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GENERALIZED MAILLET DETERMINANT

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§ 1. Introduction

In this paper, we shall study a generalization of the Maillet determinant. Let p be an odd prime, and $G = (Z/pZ)^*$. We shall identify any integer and its image in G if there is no fear of confusion. For any integer a, let $R(a)$ denote an integer satisfying

$$R(a) \equiv a \bmod p\,, \qquad 0 \leqq R(a) < p\,.$$

Maillet studied the following determinant

$$D_p = \det\,(R(ab^{-1}))_{1\leq a,b\leq (p-1)/2}$$

which is called the Maillet determinant and he raised the question whether $D_p \neq 0$ for all p. Carlitz-Olson [1] proved that the Maillet determinant is not zero by showing the following formula:

$$D_p = \pm p^{(p-3)/2}h^-$$

where h^- is the first factor of the class number of $\boldsymbol{Q}(\zeta_p)$, ζ_p the primitive p-th root of unity.

Carlitz considered a generalization of D_p in [2]. We consider another generalization of the determinant D_p. Let S be a subset of G. S is called a CM-type if

$$S \cup (-S) = G\,, \qquad S \cap (-S) = \phi\,.$$

Clearly $\{1, 2, \cdots, (p-1)/2\}$ is a CM-type. For any CM-type S, we define a determinant D_S by

$$D_S = \det\,(R(ab^{-1}))_{a,b\in S}\,.$$

We call D_S the generalized Maillet determinant for S. Since $D_S = D_{-S}$, we may only consider CM-types which contain 1.

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Let χ be a Dirichlet character mod p. For any CM-type S, we define $c_\chi = c_\chi(S)$ by

$$c_\chi = \sum_{a \in S} \chi(a) .$$

Let $B_{1,\chi}$ denote the first generalized Bernoulli number. If χ is odd, $B_{1,\chi} \neq 0$. Therefore, we can define a rational number A_S by

$$A_S = \frac{2}{p-1} \sum_{\chi \cdot \mathrm{odd}} c_\chi c_{\bar{\chi}} B_{1,\chi}^{-1} .$$

Then we have

THEOREM. *For any CM-type S which contains* 1, *we have*

$$D_S = -\frac{1}{2}(-p)^{(p-3)/2}(1 + A_S)h^- .$$

We shall prove this theorem and see the connection between our theorem and Carlitz-Olson's formula.

§ 2. Proof of the theorem

We need the following lemma, which is well-known as the Dedekind determinant [3]:

LEMMA. *Let S be a CM-type, and f be an odd function on G. Then the determinant $D(f) = \det(f(ab^{-1}))_{a,b \in S}$ is independent of S, and*

$$D(f) = \sum_{\chi:\mathrm{odd}} \frac{1}{2} \sum_{a \in G} \chi(a) f(a) .$$

We define the determinant $D_S(x)$ as follows:

$$D_S(x) = \det(R(ab^{-1}) + x)_{a,b \in S} .$$

Since $R(a) - (p/2)$ is an odd function, by Lemma

$$\begin{aligned} D_S\left(-\frac{p}{2}\right) &= \sum_{\chi:\mathrm{odd}} \frac{1}{2} \sum_{a \in G} \chi(a)\left(R(a) - \frac{p}{2}\right) \\ &= \sum_{\chi:\mathrm{odd}} \frac{p}{2} B_{1,\chi} = -\frac{1}{2}(-p)^{(p-3)/2} h^- . \end{aligned}$$

And so, it suffices to show that

$$\left[A_S\left(x + \frac{p}{2}\right) + \frac{p}{2}\right] D_S = \frac{p}{2}(1 + A_S) D_S(x) .$$

Now, it is clear that

$$(p-1)R(a)=\sum_{\chi\in\hat{G}}\bar{\chi}(a)\sum_{b\in G}\chi(b)R(b),$$

where $\hat{G}$ denotes the character group of G. If χ is not trivial, then

$$\sum_{b\in G}\chi(b)R(b)=pB_{1,\chi}.$$

Therefore,

$$(p-1)R(a)=\frac{p(p-1)}{2}+p\sum_{\chi:\mathrm{odd}}\bar{\chi}(a)B_{1,\chi}$$

because $B_{1,\chi}=0$ for any non-trivial even character χ. We define the rational number $A(a)$ as follows:

$$A(a)=\sum_{\chi:\mathrm{odd}}c_{\bar{\chi}}\chi(a)B_{1,\chi}^{-1}.$$

Then it is clean that $A_S=\sum_{a\in S}A(a)$. For $b\in S$,

$$\begin{aligned}&\sum_{a\in S}A(a)(R(ab^{-1})+x)\\&=\sum_{a\in S}A(a)\Big(x+\frac{p}{2}+\frac{p}{p-1}\sum_{\chi:\mathrm{odd}}\bar{\chi}(ab^{-1})B_{1,\chi}\Big)\\&=\frac{p-1}{2}A_S\Big(x+\frac{p}{2}\Big)+\frac{p}{p-1}\sum_{\chi:\mathrm{odd}}\sum_{a\in S}A(a)\bar{\chi}(ab^{-1})B_{1,\chi}.\end{aligned}$$

And then

$$\sum_{\chi:\mathrm{odd}}\sum_{a\in S}A(a)\bar{\chi}(ab^{-1})B_{1,\chi}=\sum_{\chi:\mathrm{odd}}\sum_{\psi:\mathrm{odd}}c_{\bar{\psi}}B_{1,\psi}^{-1}\chi(b)B_{1,\chi}\sum_{a\in S}\psi\bar{\chi}(a).$$

We have

$$\sum_{a\in S}\psi\bar{\chi}(a)=\begin{cases}\dfrac{p-1}{2} & \text{if } \chi=\psi\\[1ex] 0 & \text{if } \chi\neq\psi\end{cases}$$

because $\psi\bar{\chi}$ is even and S is a complete system of representatives of $G/(\pm 1)$. Hence

$$\begin{aligned}\sum_{\chi:\mathrm{odd}}\sum_{a\in S}A(a)\bar{\chi}(ab^{-1})B_{1,\chi}&=\frac{p-1}{2}\sum_{\chi:\mathrm{odd}}c_{\bar{\chi}}\chi(b)\\&=\frac{p-1}{2}\sum_{a\in S}\sum_{\chi:\mathrm{odd}}\chi(a^{-1}b)=\Big(\frac{p-1}{2}\Big)^2.\end{aligned}$$

Consequently,

$$\sum_{a\in S} A(a)(R(ab^{-1}) + x) = \frac{p-1}{2}\left[A_S\left(x + \frac{p}{2}\right) + \frac{p}{2}\right].$$

Therefore, there exists an integer $a_i \in S$ such that $A(a_i) \neq 0$.

Now we put the matrix $M(x) = (m_{a,b})_{a,b\in S}$ as follows:

$$m_{a,b} = \begin{cases} R(ab^{-1}) + x & \text{if } a \neq a_i \\ 1 & \text{if } a = a_i\,. \end{cases}$$

Then, by some properties of matrices, we have

$$A(a_i)D_S(x) = \frac{p-1}{2}\left[A_S\left(x + \frac{p}{2}\right) + \frac{p}{2}\right] \det M(x)\,,$$

and

$$A(a_i)D_S = \frac{p(p-1)}{4}(1 + A_S) \det M(0)\,.$$

Since $\det M(x) = \det M(0) \neq 0$ and $A(a_i) \neq 0$,

$$\left[A_S\left(x + \frac{p}{2}\right) + \frac{p}{2}\right] = \frac{P}{2}(1 + A_S)D_S(x)\,.$$

This completes the proof of the theorem.

In the rest of this paper, we shall calculate A_S in more convenient form, and show that Carlitz-Olson's formula follows easily from our theorem. Let $Z[G]$ be the group ring of G, and $Z[G]^- = \{\alpha \in Z[G] \mid \sigma_{-1}\alpha = -\alpha\}$ where σ_a is the image in G of an integer a. Let S be a subset of G. We define the element $s(S)$ in $Z[G]$ by $s(S) = \sum_{\sigma\in S}\sigma$. We put the element

$$\theta' = \sum_{\sigma\in G} \frac{R(\sigma)}{p}\sigma^{-1} \qquad \text{in } \boldsymbol{Q}[G]\,,$$

and the ideal of $Z[G]$

$$\varphi' = \theta' Z[G] \cap Z[G]\,.$$

Then the Stickelberger element is defined by

$$\theta = \sum_{\sigma\in G}\left(\frac{R(\sigma)}{p} - \frac{1}{2}\right)\sigma^{-1} = \varepsilon^-\theta'$$

where $\varepsilon^- = \frac{1}{2}(1 - \sigma_{-1})$. And the Stickelberger ideal is defined by

$$\varphi = \theta Z[G] \cap Z[G] = \{\alpha \in \varphi' \mid \sigma_{-1}\alpha = -\alpha\}\,.$$

Moreover, by [4] we have the formula

$$(Z[G]^-: \varphi) = h^- .$$

For the CM-type $S_0 = \{1, 2, \cdots, (p-1)/2\}$

$$s(S_0) = (\sigma_{-1} + \sigma_2 - 1)\theta' ,$$

and so $s(S_0) \in \varphi'$. Therefore, for any CM-type S, $h^-\{s(S) - s(S_0)\} \in \varphi$ because $s(S)$-$s(S_0) \in Z[G]^-$, and then $h^- s(S) \in \varphi'$. Therefore,

$$ks(S) = \theta'\alpha$$

for some $\alpha = \alpha_S \in Z[G]$ and some integer $k | h^-$. Then we have

PROPOSITION. *For any CM-type S,*

$$A_S = \frac{1}{k}(\sum_{\sigma \in S} n_\sigma - \sum_{\tau \notin S} n_\tau)$$

where $ks(S) = \theta'\alpha$, $\alpha = \sum_{\alpha \in G} n_\sigma \sigma$, $n_\sigma \in Z$.

Proof. We extend a character χ to a function on $Q[G]$ by

$$\chi(\alpha) = \sum_{\sigma \in G} n_\sigma \chi(\sigma) \qquad \text{for } \alpha = \sum_{\sigma \in G} n_\sigma \sigma \in Q[G] .$$

Then $c_\chi = \chi(s(S))$, and $B_{1,\chi} = \chi(\theta')$ for any non trivial character χ. Hence

$$kc_\chi = B_{1,\chi}\chi(\sigma)$$

and so,

$$A_S = \frac{2}{(p-1)k} \sum_{\chi:\text{odd}} c_{\bar{\chi}} \chi(\alpha) .$$

For $\sigma \in G$,

$$\sum_{\chi:\text{odd}} \chi(\sigma) = \begin{cases} \dfrac{p-1}{2} & \text{for } \sigma = 1 , \\ -\dfrac{p-1}{2} & \text{for } \sigma = \sigma_{-1} , \\ 0 & \text{otherwise} . \end{cases}$$

Therefore,

$$\sum_{\chi:\text{odd}} c_{\bar{\chi}} \chi(\sigma) = \sum_{\chi:\text{odd}} \sum_{\tau \in S} \chi(\sigma\tau^{-1}) = \begin{cases} \dfrac{p-1}{2} & \text{for } \sigma \in S \\ -\dfrac{p-1}{2} & \text{for } \sigma \notin S \end{cases}$$

Consequently,

$$A_S = \frac{1}{k}\left(\sum_{\sigma \in S} n_\sigma - \sum_{\tau \notin S} n_\tau\right).$$

This completes the proof.

By the proposition and the theorem, for S_0 we have Carlitz-Olson's formula.

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