{Ker}\,\lambda_{K/F}$ by class field theory. It is obvious by the definition that $N_{F/k}(P_F(K)) \subset P_k(K)$. Therefore we have

PROPOSITION 1. $S_k(K) \subset S_F(K)$.

PROPOSITION 2. *Suppose that F/k is a cyclic extension of the maximal degree contained in K. Then $\lambda_{F/k}(C\ell(k))$ is contained in $N_{K/F}(C\ell(K))$.*

Proof. Let c be an element of $\lambda_{F/k}(C\ell(k))$, and take $a \in C\ell(k)$ so that $c = \lambda_{F/k}(a)$. Then $N_{F/k}(c) = a^{[F:k]}$. By the choice of F, the degree $[F: k]$ coincides with the exponent of the abelian group $C\ell(k)/N_{K/k}(C\ell(K))$ which is isomorphic to $\mathrm{Gal}\,(K/k)$. Therefore $N_{F/k}(c) \in N_{K/k}(C\ell(K))$. Take $b \in C\ell(K)$ so that $N_{F/k}(c) = N_{K/k}(b)$. Then we have $c \cdot N_{K/F}(b)^{-1} \in \mathrm{Ker}\, N_{F/k}$. Since K is contained in $\tilde{k} \cdot F = \tilde{k}$, we see that $N_{K/F}(C\ell(K))$ contains $\mathrm{Ker}\, N_{F/k}$. Therefore $c = \lambda_{F/k}(a)$ belongs to $N_{K/F}(C\ell(K))$. Q.E.D.

If F/k satisfies the condition of the proposition, then $\lambda_{K/k}(C\ell(k))$ is contained in $\lambda_{K/F} \circ N_{K/F}(C\ell(K))$. Therefore it is a subgroup of

$$\{\lambda_{K/F} \circ N_{K/F}(C\ell(K))\}^{\mathrm{Gal}(K/k)} \stackrel{\mathrm{def}}{=} \{c \in \lambda_{K/F} \circ N_{K/F}(C\ell(K)) \mid c^\sigma = c \text{ for } {}^\forall\sigma \in \mathrm{Gal}\,(K/k)\} .$$

We now state our main theorem, the proof of which will be given in the next section.

THEOREM 3. *Let the notation and the assumptions be as above. Suppose that F/k is a cyclic extension of the maximal degree contained in K. Then we have*

$$[C(K/k) \cap S_F(K) \cdot K: S_k(K)] = [K: k] \cdot [\{\lambda_{K/F} \circ N_{K/F}(C\ell(K))\}^{\mathrm{Gal}(K/k)} : \lambda_{K/k}(C\ell(k))] .$$

COROLLARY 1. *Let the situation be as in the theorem. If $C(K/k) \cap S_F(K) \subset \tilde{k}$, then $[K: k]$ divides $|P_k(K)|$.*

Since $|P_k(K)| = [\tilde{k}: S_k(K)]$, this is obvious by the theorem. Theorems 1 and 2 in Section 1 are also immediate consequences of this theorem.

COROLLARY 2. *Suppose that there exist subfields F and F' of K which satisfy the conditions (1)~(3): (1) F/k is a cyclic extension of the maximal degree contained in K; (2) $K = F \cdot F'$ and $F \cap F' = k$; (3) $\tilde{F} \cap \tilde{F}' = \tilde{k}$. Then $[K: k]$ divides $|P_k(K)|$.*

The proof will also be given in the next section.

§ 3. The proof of Theorem 3

Let K, k and F be as in Theorem 3, and put $G = \mathrm{Gal}(\tilde{K}/k)$, $A = \mathrm{Gal}(\tilde{K}/K)$ and $H = \mathrm{Gal}(\tilde{K}/F)$. The commutator group $[G, G]$ of G is equal to $\mathrm{Gal}(\tilde{K}/\tilde{k})$, and contained in A. By the choice of F, we see that G/H is cyclic. Take $\xi \in G$ so that $G = \langle \xi \rangle \cdot H$. Note that $[F\colon k]$ is the exponent of the abelian group $G/A \cong \mathrm{Gal}(K/k)$. It follows from the definition that $\mathrm{Gal}(\tilde{K}/C(K/k))$ is equal to

$$[G, A] = \langle g^{-1}a^{-1}ga \mid g \in G,\ a \in A \rangle .$$

Let $V_{G\to A}\colon G \to A$ and $V_{H\to A}\colon H \to A$ be the transfers of G and H to the abelian subgroup A, respectively. They induce homomorphisms $\bar{V}_{G\to A}\colon G/[G, G] \to A$ and $\bar{V}_{H\to A}\colon H/[H, H] \to A$. The transfer $V_{G\to H}\colon G \to H/[H, H]$ of G to H also induces a homomorphism $\bar{V}_{G\to H}\colon G/[G, G] \to H/[H, H]$. As is well known, we have $V_{G\to A} = \bar{V}_{H\to A} \circ V_{G\to H}$.

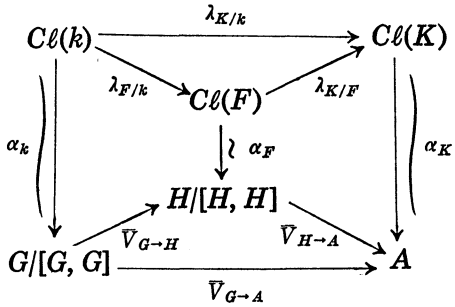
Denote the Artin maps of class field theory for k, F and K by α_k, α_F and α_K, respectively. They are isomorphisms of the following groups:

$$\alpha_k\colon C\ell(k) \xrightarrow{\sim} \mathrm{Gal}(\tilde{k}/k) = G/[G, G] \ ;$$

$$\alpha_F\colon C\ell(F) \xrightarrow{\sim} \mathrm{Gal}(\tilde{F}/F) = H/[H, H] \ ;$$

$$\alpha_K\colon C\ell(K) \xrightarrow{\sim} \mathrm{Gal}(\tilde{K}/K) = A \ .$$

By Artin [1], we have the commutative diagram,



Therefore $\mathrm{Gal}(\tilde{K}/S_k(K)) = \mathrm{Ker}\ V_{G\to A}$ and $\mathrm{Gal}(\tilde{K}/S_F(K)) = \mathrm{Ker}\ V_{H\to A}$. Hence we have

LEMMA 1.

$$[C(K/k) \cap S_F(K)\cdot K\colon S_k(K)] = [\mathrm{Ker}\ V_{G\to A}\colon [G, A]\cdot(A \cap \mathrm{Ker}\ V_{H\to A})] \ .$$

We also have

$$\alpha_K \circ \lambda_{K/k}(C\ell(k)) = V_{G\to A}(G) ,$$

and

$$\alpha_K \circ \lambda_{K/F} \circ N_{K/F}(C\ell(K)) = V_{H\to A}(A)$$

because $\alpha_F \circ N_{K/F} \circ \alpha_K^{-1}(a) = a \bmod [H, H]$ for $a \in A$, as is well known by class field theory. Since A is a normal abelian subgroup of G, the action of G on A through inner automorphisms defines the structure of (G/A)-module on A. Then α_K is a (G/A)-isomorphism. Therefore we have

$$\alpha_K(\{\lambda_{K/F} \circ N_{K/F}(C\ell(K))\}^{\mathrm{Gal}(K/k)}) = V_{H\to A}(A) \cap Z(G)$$

where $Z(G)$ is the center of G, and the following lemma.

LEMMA 2.

$$[\{\lambda_{K/F} \circ N_{K/F}(C\ell(K))\}^{\mathrm{Gal}(K/k)}\colon \lambda_{K/k}(C\ell(k))] = [V_{H\to A}(A) \cap Z(G)\colon V_{G\to A}(G)] .$$

For the completeness, let us show an elementary fact on transfers which we need.

PROPOSITION 3. *Let* $\mathfrak{G}$ *and* $\mathfrak{G}_1$ *be groups in general,* $\mathfrak{H}$ *a subgroup of* $\mathfrak{G}$ *of finite index and* $\varphi\colon \mathfrak{G} \to \mathfrak{G}_1$ *a homomorphism. Suppose that* $\operatorname{Ker}\varphi \subset \mathfrak{H}$. *Then we have* $\bar{\varphi} \circ V_{\mathfrak{G}\to\mathfrak{H}} = V_{\varphi(\mathfrak{G})\to\varphi(\mathfrak{H})} \circ \varphi$ *where* $\bar{\varphi}\colon \mathfrak{H}/[\mathfrak{H}, \mathfrak{H}] \to \varphi(\mathfrak{H})/[\varphi(\mathfrak{H}), \varphi(\mathfrak{H})]$ *is the homomorphism induced by* φ.

Proof. Take a set of representatives $\{R_i \mid i = 1, \cdots, [\mathfrak{G}\colon \mathfrak{H}]\}$ of the cosets of $\mathfrak{G} \bmod \mathfrak{H}$, i.e. $\mathfrak{G} = \bigcup_i \mathfrak{H}\cdot R_i$ (disjoint). Since $\operatorname{Ker}\varphi \subset \mathfrak{H}$, we have $\varphi(\mathfrak{G}) = \bigcup_i \varphi(\mathfrak{H})\cdot\varphi(R_i)$ (disjoint). Furthermore we see that $R_i \cdot G = H_i(G)\cdot R_{i'}$ with $H_i(G) \in \mathfrak{H}$ if and only if $\varphi(R_i)\cdot\varphi(G) = \varphi(H_i(G))\cdot\varphi(R_{i'})$ with $\phi(H_i(G)) \in \varphi(\mathfrak{H})$ for each $G \in \mathfrak{G}$. Then we see the proposition at once by Huppert [2, 1.4, b)].

COROLLARY. *Let* $\mathfrak{G}$ *be a group and* $\mathfrak{A}$ *a normal abelian subgroup of* $\mathfrak{G}$ *of finite index. Then* $V_{\mathfrak{G}\to\mathfrak{A}}(\mathfrak{G}) \subset Z(\mathfrak{G})$.

Proof. For $x \in \mathfrak{G}$, let $\varphi\colon \mathfrak{G} \rightleftarrows \mathfrak{G}$ be the inner automorphism of $\mathfrak{G}$ defined by x. Since $\mathfrak{A}$ is normal in $\mathfrak{G}$ and abelian, we have, for $g \in \mathfrak{G}$, $x^{-1}\cdot V_{\mathfrak{G}\to\mathfrak{A}}(g)\cdot x = \varphi(V_{\mathfrak{G}\to\mathfrak{A}}(g)) = V_{\mathfrak{G}\to\mathfrak{A}}(\varphi(g)) = V_{\mathfrak{G}\to\mathfrak{A}}(x^{-1}\cdot g\cdot x) = V_{\mathfrak{G}\to\mathfrak{A}}(x)^{-1}$. $V_{\mathfrak{G}\to\mathfrak{A}}(g)\cdot V_{\mathfrak{G}\to\mathfrak{A}}(x) = V_{\mathfrak{G}\to\mathfrak{A}}(g)$. This is true for every $x \in \mathfrak{G}$. Therefore, $V_{\mathfrak{G}\to\mathfrak{A}}(g)$ belongs to $Z(\mathfrak{G})$. Q.E.D.

It has already been proved as a part of Lemma 2 that $V_{G\to A}(G)$ lies in $V_{H\to A}(A) \cap Z(G)$. We have just shown group theoretically that $V_{G\to A}(G)$

$\subset Z(G)$. We also give a group theoretic proof to the fact that $V_{G\to A}(G) \subset V_{H\to A}(A)$. This fact corresponds to Proposition 2 in the preceding section.

PROPOSITION 4. *Let $G \supset H \supset A$ be as above. Namely, H and A are normal subgroups of G, A is abelian containing $[G, G]$, and $[G: H]$ coincides with the exponent of the abelian group G/A. Then $V_{G\to A}(G) \subset V_{H\to A}(A)$.*

Proof. For $g \in G$, we have $V_{G\to A}(g) = \overline{V}_{H\to A} \circ V_{G\to H}(g)$. By Huppert [2, IV, 1.7] for example, we easily see that $V_{G\to H}(g) \equiv g^{[G:H]}[H, H] \bmod [G, G]$. Since $[G: H]$ is the exponent of G/A, we see $V_{G\to H}(g) \in A/[H, H]$. Hence we have $V_{G\to A}(g) \in V_{H\to A}(A) = \overline{V}_{H\to A}(A/[H, H])$. Q.E.D.

Let us continue the proof of Theorem 3. Put

$$q = \frac{[\mathrm{Ker}\ V_{G\to A}: [G, A]\cdot(A \cap \mathrm{Ker}\ V_{H\to A})]}{[G: A]\cdot[V_{H\to A}(A) \cap Z(G): V_{G\to A}(G)]}\ .$$

Then by Lemmas 1 and 2, it is sufficient to show that $q = 1$ since $[K: k] = [G: A]$. Multiplying both of the numerator and the denominator of q by $|V_{G\to A}(G)| = [G: \mathrm{Ker}\ V_{G\to A}]$, we have

$$\begin{aligned} q &= \frac{[G: [G, A]\cdot(A \cap \mathrm{Ker}\ V_{H\to A})]}{[G: A]\cdot|V_{H\to A}(A) \cap Z(G)|} \\ &= \frac{[A: [G, A]\cdot(A \cap \mathrm{Ker}\ V_{H\to A})]}{|V_{H\to A}(A) \cap Z(G)|}\ . \end{aligned}$$

Since $G = \langle\xi\rangle\cdot H$ and $V_{H\to A}(A) \subset Z(H)$, we have $V_{H\to A}(A) \cap Z(G) = V_{H\to A}(A) \cap C_A(\xi)$ where $C_A(\xi)$ is the centralizer of ξ in A.

LEMMA 3. *The map φ: $A \to A$ defined by $\varphi(a) = [\xi, a] = \xi^{-1}a^{-1}\xi a$ for $a \in A$ is an endomorphism of A with* $\mathrm{Ker}\,\varphi = C_A(\xi)$.

Proof. For a, $b \in A$, we have

$$\begin{aligned} [\xi, a\cdot b] &= [\xi, b]\cdot[\xi, a]^b \\ &= [\xi, b]\cdot[\xi, a]\cdot[[\xi, a], b]\ . \end{aligned}$$

Since A is normal in G and abelian, we have $[[\xi, a], b] = 1$, and

$$[\xi, a\cdot b] = [\xi, a]\cdot[\xi, b]\ .$$

This shows that φ: $A \to A$ is a well defined homomorphism. It is obvious that $\mathrm{Ker}\,\varphi = C_A(\xi)$. Q.E.D.

LEMMA 4. $[G, A] = [\xi, A]\cdot[H, A] = \varphi(A)\cdot[H, A]$.

Proof. For $x, y \in G$ and $a \in A$, we have

$$\begin{aligned}[x\cdot y, a] &= [x, a]^y\cdot[y, a]\\ &= [x, a]\cdot[[x, a], y]\cdot[y, a]\\ &= [x, a]\cdot[y, [x, a]]^{-1}\cdot[y, a]\\ &= [x, a]\cdot[y, [x, a]^{-1}\cdot a]\end{aligned}$$

because A is normal in G and abelian. Since $G = \langle\xi\rangle\cdot H$, we have the desired result.

Put $\psi = V_{H\to A}|_A\colon A \to A$. Then this is an endomorphism of A with $\operatorname{Ker}\psi = A \cap \operatorname{Ker} V_{H\to A}$. Since $\operatorname{Ker}\psi$ contains $[H, A]$, we have

$$\begin{aligned}q &= \frac{[A\colon \operatorname{Im}\varphi\cdot\operatorname{Ker}\psi]}{|\operatorname{Im}\psi \cap \operatorname{Ker}\varphi|}\\ &= \frac{[A\colon \operatorname{Im}\varphi]}{|\operatorname{Im}\psi \cap \operatorname{Ker}\varphi|\cdot[\operatorname{Im}\varphi\cdot\operatorname{Ker}\psi\colon \operatorname{Im}\varphi]}.\end{aligned}$$

Since $[A\colon \operatorname{Im}\varphi] = |\operatorname{Ker}\varphi|$ and $[\operatorname{Ker}\varphi\colon \operatorname{Im}\psi \cap \operatorname{Ker}\varphi] = [\operatorname{Im}\psi\cdot\operatorname{Ker}\varphi\colon \operatorname{Im}\psi]$, we finally obtain

$$q = \frac{[\operatorname{Im}\psi\cdot\operatorname{Ker}\varphi\colon \operatorname{Im}\psi]}{[\operatorname{Im}\varphi\cdot\operatorname{Ker}\psi\colon \operatorname{Im}\varphi]}.$$

LEMMA 5. *We have $\varphi\circ\psi = \psi\circ\varphi$. Therefore $q = 1$.*

Proof. For $a \in A$, we have $(\varphi\circ\psi)(a) = [\xi, V_{H\to A}(a)] = \xi^{-1}\cdot V_{H\to A}(a^{-1})\cdot\xi\cdot V_{H\to A}(a)$. Since H is a normal subgroup of G, the inner automorphism of G defined by ξ induces an automorphism of H, which maps A onto itself. Therefore we have $\xi^{-1}\cdot V_{H\to A}(a^{-1})\cdot\xi = V_{H\to A}(\xi^{-1}a^{-1}\xi)$ by Proposition 3 for $\mathfrak{G} = H$ and $\mathfrak{H} = A$. Hence we have

$$\begin{aligned}(\varphi\circ\psi)(a) &= V_{H\to A}(\xi^{-1}a^{-1}\xi)\cdot V_{H\to A}(a)\\ &= V_{H\to A}(\xi^{-1}a^{-1}\xi a) = (\psi\circ\varphi)(a).\end{aligned}$$

Thus we have shown that $\varphi\circ\psi = \psi\circ\varphi$.

Now put $B = \operatorname{Im}(\varphi\circ\psi) = \operatorname{Im}(\psi\circ\varphi)$. Then $\varphi(B) \subset B$ and $\psi(B) \subset B$. Therefore φ and ψ induce endomorphisms of $\bar{A} = A/B$, which we denote by $\bar\varphi$ and $\bar\psi$ respectively. Then $\bar\varphi\circ\bar\psi = \bar\psi\circ\bar\varphi =$ trivial. By Herbrand's lemma (see Huppert [2, III, 19.4]), we have

$$[\operatorname{Ker}\bar\varphi\colon \operatorname{Im}\bar\psi] = [\operatorname{Ker}\bar\psi\colon \operatorname{Im}\bar\varphi].$$

Let us show that $\operatorname{Ker}\bar{\varphi} = (\operatorname{Ker}\varphi\cdot\operatorname{Im}\psi)/B$. In fact, suppose that $\varphi(a)\in B$ for $a\in A$. Take $b\in A$ so that $\varphi(a)=\varphi(\psi(b))$. Then $a\cdot\psi(b)^{-1}\in\operatorname{Ker}\varphi$. Therefore $a=(a\cdot\psi(b)^{-1})\cdot\psi(b)\in\operatorname{Ker}\varphi\cdot\operatorname{Im}\psi$. It is obvious that φ maps $\operatorname{Ker}\varphi\cdot\operatorname{Im}\psi$ into B. Thus we have $\operatorname{Ker}\bar{\varphi}=(\operatorname{Ker}\varphi\cdot\operatorname{Im}\psi)/B$. By the same way, we also have $\operatorname{Ker}\bar{\psi}=(\operatorname{Ker}\psi\cdot\operatorname{Im}\varphi)/B$. Since both of $\operatorname{Im}\varphi$ and $\operatorname{Im}\psi$ contain B, we have $q=1$ by the above equality. Q.E.D.

The proof of Theorem 3 is also completed.

Proof of Corollary 2 *to Theorem* 3. Suppose that F and F' are given as in the corollary. Using the same notation as above, we may assume that $H=\operatorname{Gal}(\tilde{K}/F)$ and $\langle\xi\rangle\cdot A=\operatorname{Gal}(\tilde{K}/F')$. Then $[H,H]=\operatorname{Gal}(\tilde{K}/\tilde{F})$ and $[\langle\xi\rangle A,\langle\xi\rangle A]=\operatorname{Gal}(\tilde{K}/\tilde{F}')$. Since A is abelian, we have $[\langle\xi\rangle A,\langle\xi\rangle A]=[\xi,A]$. Therefore $\operatorname{Gal}(\tilde{K}/\tilde{F}\cap\tilde{F}')=[\xi,A]\cdot[H,H]$. By the assumption (3), we have $[G,G]=[\xi,A]\cdot[H,H]$. Since $\operatorname{Ker}V_{H\to A}$ contains $[H,H]$, we see $[G,G]$ lie in $[G,A]\cdot(A\cap\operatorname{Ker}V_{H\to A})=\operatorname{Gal}(\tilde{K}/C(K/k)\cap S_F(K)\cdot K)$. This shows that $\tilde{k}$ contains $C(K/k)\cap S_F(K)\cdot K$. Therefore Corollary 2 follows from Corollary 1 to Theorem 3. The proof is completed.

§ 4. The adelic version

Let $k_A^\times$ be the idele group of k, $k_{\infty+}^\times$ the connected component of the unity of the Archimedean part of $k_A^\times$ and $k^\#$ the closure of $k^\times\cdot k_{\infty+}^\times$ in $k_A^\times$. Let K be an abelian extension of k of finite degree. (K/k is not necessarily unramified.) Put $\mathfrak{g}=\operatorname{Gal}(K/k)$, and let $K_A^{\Delta\mathfrak{g}}$ be the closed subgroup of the idele group $K_A^\times$ of K defined by

$$K_A^{\Delta\mathfrak{g}}=\langle x^{1-\sigma}\,|\,x\in K_A^\times,\ \sigma\in\mathfrak{g}\rangle\ .$$

Let $N_{K/k}\colon K_A^\times\to k_A^\times$ be the norm map. We consider $k_A^\times$ a subgroup of $K_A^\times$ naturally.

Suppose that we are given a subfield F of K such that F is cyclic over k of the maximal degree. The idele group $F_A^\times$ is also considered a subgroup of $K_A^\times$. Let $N_{K/F}\colon K_A^\times\to F_A^\times$ be the norm map of K over F.

THEOREM 4. *Let the notation and the assumptions be as above. Let U be an open subgroup of $K_A^\times$, and suppose that $U\supset K^\times\cdot K_{\infty+}^\times$ and that $U^\sigma=U$ for each $\sigma\in\mathfrak{g}$. Put*

$$\{N_{K/F}(K_A^\times)\cdot U/U\}^{\mathfrak{g}}=\{c\in N_{K/F}(K_A^\times)\cdot U/U\,|\,c^\sigma=c\ \textit{for}\ \forall\sigma\in\mathfrak{g}\}\ .$$

Then we have

$$[k_A^\times \cap U\colon k^\times \cdot N_{K/k}(K_A^\times) \cap U]\cdot[N_{K/k}^{-1}(k_A^\times \cap U)\colon K_A^{\Delta\mathfrak{g}}\cdot N_{K/F}^{-1}(F_A^\times \cap U)]$$
$$= [K\colon k]\cdot[\{N_{K/F}(K_A^\times)\cdot U/U\}^{\mathfrak{g}}\colon k_A^\times\cdot U/U] .$$

Proof. Let k_{ab}, K_{ab} and F_{ab} be the maximal abelian extensions of k, K and F, respectively, in the algebraic closure of k. The Artin maps of k, K and F are open continuous surjective homomorphisms

$$\alpha_k\colon k_A^\times \longrightarrow \mathrm{Gal}\,(k_{\text{ab}}/k) ,$$
$$\alpha_K\colon K_A^\times \longrightarrow \mathrm{Gal}\,(K_{\text{ab}}/K) ,$$

and

$$\alpha_F\colon F_A^\times \longrightarrow \mathrm{Gal}\,(F_{\text{ab}}/F) ,$$

respectively, the kernels of which are $k^\#$, $K^\#$ and $F^\#$. Let $\bar{K}$ be the subfield of K_{ab} corresponding to the open subgroup $\alpha_K(U)$ of $\mathrm{Gal}\,(K_{\text{ab}}/K)$. Then $\bar{K}$ is normal over k. Put $G = \mathrm{Gal}\,(\bar{K}/k)$, $A = \mathrm{Gal}\,(\bar{K}/K)$ and $H = \mathrm{Gal}\,(\bar{K}/F)$. Then A and H are normal in G. Furthermore A is abelian and contains $[G, G]$. We have the following commutative diagram whose three columns are exact:

$$\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
k^\times\cdot N_{K/k}(U) & \hookrightarrow & F^\times\cdot N_{K/F}(U) & \hookrightarrow & U \\
\downarrow & & \downarrow & & \downarrow \\
k_A^\times & \hookrightarrow & F_A^\times & \hookrightarrow & K_A^\times \\
\bar{\alpha}_k\downarrow & & \bar{\alpha}_F\downarrow & & \downarrow\bar{\alpha}_K \\
G/[G, G] & \xrightarrow[V_{G\to H}]{} & H/[H, H] & \xrightarrow[V_{H\to A}]{} & A \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}$$

Here $\bar{\alpha}_k$, $\bar{\alpha}_F$ and $\bar{\alpha}_K$ are the homomorphisms naturally induced from α_k, α_F and α_K, respectively. (Cf. [3, Proposition 3] for example.) Therefore we have homomorphisms

$$\bar{\alpha}_k\colon k_A^\times \cap U/k^\times\cdot N_{K/k}(U) \xrightarrow{\sim} \mathrm{Ker}\ V_{G\to A}/[G, G] ,$$

$$\bar{\alpha}_F\colon F_A^\times \cap U/F^\times\cdot N_{K/F}(U) \xrightarrow{\sim} \mathrm{Ker}\ V_{H\to A}/[H, H] .$$

Furthermore, we have, for $x \in K_A^\times$,

$$\bar{\alpha}_F \circ N_{K/F}(x) = \bar{\alpha}_K(x) \cdot [H, H] , \quad \text{and} \quad \bar{\alpha}_F(N_{K/F}(K_A^\times)) = A/[H, H]$$

by class field theory. This shows that

$$V_{H \to A}(A) = \bar{\alpha}_K(N_{K/F}(K_A^\times)) \simeq N_{K/F}(K_A^\times) \cdot U/U .$$

Note that these isomorphisms are ones of $\mathfrak{g}$-modules for $\mathfrak{g} = \mathrm{Gal}\,(K/k) = G/A$.

Let us now interpret the equality

$$q = \frac{[\mathrm{Ker}\ V_{G \to A}\colon\ [G, A] \cdot (A \cap \mathrm{Ker}\ V_{H \to A})]}{[G\colon A] \cdot [V_{H \to A}(A) \cap Z(G)\colon\ V_{G \to A}(G)]} = 1$$

which was proved in the previous section. In the similar way there, we have $[G\colon A] = [K\colon k]$ and

$$[V_{H \to A}(A) \cap Z(G)\colon\ V_{G \to A}(G)] = [\{N_{K/F}(K_A^\times) \cdot U/U\}^{\mathfrak{g}}\colon\ k_A^\times \cdot U/U]$$

in the present situation. As for the numerator, it is equal to

$$[\mathrm{Ker}\ V_{G \to A}\colon\ A \cap \mathrm{Ker}\ V_{G \to A}] \cdot [A \cap \mathrm{Ker}\ V_{G \to A}\colon\ [G, A] \cdot (A \cap \mathrm{Ker}\ V_{H \to A})] .$$

We have

$$\begin{aligned} &[\mathrm{Ker}\ V_{G \to A}\colon\ A \cap \mathrm{Ker}\ V_{G \to A}] \\ &\quad = [\mathrm{Ker}\ V_{G \to A}/[G, G]\colon\ (A \cap \mathrm{Ker}\ V_{G \to A})/[G, G]] \\ &\quad = [k_A^\times \cap U\colon\ k^\times \cdot N_{K/k}(K_A^\times) \cap U] \end{aligned}$$

because the subgroup $A/[G, G]$ of $G/[G, G]$ is equal to $\bar{\alpha}_k(N_{K/k}(K_A^\times))$. Furthermore, we also have

$$\bar{\alpha}_K(N_{K/k}^{-1}(k_A^\times \cap U)) = A \cap \mathrm{Ker}\ V_{G \to A} ,$$

and

$$\bar{\alpha}_K(N_{K/F}^{-1}(F_A^\times \cap U)) = A \cap \mathrm{Ker}\ V_{H \to A}$$

because, by class field theory, the following diagrams are commutative:

$$\begin{array}{ccc} K_A^\times & \xrightarrow{N_{K/k}} & k_A^\times \\ \bar{\alpha}_K \downarrow & & \downarrow \bar{\alpha}_k \\ A & & \\ \cap \downarrow & & \downarrow \\ G & \longrightarrow & G/[G, G] \end{array} \qquad \begin{array}{ccc} K_A^\times & \xrightarrow{N_{K/F}} & F_A^\times \\ \bar{\alpha}_K \downarrow & & \downarrow \bar{\alpha}_F \\ A & & \\ \cap \downarrow & & \downarrow \\ H & \longrightarrow & H/[H, H] \end{array}$$

where the homomorphisms of the last row are the natural projections. Since $\bar{\alpha}_K(K_A^{\Delta\mathfrak{g}}) = [G, A]$ and $N_{K/F}^{-1}(F_A^\times \cap U) \supset U = \operatorname{Ker} \bar{\alpha}_K$, we finally have

$$\begin{aligned}[A \cap \operatorname{Ker} V_{G\to A}&: [G, A]\cdot(A \cap \operatorname{Ker} V_{H\to A})] \\ &= [N_{K/k}^{-1}(k_A^\times \cap U): K_A^{\Delta\mathfrak{g}}\cdot N_{K/k}^{-1}(F_A^\times \cap U)] .\end{aligned}$$

The equality, $q = 1$, now gives the equality of the theorem at once.

COROLLARY. *Let K/k be an abelian extension of finite degree with $\mathfrak{g} = \operatorname{Gal}(K/k)$. Let U be an open subgroup of $K_A^\times$ which contains $K^\times \cdot K_{\infty+}^\times$ and satisfies that $U^\sigma = U$ for each $\sigma \in \mathfrak{g}$. If $U\cdot K_A^{\Delta\mathfrak{g}}$ contains $N_{K/k}^{-1}(k^\#)$, then $[K: k]$ divides $[k_A^\times \cap U: k^\times \cdot N_{K/k}(U)]$.*

Proof. We have $k^\times \cdot N_{K/k}(K_A^\times) = k^\# \cdot N_{K/k}(K_A^\times)$ and $k^\times \cdot N_{K/k}(U) = k^\# \cdot N_{K/k}(U)$. If, therefore, $U\cdot K_A^{\Delta\mathfrak{g}}$ contains $N_{K/k}^{-1}(k^\#)$, we have

$$\begin{aligned}[N_{K/k}^{-1}&(k_A^\times \cap U): K_A^{\Delta\mathfrak{g}}\cdot U] \\ &= [k^\times \cdot N_{K/k}(K_A^\times) \cap U: k^\times \cdot N_{K/k}(U)] .\end{aligned}$$

Therefore $[k_A^\times \cap U: k^\times \cdot N_{K/k}(K_A^\times) \cap U] \cdot [N_{K/k}^{-1}(k_A^\times \cap U): K_A^{\Delta\mathfrak{g}} \cdot U]$ is equal to $[k_A^\times \cap U: k^\times \cdot N_{K/k}(U)]$. Since $K_A^{\Delta\mathfrak{g}}\cdot U$ is a subgroup of $K_A^{\Delta\mathfrak{g}}\cdot N_{K/k}^{-1}(F_A^\times \cap U)$, we have the corollary from the theorem at once.

Remark 1. If K/k is unramified and $U = O^\times(K_A) =$ the unit group of the adele ring K_A, then Theorem 4 is equivalent to Theorem 3, and the corollary to the one to Theorem 1 in Section 1.

Remark 2. Let L be the abelian extension of K corresponding to U in the corollary. Then the maximal central extension L^* of K/k contained in L corresponds to $U\cdot K_A^{\Delta\mathfrak{g}}$. Therefore the condition, $U\cdot K_A^{\Delta\mathfrak{g}} \supset N_{K/k}^{-1}(k^\#)$, is equivalent to the one that L^* is contained in $K\cdot k_{\mathrm{ab}}$, i.e. that L^* reduces to its genus field $L \cap K\cdot k_{\mathrm{ab}}$.

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Department of Mathematics
College of General Education
Nagoya University
Chikusa-ku, Nagoya 464
Japan