G. Tamone
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# A CONNECTION BETWEEN BLOWING-UP AND GLUINGS IN ONE-DIMENSIONAL RINGS

GRAZIA TAMONE

## Introduction

Let $C$ be an affine curve, contained on a non-singular surface $X$ as a closed 1-dimensional subscheme. If $P$ is a closed point on $C$, the blowing-up $C'$ of $C$ with center $P$ (induced by the blowing-up of $X$ with center $P$) is an affine curve. It is known that there is a sequence:

$$(\cdot) \qquad \bar{C} = C_k \longrightarrow C_{k-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 = C\,,$$

where $\bar{C}$ is the normalization of $C$, and each $C_{i+1}$ is the blowing-up of $C_i$ with center a singular point $P_i$ on $C_i$ ($i = 0, \cdots, k-1$).

The sequence $(\cdot)$ induces a sequence of rings:

$$(*) \qquad R = R_0 \subset R_1 \subset \cdots \subset R_{k-1} \subset R_k = \bar{R}\,,$$

where, for $j = 0, \cdots, k$, $R_j$ is the coordinate ring of $C_j$; for each $i = 0, \cdots, k-1$, $R_{i+1}$ is called the ring "obtained from $R_i$ by blowing-up the maximal ideal of $R_i$ corresponding to $P_i$".

On the other hand, there is also a sequence between $R$ and $\bar{R}$:

$$(**) \qquad R = B_n \subset B_{n-1} \subset \cdots \subset B_1 \subset B_0 = \bar{R}\,,$$

where each $B_{i+1}$ (i $= 0, \cdots, n-1$) is a "gluing of primary ideals of $B_i$ over a prime ideal of $R$" (see [6]).

In this paper we wonder under what assumptions a sequence $(*)$ is also a sequence $(**)$ of gluings between $R$ and $\bar{R}$; in this case, the method of "gluing" defined in [6] is "inverse" of the process of "blowing-up" used to obtain the desingularization of $C$. We give necessary and sufficient conditions on $(*)$ in order that $(*)$ is also a sequence of gluings like $(**)$; then, we show some classes of rings satisfying the required condition, in particular the rings considered in the last theorem of [7].

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## §1.

Let $C$ be an affine curve, $P_1, \cdots, P_n$ the singular points on $C$, $R$ the coordinate ring of $C$. For $i = 1, \cdots, n$, the maximal ideal of $R$ corresponding to $P_i$ is a prime ideal belonging to the conductor $\mathfrak{b}$ of $R$ in $\bar{R}$. Then, if $\mathrm{Ass}(R/\mathfrak{b}) = \{\mathfrak{m}_1, \cdots, \mathfrak{m}_n\}$, and $S = R - \bigcup \mathfrak{m}_i$, the ring $A = S^{-1}R$ is semilocal, and its maximal ideals are exactly $\mathfrak{m}_1 A, \cdots, \mathfrak{m}_n A$, so that the maximal ideals of $A$ correspond to the singular points of $C$. Besides, if $R'$ is the coordinate ring of the blowing-up of $C$ with center $P_i$ $(i = 1, \cdots, n)$, the ring "obtained from $A$ by blowing-up $\mathfrak{m}_i A$" is canonically isomorphic to $S^{-1}R'$ ([4], p. 663). Owing to these facts, we can consider $A$ instead of $R$ without loss of generality.

Since $A$ is semilocal, the ring "obtained from $A$ by blowing-up a maximal ideal $\mathfrak{m}$" can be described in various ways, according to [4] and [5]. In fact, if $A$ is a semilocal 1-dimensional Cohen-Macaulay ring, the ring obtained by blowing-up $\mathfrak{m} \in \mathrm{Spm}(A)$ coincides with the "first neighbourhood of $A$": $\Lambda = \{b/a \mid b \in \mathfrak{m}^s,\ a \text{ is superficial of degree } s\}$, defined in [5], Chapter XII. This ring can also be written as $A[z_1/x, \cdots, z_t/x]$, where $\{z_1, \cdots, z_t\}$ is a set of generators of $\mathfrak{m}$, $x \in \mathfrak{m}$ is $\mathfrak{m}$-transversal; besides, this ring coincides with $\mathfrak{m}^n : \mathfrak{m}^n = \{a \in \bar{A} \mid a\mathfrak{m}^n \subset \mathfrak{m}^n\}$ for all sufficiently large $n$ (see [4], Proposition 1.1, Definition 1.7, Lemma 1.8, and [2], Corollary 3.5).

In this paper, unless we give further notice, $A$ will mean a semilocal 1-dimensional Cohen-Macaulay ring. Besides, we shall denote the "embedding dimension" and the "multiplicity" of a local ring $S$ respectively by: $\mathrm{emdim}(S)$ and $e(S)$.

First of all, we prove some lemmas we need to study some conductors which we are interested in.

LEMMA 1.1. *Let $\mathfrak{p}$ be a maximal ideal in $A$, $\Lambda$ be the ring obtained from $A$ by blowing-up $\mathfrak{p}$. If $A \neq \Lambda$, the conductor of $A$ in $\Lambda$ is a $\mathfrak{p}$-primary ideal.*

*Proof.* Let $\mathfrak{a}$ be the conductor of $A$ in $\Lambda$. As seen before, $\Lambda = \mathfrak{p}^n : \mathfrak{p}^n$ for a suitable $n$, so, for each $x \in \mathfrak{p}$, $y \in \Lambda$ we have: $yx^n \in \mathfrak{p}^n$, thence $x^n\Lambda \subset A$. It follows: $x^n \in \mathfrak{a}$ for each $x \in \mathfrak{p}$, so $\mathfrak{p} \subset \sqrt{\mathfrak{a}}$. Now, $\mathfrak{p}$ is maximal and $\mathfrak{a}$ is a proper ideal, then we have $\mathfrak{p} = \sqrt{\mathfrak{a}}$ and $\mathfrak{a}$ is $\mathfrak{p}$-primary.

COROLLARY 1.2. *Let $\mathfrak{b}$ be the conductor of $A$ in $\bar{A}$, $\Lambda$ be the ring obtained from $A$ by blowing-up a maximal ideal $\mathfrak{p}$ belonging to $\mathfrak{b}$. If $\mathfrak{p}$ coincides with the $\mathfrak{p}$-primary component of $\mathfrak{b}$, the conductor of $A$ in $\Lambda$ is $\mathfrak{p}$.*

*Proof.* We first have: $A \neq \Lambda$: in fact, $A = \Lambda$ implies $\mathfrak{p} = \mathfrak{p}\Lambda = x\Lambda = xA$ for some regular element $x \in A$ ([4], Proposition 1.1, (ii)), so $A_{\mathfrak{p}}$ is regular, then $A_{\mathfrak{p}} = \overline{A_{\mathfrak{p}}}$ while $\mathfrak{p} \in \mathrm{Ass}(A/\mathfrak{b})$. So, $A \subsetneqq \Lambda \subset \overline{A}$; then, if $\mathfrak{a}$ is the conductor of $A$ in $\Lambda$, we have $\mathfrak{b} \subset \mathfrak{a}$, and also $\sqrt{\mathfrak{a}} = \mathfrak{p}$ (Lemma 1.1). Let $\mathfrak{q}$ be the $\mathfrak{p}$-primary ideal belonging to $\mathfrak{b}$; the reduced primary decomposition of $\mathfrak{b}$ is like this: $\mathfrak{b} = \mathfrak{q} \cap (\cap \mathfrak{q}_j)$. Then, if $\mathfrak{p} = \mathfrak{q}$, owing to the above facts we have: $\mathfrak{p} \cap (\cap \mathfrak{q}_j) \supset \mathfrak{a} \cap (\cap \mathfrak{q}_j) \supset \mathfrak{b} = \mathfrak{q} \cap (\cap \mathfrak{q}_j) = \mathfrak{p} \cap (\cap \mathfrak{q}_j)$, hence:

$$(\cdot) \qquad \mathfrak{p} \cap (\cap \mathfrak{q}_j) = \mathfrak{a} \cap (\cap \mathfrak{q}_j)\,, \quad \text{with} \quad \sqrt{\mathfrak{a}} = \mathfrak{p}\,.$$

It follows that the two sides of $(\cdot)$ are two reduced primary decompositions of the same ideal $\mathfrak{b}$, whose primary components are all isolated; then, owing to the uniqueness of these components, we have, in particular, $\mathfrak{p} = \mathfrak{a}$.

*Remarks.* 1) In general, if $\mathfrak{p}$ doesn't coincide with the $\mathfrak{p}$-primary component of $\mathfrak{b}$, one has: $\mathfrak{p} \neq \mathfrak{a}$. As an example, let us consider the ring $A = k[\![t^3, t^5]\!]$. The conductor $\mathfrak{b}$ of $A$ in $\overline{A} = k[\![t]\!]$ is $\mathfrak{p}$-primary, where $\mathfrak{p} = (t^3, t^5)$. We have: $\Lambda = A[t^3/t^3, t^5/t^3]$ ([4], Definition 1.7, Lemma 1.8, and the beginning of Section 1) $= k[\![t^2, t^3]\!]$. Let $\mathfrak{a}$ be the conductor of $A$ in $\Lambda$. One ean easily show that $\mathfrak{a} \neq \mathfrak{p}$, seeing that $t^5 \in \mathfrak{p}$, $t^5 \notin \mathfrak{a}$ because $t^5t^2 = t^7 \notin A$.

2) The inverse of Corollary 1.2 is not true, i.e. in some cases the conductor of $A$ in $\Lambda$ is $\mathfrak{p}$, but $\mathfrak{p}$ is not a primary ideal belonging to $\mathfrak{b}$. For example, if $A = k[\![t^2, t^5]\!]$, we have: $\overline{A} = k[\![t]\!]$, $\mathfrak{b} = (t^4, t^5)$ is $(t^2, t^5)$-primary, and $\mathfrak{b} \neq (t^2, t^5)$. One has: $\Lambda = A[t^2/t^2, t^5/t^2]$ ([4], Proposition 1.1, Definition 1.7, Lemma 1.8) $= k[\![t^2, t^3]\!]$. Now, we show the conductor $\mathfrak{a}$ of $A$ in $\Lambda$ is $(t^2, t^5)$. Owing to the maximality of $(t^2, t^5)$ it is enough to prove: $(t^2, t^5) \subset \mathfrak{a}$. So, for each $x \in (t^2, t^5)$, we must prove $x\Lambda \subset A$. Let $x \in (t^2, t^5)$, $y \in \Lambda$; then, $x = t^2 \sum a_{ij} t^{2i} t^{5j} + t^5 \sum b_{hk} t^{2h} t^{5k}$, $y = \sum c_{pq} t^{2p} t^{3q}$. So, $xy = \sum c_{pq} t^{2p}(xt^{3q})$. Now, $xt^{3q} = (\sum a_{ij} t^{2i} t^{5j}) t^{3q+2} + (\sum b_{hk} t^{2h} t^{5k}) t^{3q+5} = \sum a_{ij} t^{2i+5j+3q+2} + \sum b_{hk} t^{2h+5k+3q+5}$, and we have: $2i + 5j + 3q + 2 \geqslant 4$, or $=2$ for $i, j, q \in N$, $2h + 5k + 3q + 5 \geqslant 7$, or $=5$ for $h, k, q \in N$. So, $xt^{3q} \in A$. Then, $xy = \sum c_{pq} t^{2p}(xt^{3q}) \in A$, since also $t^{2p} \in A$ for each $p$.

COROLLARY 1.3. *Let $\mathfrak{p}, \Lambda$ be as in Lemma* 1.1. *If $\mathfrak{p}'$ is a prime ideal of $A$, and $\mathfrak{p}' \neq \mathfrak{p}$, there is a unique prime in $\Lambda$ over $\mathfrak{p}'$.*

*Proof.* Owing to Lemma 1.1, the conductor $\mathfrak{a}$ of $A$ in $\Lambda$ is such that $\sqrt{\mathfrak{a}} = \mathfrak{p}$; then, if $\mathfrak{p}' \neq \mathfrak{p}$, one has $\mathfrak{p}' \not\supset \mathfrak{a}$ (otherwise $\mathfrak{p}' \supset \mathfrak{p}$, and this implies

$\mathfrak{p}' = \mathfrak{p}$). It follows: $A_{\mathfrak{p}'} = \Lambda_{A-\mathfrak{p}'}$, so there is a unique prime ideal in $\Lambda$ over $\mathfrak{p}'$ (since there is one-to-one correspondence between $\{\mathfrak{P} \in \operatorname{Spec} \Lambda / \mathfrak{P} \cap A = \mathfrak{p}'\}$ and $\operatorname{Spec}(\Lambda_{A-\mathfrak{p}'}/\mathfrak{p}'\Lambda_{A-\mathfrak{p}'}) = \operatorname{Spec}(A_{\mathfrak{p}'}/\mathfrak{p}'A_{\mathfrak{p}'}) = \operatorname{Spec}(k(\mathfrak{p}'))$.

The next lemma holds in the general case: so, the rings considered here are not necessarily of the above type.

LEMMA 1.4. *Let $A, B, C$ be rings such that $A \subset B \subset C$, and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{b}'$ be respectively the conductor of $A$ in $B$, of $A$ in $C$, of $B$ in $C$. Then, $\mathfrak{a}\mathfrak{b}' \subset \mathfrak{b}$ in $B$.*

*Proof.* For each $x \in \mathfrak{a}$, $y \in \mathfrak{b}'$, $c \in C$ we have (in $B$): $(xy)c = x(yc)$, where $yc \in B$, since $y \in \mathfrak{b}'$; so, $x(yc) \in A$ because $x \in \mathfrak{a}$. Then, $(xy)c \in A$, so that $xy \in \mathfrak{b}$. It follows that $\mathfrak{a}\mathfrak{b}' \subset \mathfrak{b}$.

LEMMA 1.5. *Under the assumptions of Corollary* 1.2, *let $\mathfrak{a}, \mathfrak{b}'$ be respectively the conductor of $A$ in $\Lambda$ and of $\Lambda$ in $\bar{A}$. If $\mathfrak{p}_i \in \operatorname{Ass}(A/\mathfrak{b}) - \{\mathfrak{p}\}$, and $S = A - \mathfrak{p}_i$ we have $\mathfrak{b}S^{-1}\Lambda = \mathfrak{b}'S^{-1}\Lambda$.*

*Proof.* We have $\mathfrak{b}\Lambda \subset \mathfrak{b}'$, since $(\mathfrak{b}\Lambda)\bar{A} \subset \mathfrak{b}\bar{A} \subset A \subset \Lambda$, so $\mathfrak{b}S^{-1}\Lambda \subset \mathfrak{b}'S^{-1}\Lambda$. On the other hand, in $\Lambda$ one has $\mathfrak{a}\mathfrak{b}' \subset \mathfrak{b}$ (Lemma 1.4), so $(\mathfrak{a}S^{-1}\Lambda)(\mathfrak{b}'S^{-1}\Lambda) = (\mathfrak{a}\mathfrak{b}')S^{-1}\Lambda \subset \mathfrak{b}S^{-1}\Lambda$, hence $\mathfrak{b}'S^{-1}\Lambda \subset \mathfrak{b}S^{-1}\Lambda$ because $\mathfrak{a}S^{-1}\Lambda = S^{-1}\Lambda$ owing to the assumptions and Lemma 1.1.

Using the above results, we can prove some facts concerning the conductor of $\Lambda$ in $\bar{A}$. We assume that $\bar{A}$ is a finite $A$-module.

PROPOSITION 1.6. *Let $\mathfrak{b}$ be the conductor of $A$ in $\bar{A}$, and* $\operatorname{Ass}(A/\mathfrak{b}) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. *Let $\Lambda_j$ be the ring obtained from $A$ by blowing-up $\mathfrak{p}_j$ $(1 \leqslant j \leqslant n)$, $\mathfrak{b}_j$ be the conductor of $\Lambda_j$ in $\bar{A}$. The following facts hold:*

1) *for each $i \in \{1, \cdots, \hat{j}, \cdots, n\}$ there is a unique prime ideal $\mathfrak{P}_i$ in $\Lambda_j$ over $\mathfrak{p}_i$, and $\{\mathfrak{P}_1, \cdots, \hat{j}, \cdots, \mathfrak{P}_n\} \in \operatorname{Ass}(\Lambda_j/\mathfrak{b}_j)$*

2) *for each prime ideal $\mathfrak{P}$ in $\Lambda_j$ such that $\mathfrak{P} \cap A \neq \mathfrak{p}_i$ $(i = 1, \cdots, n)$ we have: $\mathfrak{P} \notin \operatorname{Ass}(\Lambda_j/\mathfrak{b}_j)$.*

*Proof.* 1) For each $i \in \{1, \cdots, \hat{j}, \cdots, n\}$ we have $\mathfrak{p}_i \neq \mathfrak{p}_j$, so (Corollary 1.3) there is a unique prime in $\Lambda_j$ over $\mathfrak{p}_i$, say $\mathfrak{P}_i$. For each $\mathfrak{P}_i \in \{\mathfrak{P}_1, \cdots, \hat{j}, \cdots, \mathfrak{P}_n\}$ we have $\mathfrak{P}_i \cap A = \mathfrak{p}_i \supset \mathfrak{b}$, so $\mathfrak{P}_i \supset \mathfrak{b}\Lambda_j$, thence if $S = A - \mathfrak{p}_i$, the ideal $\mathfrak{P}_i S^{-1}\Lambda_j$ is proper, and contains $\mathfrak{b}S^{-1}\Lambda_j$. Now, owing to Lemma 1.5, $\mathfrak{b}S^{-1}\Lambda_j = \mathfrak{b}_j S^{-1}\Lambda_j$. Then, we have: $\mathfrak{P}_i S^{-1}\Lambda_j \supset \mathfrak{b}_j S^{-1}\Lambda_j$; this implies $\mathfrak{P}_i S^{-1}\Lambda_j$ is in $\operatorname{Ass}(S^{-1}\Lambda_j/\mathfrak{b}_j S^{-1}\Lambda_j)$, hence $\mathfrak{P}_i \in \operatorname{Ass}(\Lambda_j/\mathfrak{b}_j)$.

2) Let $\mathfrak{P} \in \operatorname{Spec}(\Lambda_j)$ be such that $\mathfrak{p} = \mathfrak{P} \cap A \neq \mathfrak{p}_i$ for $i = 1, \cdots, n$. Then, $\mathfrak{p} \not\supset \mathfrak{b}$, so $A_{\mathfrak{p}} = \bar{A}_{A-\mathfrak{p}}$; it follows: $A_{\mathfrak{p}} \subset (\Lambda_j)_{A-\mathfrak{p}} \subset \bar{A}_{A-\mathfrak{p}} = A_{\mathfrak{p}}$, so $(\Lambda_j)_{A-\mathfrak{p}}$

$= \overline{A}_{A-\mathfrak{p}}$. Hence, the conductor $\mathfrak{b}_j(A_j)_{A-\mathfrak{p}}$ is not a proper ideal, so $\mathfrak{P} \notin \operatorname{Ass}(A_j/\mathfrak{b}_j)$ (otherwise $\mathfrak{P}(A_j)_{A-\mathfrak{p}}$, which is a proper ideal, would contain $\mathfrak{b}_j(A_j)_{A-\mathfrak{p}} = (A_j)_{A-\mathfrak{p}}$).

PROPOSITION 1.7. *Under the assumptions of Proposition* 1.6, *if* $\mathfrak{p}_j$ *coincides with the* $\mathfrak{p}_j$*-primary ideal belonging to* $\mathfrak{b}$, *then*:

$$\{\mathfrak{P} \in \operatorname{Spec} A_j \mid \mathfrak{P} \cap A = \mathfrak{p}_j\} \not\subset \operatorname{Ass}(A_j/\mathfrak{b}_j)\ , \quad \textit{so}$$
$$\{\mathfrak{P} \in \operatorname{Spec} A_j \mid \mathfrak{P} \cap A = \mathfrak{p}_j\} \cap \operatorname{Ass}(A_j/\mathfrak{b}_j) \quad \textit{is empty}\ .$$

*Proof.* Let $S = A - \mathfrak{p}_j$; then, $\overline{S^{-1}A} = S^{-1}\overline{A}$, and the ring obtained from $S^{-1}A$ by blowing-up $\mathfrak{p}_jS^{-1}A$ is canonically isomorphic to $S^{-1}A_j$ (see the beginning of Section 1). Since $\mathfrak{p}_j$ equals the $\mathfrak{p}_j$-primary component of $\mathfrak{b}$, the conductor of $A$ in $A_j$ is $\mathfrak{p}_j$ (Corollary 1.2), so $\mathfrak{p}_jS^{-1}A_j \subset S^{-1}A$, then $\mathfrak{p}_jS^{-1}A_j = \mathfrak{p}_jS^{-1}A$. It follows: $S^{-1}A_j = \{x \in S^{-1}\overline{A} \mid x\mathfrak{p}_jS^{-1}A \subset \mathfrak{p}_jS^{-1}A\}$ ([4], Proposition 1.1 (i), Definition 1.3); besides, the conductor of $S^{-1}A$ in $S^{-1}\overline{A}$ is $\mathfrak{p}_jS^{-1}A$. All this allows us to prove: $S^{-1}A_j = S^{-1}\overline{A}$. Indeed, for each $x \in S^{-1}\overline{A}$ we have: $x(\mathfrak{p}_jS^{-1}A) \subset \mathfrak{p}_j\overline{S^{-1}A} \subset S^{-1}A$, so $x(\mathfrak{p}_jS^{-1}A) \subset \mathfrak{p}_jS^{-1}\overline{A} \cap S^{-1}A = \mathfrak{p}_jS^{-1}A$, then $x \in S^{-1}A_j$. Now, let $\mathfrak{P} \in \operatorname{Spec} A_j$ be such that $\mathfrak{P} \cap A = \mathfrak{p}_j$; if $\mathfrak{P} \in \operatorname{Ass}(A_j/\mathfrak{b}_j)$, we have $\mathfrak{P}S^{-1}A_j \in \operatorname{Ass}(S^{-1}A_j/\mathfrak{b}_jS^{-1}A_j)$, while $\mathfrak{P}S^{-1}A_j$ is a proper ideal, and $\mathfrak{b}_jS^{-1}A_j$ is not a proper ideal, since $S^{-1}A_j = S^{-1}\overline{A}$. So, the result follows.

*Remark.* There are examples of rings $A$ such that $\mathfrak{p}_j$ doesn't equal the $\mathfrak{p}_j$-primary component of $\mathfrak{b}$, and $\operatorname{Ass}(A_j/\mathfrak{b}_j)$ contains a prime ideal $\mathfrak{P}$ such that $\mathfrak{P} \cap A = \mathfrak{p}_j$. The ring $A = k[\![t^3, t^5]\!]$ and the ideal $\mathfrak{p}_j = (t^3, t^5)$ considered in remark 1) after Corollary 1.2 are an example of that. In fact, $A_j = k[\![t^2, t^3]\!]$, and the conductor $\mathfrak{b}_j$ is $\mathfrak{P} = (t^2, t^3)$; it is easily seen that $\mathfrak{P} \cap A = (t^3, t^5) = \mathfrak{p}_j$.

From Proposition 1.6 and Proposition 1.7 it follows immediately:

COROLLARY 1.8. *Under the assumptions of Proposition* 1.6, *if* $\mathfrak{p}_j$ *coincides with the* $\mathfrak{p}_j$*-primary component of* $\mathfrak{b}$, *then* $\operatorname{Ass}(A_j/\mathfrak{b}_j) = \{\mathfrak{P}_1, \cdots, \hat{j}, \cdots, \mathfrak{P}_n\}$ *where* $\mathfrak{P}_i$ *is the only prime ideal in* $A_j$ *over* $\mathfrak{p}_i$, *for* $i = 1, \cdots, \hat{j}, \cdots, n$.

The following proposition shows another connection between the properties of the conductors $\mathfrak{b}$ and $\mathfrak{b}_j$.

PROPOSITION 1.9. *Let* $A, \mathfrak{p}_j, A_j$ *be as in Proposition* 1.6, *and* $\mathfrak{P}_i$ *be the only prime ideal in* $A_j$ *over* $\mathfrak{p}_i$, *for* $i = 1, \cdots, \hat{j}, \cdots, n$. *If* $\mathfrak{p}_i$ *coincides with the* $\mathfrak{p}_i$*-primary component of* $\mathfrak{b}$, *then* $\mathfrak{P}_i$ *coincides with the* $\mathfrak{P}_i$*-primary ideal belonging to* $\mathfrak{b}_j$.

*Proof.* Let $S = A - \mathfrak{p}_i$, $\mathfrak{a}$ be the conductor of $A$ in $A_j$. Since $\mathfrak{p}_i \neq \mathfrak{p}_j$, we have: $\mathfrak{p}_i \not\supset \mathfrak{a}$, because $\mathfrak{a}$ is $\mathfrak{p}_j$-primary (Lemma 1.1) and $\mathfrak{p}_j$ is maximal; so, $S^{-1}A = S^{-1}A_j$. Moreover, $\mathfrak{b}S^{-1}A_j = \mathfrak{b}_jS^{-1}A_j$, owing to Lemma 1.5. So, $\mathfrak{b}_jS^{-1}A_j = \mathfrak{b}S^{-1}A_j = \mathfrak{b}S^{-1}A$, and this last ideal coincides with $\mathfrak{p}_iS^{-1}A$ because of the assumptions on $\mathfrak{p}_i$. Now, if $\mathfrak{Q}_i$ is the $\mathfrak{P}_i$-primary component of $\mathfrak{b}_j$, we have: $\mathfrak{b}_jS^{-1}A_j = \mathfrak{Q}_iS^{-1}A_j$. Then, $\mathfrak{Q}_iS^{-1}A_j = \mathfrak{p}_iS^{-1}A$. It follows that $\mathfrak{Q}_iS^{-1}A_j$ is a prime ideal; so, it coincides with its own radical $\mathfrak{P}_iS^{-1}A_j$. Thence, $\mathfrak{Q}_i = \mathfrak{P}_i$, because $\mathfrak{Q}_i$ is $\mathfrak{P}_i$-primary.

From Corollary 1.8 and Proposition 1.9 we get the following

COROLLARY 1.10. *Let $A, A_j$ be as in Proposition* 1.6 *and let $\mathfrak{P}_i$ be the only prime ideal in $A_j$ over $\mathfrak{p}_i$, for $i \in \{1, \cdots, \hat{j}, \cdots, n\}$. If $\mathfrak{b} = {}^{A}\sqrt{\mathfrak{b}} = \bigcap_{i=1}^{n} \mathfrak{p}_i$ then $\mathfrak{b}_j = {}^{A_j}\sqrt{\mathfrak{b}_j} = \bigcap_{i \neq j} \mathfrak{P}_i$.*

§ 2.

Now, let

$$(*) \qquad A = A_0 \subset A_1 \subset \cdots \subset A_{k-1} \subset A_k = \bar{A}$$

be a sequence where each $A_{j+1}$ is the ring obtained from $A_j$ by blowing-up a prime ideal $\mathfrak{P}_j$ in $A_j$ ($j = 0, \cdots, k-1$). We want to find necessary and sufficient conditions in order that $(*)$ is also a sequence

$$(**) \qquad A = B_n \subset B_{n-1} \subset \cdots \subset B_1 \subset B_0 = \bar{A}\,,$$

where each $B_{j+1}$ is the gluing, over a prime ideal $\mathfrak{p}$ of $A$, of the primary ideals belonging to $\mathfrak{p}B_j$ ($j = 0, \cdots, n-1$). Now, $A_j$ in $(*)$ is the gluing, over a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, of the primary ideals of $\mathfrak{p}A_{j+1}$, if and only if $A_j$ is the gluing, over $\mathfrak{P}_j \cap A$, of the primary ideals of $(\mathfrak{P}_j \cap A)A_{j+1}$. In fact, if $A_j$ is the gluing, over a prime $\mathfrak{p}'$ of $A$, of the primary ideals of $\mathfrak{p}'A_{j+1}$, we have: $A_j = A + \mathfrak{p}'A_{j+1}$, and $\mathfrak{P}' = \mathfrak{p}'A_{j+1}$ is a maximal ideal (see [7], Lemma 1.2, 1)); besides, $\mathfrak{P}'$ is the conductor of $A_j$ in $A_{j+1}$ (since $\mathfrak{P}'A_{j+1} = \mathfrak{p}'A_{j+1} \subset A_j$, and $\mathfrak{P}'$ is maximal). Now, since $A_{j+1}$ is obtained from $A_j$ by blowing-up $\mathfrak{P}_j$, the conductor $\mathfrak{a}$ of $A_j$ in $A_{j+1}$ is such that $\sqrt{\mathfrak{a}} = \mathfrak{P}_j$ (Lemma 1.1). Then, we have: $\mathfrak{a} = \mathfrak{P}'$, $\sqrt{\mathfrak{P}'} = \sqrt{\mathfrak{a}} = \mathfrak{P}_j$, so $\mathfrak{P}_j = \mathfrak{P}'$. It follows: $\mathfrak{p}' = \mathfrak{P}' \cap A = \mathfrak{P}_i \cap A$, so $A_j$ is the gluing, over $\mathfrak{P}_j \cap A$, of the primary ideals belonging to $(\mathfrak{P}_j \cap A)A_{j+1}$. On the contrary, if each $A_j$ is the gluing, over $\mathfrak{P}_j \cap A$, of the primary ideals belonging to $(\mathfrak{P}_j \cap A)A_{j+1}$, then obviously $(*)$ is a sequence like $(**)$. So, our problem is to require conditions in order that each $A_j$ is the gluing, over $\mathfrak{p} = \mathfrak{P}_j \cap A$, of the primary ideals of $\mathfrak{p}A_{j+1}$. We note that the property we are interested in implies the following (weaker) one: for $j = 0, \cdots, k-1$, $A_j$ is the gluing, over $\mathfrak{P}_j$, of the primary ideals of $\mathfrak{P}_j A_{j+1}$, owing to the equality $\mathfrak{P}_j = \mathfrak{p}A_{j+1}$ and [7], Lemma 1.2, 1), 2). This last property can be characterized through certain properties of $A_j$, as we show in the following lemma, which therefore gives a necessary condition for the property of (*) we are studying. The following lemma is also a generalization of Lemma 1.3 of [7].

LEMMA 2.1. *Let $\mathfrak{p}$ be a maximal ideal of $A$, $\Lambda$, $A'$ respectively be the ring obtained from $A$ by blowing-up $\mathfrak{p}$, and the gluing, over $\mathfrak{p}$, of the primary ideals belonging to $\mathfrak{p}\Lambda$. Then the following conditions are equivalent*:

1) *the rings $A, A'$ coincide.*
2) $\operatorname{emdim}(A_{\mathfrak{p}}) = e(A_{\mathfrak{p}})$.
3) *the conductor of $A$ in $\Lambda$ is $\mathfrak{p}$.*

*Proof.* We put $S = A - \mathfrak{p}$, and we remember that $S^{-1}\Lambda$ is the ring obtained from $S^{-1}A$ by blowing-up $\mathfrak{p}S^{-1}A$. We have:

$1) \Rightarrow 2)$ The gluing over $\mathfrak{p}S^{-1}A$ of the primary ideals of $\mathfrak{p}S^{-1}\Lambda$ is $B = S^{-1}A + \mathfrak{p}S^{-1}\Lambda$ ([7], Lemma 1.2, 1)). Now, $\mathfrak{p}S^{-1}\Lambda \subset S^{-1}A$, since $\mathfrak{p}\Lambda \subset A'$ ([7], Lemma 1.2, 1)) $\subset A$; then, $B \subset S^{-1}A$, so it is enough to apply [7], Lemma 1.3, $1) \Rightarrow 2)$.

$2) \Rightarrow 3)$ Owing to [7], Lemma 1.3, $2) \Rightarrow 3)$, the conductor of $S^{-1}A$ in $S^{-1}\Lambda$ is $\mathfrak{p}S^{-1}A$. Let $\mathfrak{a}$ be the conductor of $A$ in $\Lambda$; we have $\sqrt{\mathfrak{a}} = \mathfrak{p}$ (Lemma 1.1). Then, $\mathfrak{p}S^{-1}A = \mathfrak{a}S^{-1}A$, where $\mathfrak{a}$ is $\mathfrak{p}$-primary; it follows: $\mathfrak{p} = \mathfrak{a}$.

$3) \Rightarrow 1)$ We have: $A' = A + \mathfrak{p}\Lambda$ ([7], Lemma 1.2, 1)) $\subset A$, since $\mathfrak{p}$ is the conductor; so, $A' = A$.

Owing to this lemma and the above remarks we have: the condition "$\operatorname{emdim}((A_j)_{\mathfrak{P}_j}) = e((A_j)_{\mathfrak{P}_j})$ for each $A_j$ in (*)" is necessary to get the property of (*) we are studying, but it is not sufficient (consider for example $A = k[\![t^3, t^5, t^7]\!]$: the sequence (*) is $A \subset k[\![t^2, t^3]\!] \subset k[\![t]\!] = \overline{A}$, where $\operatorname{emdim}((A_j)_{\mathfrak{P}_j}) = e((A_j)_{\mathfrak{P}_j})$ for each $A_j$, $\mathfrak{P}_j$, and (*) doesn't coincide with (**), as Proposition 3.2 of [7] shows). The following results allow us to find also sufficient conditions for the property of (*) we are interested in.

The next lemma holds in the general case, not only for semilocal one-dimensional rings.

LEMMA 2.2. *Let $A \subset B$ be rings, $\mathfrak{p}$ a maximal ideal in $A$, $A'$ be a ring between $A$ and $B$, such that $A' \subset A + \mathfrak{p}B$. If $\mathfrak{p}'$ is a prime ideal in*

*$A'$ over $\mathfrak{p}$ and $\mathfrak{p}B \neq B$, then $\mathfrak{p}'B = \mathfrak{p}B$.*

*Proof.* The ideal $\mathfrak{p}B$ is maximal in $A + \mathfrak{p}B$, since $A + \mathfrak{p}B/\mathfrak{p}B \cong A/\mathfrak{p}B \cap A = A/\mathfrak{p}$, which is a field. Besides, $\mathfrak{p}B = (\mathfrak{p}A')B \subset \mathfrak{p}'B$, because $\mathfrak{p}A' \subset \mathfrak{p}'$; so, $\mathfrak{p}B \subset \mathfrak{p}'B$, and $\mathfrak{p}B$ is maximal. It follows: $\mathfrak{p}B = \mathfrak{p}'B$.

The next lemma recalls a well-known fact:

LEMMA 2.3. *Let $(A, \mathfrak{m}, k)$ be a local ring, $k = A/\mathfrak{m}$ and $M$ be a $k$-module. Then, $1_A(M) = 1_k(M)$.*

PROPOSITION 2.4. *Let $A$, $\mathfrak{p}$, $\Lambda$ be as in Lemma 2.1, $B$ be a ring between $\Lambda$ and $\overline{A}$, $\mathfrak{P}$ be a prime ideal in $B$ over $\mathfrak{p}$. Besides, let $\Lambda'$ be the ring obtained from $B$ by blowing-up $\mathfrak{P}$. Let us suppose $B$ is a finite $A$-module, $\mathfrak{P}$ is the only prime ideal in $B$ over $\mathfrak{p}$, and the residue fields $k(\mathfrak{p})$, $k(\mathfrak{P})$ are canonically isomorphic. The following conditions are equivalent:*

1) $\mathfrak{p}\Lambda' = \mathfrak{P}\Lambda'$
2) $e(A_{\mathfrak{p}}) = e(B_{\mathfrak{P}})$.

*Proof.* We put: $R = A_{\mathfrak{p}}$, $S = B_{\mathfrak{P}} = B_{A-\mathfrak{p}}$ (see, for example, [1], p. 40), $L = \Lambda_{A-\mathfrak{p}}$, $L' = \Lambda'_{A-\mathfrak{p}}$. Then, $L$ is obtained from $R$ by blowing-up $\mathfrak{p}R$, so there is $x \in R$, $x$ regular in $L$ such that $\mathfrak{p}L = xL$ ([4], Proposition 1.1), and we have: $e(R) = 1_R(R/xR)$ ([4], Remark a) p. 657) $= 1_R(L'/xL')$ ([4], Remark b) p. 657, where $J = L'$, $x$ is regular in $R$ since is regular in $L$) $= 1_R(L'/(xL)L') = 1_R(L'/(\mathfrak{p}L)L') = 1_R(L'/\mathfrak{p}L')$. On the other hand, there is also $y \in B$, $y$ regular in $\Lambda'$ and such that $\mathfrak{P}\Lambda' = y\Lambda'$ ([4] Proposition 1.1), so there is $y \in S$, $y$ regular in $L'$, such that $\mathfrak{P}L' = yL'$. Then, as before we have: $e(S) = 1_S(L'/yL') = 1_S(L'/\mathfrak{P}L')$.

Besides, $L'/\mathfrak{p}L'$ (resp. :$L'/\mathfrak{P}L'$) is an $A/\mathfrak{p} = k(\mathfrak{p})$-module (resp. :a $B/\mathfrak{P} = k(\mathfrak{P})$-module), where the scalar product, induced by the structure of $L'$, coincides with the inner product. Then, (Lemma 2.3) we have: $e(R) = 1_R(L'/\mathfrak{p}L') = 1_{k(\mathfrak{p})}(L'/\mathfrak{p}L')$, $e(S) = 1_S(L'/\mathfrak{P}L') = 1_{k(\mathfrak{P})}(L'/\mathfrak{P}L')$. Moreover, $k(\mathfrak{p}) \cong k(\mathfrak{P})$. Then, if 1) holds, in particular $\mathfrak{p}L' = \mathfrak{P}L'$, so we have: $e(R) = 1_{k(\mathfrak{p})}(L'/\mathfrak{p}L') = 1_{k(\mathfrak{p})}(L'/\mathfrak{P}L') = 1_{k(\mathfrak{P})}(L'/\mathfrak{P}L') = e(S)$, i.e. 2). On the contrary, if 2) holds, $1_{k(\mathfrak{p})}(L'/\mathfrak{p}L') = e(R) = e(S) = 1_{k(\mathfrak{P})}(L'/\mathfrak{P}L') = 1_{k(\mathfrak{p})}(L'/\mathfrak{P}L')$, so $M = L'/\mathfrak{p}L'$ and $N = L'/\mathfrak{P}L'$ are two $k(\mathfrak{p})$-vector spaces of the same dimension. On the other hand, since $\mathfrak{p}L' \subset \mathfrak{P}L'$, we have: $M/(\mathfrak{P}L'/\mathfrak{p}L')$ and $N$ are isomorphic as $k(\mathfrak{p})$-vector spaces. Then, putting $P = \mathfrak{P}L'/\mathfrak{p}L'$, it follows: $\dim_{k(\mathfrak{p})}(M) = \dim_{k(\mathfrak{p})}(N)$, and also $\dim_{k(\mathfrak{p})}(M) - \dim_{k(\mathfrak{p})}(P) = \dim_{k(\mathfrak{p})}(N)$. Therefore, $\dim_{k(\mathfrak{p})}(P) = 0$, so $\mathfrak{p}L' = \mathfrak{P}L'$; this equality implies $\mathfrak{p}\Lambda' = \mathfrak{P}\Lambda'$.

From Proposition 2.4 and Lemma 2.2 it follows

COROLLARY 2.5. *Let $A$, $B$, $A'$ as in Proposition* 2.4. *If $B$ coincides with the gluing, over $\mathfrak{p}$, of the primary ideals belonging to $\mathfrak{p}A'$, and $\mathfrak{P}$ is the only prime ideal of $B$ over $\mathfrak{p}$, the equivalent conditions of Proposition* 2.4 *are satisfied.*

*Proof.* We have: $B = A + \mathfrak{p}A'$ ([7], Lemma 1.2, 1)), so (Lemma 2.2): $\mathfrak{p}A' = \mathfrak{P}A'$, then 1) of Proposition 2.4 holds.

Using the above results and Section 1 we can find necessary and sufficient conditions in order that in $(*)$ each $A_j$ is a gluing, as required. We notice that in $(*)$ each blowing-up concerns a prime ideal $\mathfrak{P}_j \in \operatorname{Ass}(A_j/\mathfrak{b}_j)$ such that $\mathfrak{P}_j \cap A \in \operatorname{Ass}(A/\mathfrak{b})$, where $\mathfrak{b}_j, \mathfrak{b}$ are respectively the conductor of $A_j$ in $\overline{A}$ and of $A$ in $\overline{A}$. In fact, according to the definition of $(*)$, $\mathfrak{P}_j$ is an associated prime of the conductor $\mathfrak{a}$ of $A_j$ in $A_{j+1}$ (Lemma 1.1); besides, $\mathfrak{b}_j \subset \mathfrak{a}$ since $A_j \subset A_{j+1} \subset \overline{A}$. Then, $\mathfrak{P}_j \supset \mathfrak{b}_j$, so $\mathfrak{P}_j \in \operatorname{Ass}(A_j/\mathfrak{b}_j)$. This implies: $\mathfrak{p} = \mathfrak{P}_j \cap A \in \operatorname{Ass}(A/\mathfrak{b})$. In fact, putting $S = A - \mathfrak{p}$, we have $S^{-1}A_j \subsetneqq S^{-1}\overline{A}$ (otherwise $(A_j)_{\mathfrak{P}_j} = (S^{-1}A_j)_{S^{-1}A_j - \mathfrak{P}_j S^{-1}A_j} = (S^{-1}\overline{A})_{S^{-1}A_j - \mathfrak{P}_j S^{-1}A_j} = \overline{A}_{A_j - \mathfrak{P}_j}$, with $\mathfrak{P}_j \in \operatorname{Ass}(A_j/\mathfrak{b}_j)$, contradiction); then, a fortiori we have: $S^{-1}A \subsetneqq S^{-1}\overline{A}$, so $\mathfrak{p} \in \operatorname{Ass}(A/\mathfrak{b})$.

Let $\operatorname{Ass}(A/\mathfrak{b}) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$. In general (see remark after Proposition 1.7), for each $\mathfrak{p}_i \in \operatorname{Ass}(A/\mathfrak{b})$ there are in $(*)$ $n_i \geqslant 1$ rings obtained by blowing-up prime ideals which are over $\mathfrak{p}_i$. So, we write $(*)$ in such a way to point out this fact:

$$(*)' \qquad \begin{aligned} A = A_1 \subset \cdots \subset A_{j_1} \subset A_{j_1+1} \subset \cdots \subset A_{j_2} \subset A_{j_2+1} \\ \subset \cdots \subset A_{j_k} \subset A_{j_k+1} = A_n = \overline{A}\ , \end{aligned}$$

meaning that, for $i = 0, \cdots, k-1$, $A_{j_i+2}, \cdots, A_{j_{i+1}+1}$ are obtained by blowing-up respectively $\mathfrak{P}_{j_i+1} \in \operatorname{Spec}(A_{j_i+1}), \cdots, \mathfrak{P}_{j_{i+1}} \in \operatorname{Spec}(A_{j_{i+1}})$, where $\mathfrak{P}_{j_i+1} \cap A = \cdots = \mathfrak{P}_{j_{i+1}} \cap A = \mathfrak{p}_{i+1}$ (we put: $j_0 = 0$).

THEOREM 2.6. *With the above notations, we assume*: $k(\mathfrak{P}_j) = k(\mathfrak{p})$ *for each $\mathfrak{P}_j \in \operatorname{Spec} A_j$, $\mathfrak{p} \in \operatorname{Spec} A$ such that $\mathfrak{p} = \mathfrak{P}_j \cap A$. The following conditions are equivalent*:

1) *in the sequence $(*)'$ each $A_j$ is the gluing, over $\mathfrak{p} = \mathfrak{P}_j \cap A$, of the primary ideals belonging to $\mathfrak{p}A_{j+1}$ $(j = 1, \cdots, n-1)$*

2) *for $j = 1, \cdots, n-1$, $\mathfrak{P}_j$ is the only prime ideal in $A_j$ over $\mathfrak{p} = \mathfrak{P}_j \cap A$, and* $\operatorname{emdim}((A_j)_{\mathfrak{P}_j}) = e((A_j)_{\mathfrak{P}_j}) = e(A_{\mathfrak{p}})$.

*Proof.* It is enough to prove: 1) $\Leftrightarrow$ 2) for each $i = 0, \cdots, k-1$ and each $j \in \{j_i + 1, \cdots, j_{i+1}\}$.

Let us localize $(*)'$ at $S = A - \mathfrak{p}_{i+1}$. We obtain:

$$A_{\mathfrak{p}_{i+1}} = S^{-1}A_1 \subset \cdots \subset S^{-1}A_{j_i} \subset S^{-1}A_{j_i+1} \subset S^{-1}A_{j_i+2} \subset \cdots \subset S^{-1}A_{j_{i+1}} \subset \cdots \subset S^{-1}\overline{A}\ ,$$

where, for each $j$, $S^{-1}A_{j+1}$ is the ring obtained from $S^{-1}A_j$ by blowing-up $\mathfrak{P}_j S^{-1}A_j$. Now, we have: $S^{-1}A_2 = \cdots = S^{-1}A_{j_i+1} = A_{\mathfrak{p}_{i+1}}$. In fact, these rings are obtained by blowing-up prime ideals which are not over $\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}$; so, after calling $\mathfrak{a}_j$ the conductor of $A_{\mathfrak{p}_{i+1}}$ in $S^{-1}A_j$ ($j = 2, \cdots, j_i + 1$), we have: $\sqrt{\mathfrak{a}_j}$ contains a product of prime ideals $\mathfrak{P}_{a_1} \cdots \mathfrak{P}_{a_k}$, where $\mathfrak{P}_{a_1} \cap A_{\mathfrak{p}_{i+1}} \neq \mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}, \cdots, \mathfrak{P}_{a_k} \cap A_{\mathfrak{p}_{i+1}} \neq \mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}$ (see Lemma 1.1 and Lemma 1.4), so that no prime ideal belonging to $\mathfrak{a}_j$ coincides with $\mathfrak{p}_{i+1}$ for $j = 2, \cdots, j_i + 1$. Besides, $S^{-1}A_{j_{i+1}+1} = \cdots = S^{-1}A_{j_k} = S^{-1}\overline{A} = \overline{A_{\mathfrak{p}_{i+1}}}$. In fact, for $j = j_{i+1} + 1, \cdots, j_k$, because of the definition of $(*)'$, no prime ideal belonging to the conductor of $A_j$ in $\overline{A}$ lies over $\mathfrak{p}_{i+1}$, so that the conductor of $S^{-1}A_j$ in $S^{-1}\overline{A} = \overline{A_{\mathfrak{p}_{i+1}}}$ is not a proper ideal. Owing to these facts, the localization of $(*)'$ at $S$ is:

$$A_{\mathfrak{p}_{i+1}} = S^{-1}A_{j_i+1} \subset S^{-1}A_{j_i+2} \subset \cdots \subset S^{-1}A_{j_{i+1}} \subset \overline{A_{\mathfrak{p}_{i+1}}}\ ,$$

where the first blowing-up concerns $\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}$.

1) $\Rightarrow$ 2) For each $j \in \{j_i + 2, \cdots, j_{i+1}\}$, $\mathfrak{P}_j$ is the only prime ideal in $A_j$ over $\mathfrak{p}_{i+1}$ ([6], osserv. II); besides, $S^{-1}A_j$ contains the ring obtained from $A_{\mathfrak{p}_{i+1}}$ by blowing-up $\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}$, it coincides with the gluing, over $\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}$, of the primary ideals belonging to $\mathfrak{p}_{i+1}S^{-1}A_{j+1}$, and contains $\mathfrak{P}_j S^{-1}A_j$ as the only prime ideal over $\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}$. Then (Corollary 2.5) we have: $e(A_{\mathfrak{p}_{i+1}}) = e((S^{-1}A_j)_{\mathfrak{P}_j S^{-1}A_j})$, so $e(A_{\mathfrak{p}_{i+1}}) = e((A_j)_{\mathfrak{P}_j})$ because $(S^{-1}A_j)_{\mathfrak{P}_j S^{-1}A_j} = (A_j)_{\mathfrak{P}_j}$. Moreover, in $A_{j_i+1}$, $\mathfrak{P}_{j_i+1}$ is the only prime ideal over $\mathfrak{p}_{i+1}$, and we have also: $S^{-1}A_{j_i+1} = A_{\mathfrak{p}_{i+1}}$. So, $A_{\mathfrak{p}_{i+1}} = S^{-1}A_{j_i+1} = (A_{j_i+1})_{\mathfrak{P}_{j_i+1}}$, then $e(A_{\mathfrak{p}_{i+1}}) = e((A_{j_i+1})_{\mathfrak{P}_{j_i+1}})$. So, for $j \in \{j_i + 1, \cdots, j_{i+1}\}$ we have: $e(A_{\mathfrak{p}_{i+1}}) = e((A_j)_{\mathfrak{P}_j})$. On the other hand, $A_j$, being the gluing over $\mathfrak{p}_{i+1}$ of the primary ideals of $\mathfrak{p}_{i+1}A_{j+1}$, is also the gluing, over $\mathfrak{P}_j$, of the primary ideals of $\mathfrak{P}_j A_{j+1}$ ([7], Lemma 1.2, 2)); then, owing to Lemma 2.1: $\operatorname{emdim}((A_j)_{\mathfrak{P}_j}) = e((A_j)_{\mathfrak{P}_j})$. It follows: $\operatorname{emdim}((A_j)_{\mathfrak{P}_j}) = e((A_j)_{\mathfrak{P}_j}) = e(A_{\mathfrak{p}_{i+1}})$ for $j \in \{j_i + 1, \cdots, j_{i+1}\}$.

2) $\Rightarrow$ 1) Let $i \in \{0, \cdots, k-1\}$. For each $j \in \{j_i + 1, \cdots, j_{i+1}\}$, we have $\operatorname{emdim}((A_j)_{\mathfrak{P}_j}) = e((A_j)_{\mathfrak{P}_j})$, so (Lemma 2.1): $A_j$ coincides with the gluing,

over $\mathfrak{P}_j$, of the primary ideals of $\mathfrak{P}_j A_{j+1}$. Then, owing to [6], Proposition 1.5 we have: $A_j = \{x \in A_{j+1} \mid x \bmod (\mathfrak{P}_j A_{j+1}) \in f(k(\mathfrak{P}_j))\}$, where $f$ is the canonical embedding: $k(\mathfrak{P}_j) \hookrightarrow T^{-1}(A_{j+1}/\mathfrak{P}_j A_{j+1})$, $T = A_j/\mathfrak{P}_j - \{\bar{0}\}$. We want to prove: $A_j$ is the gluing, over $\mathfrak{p}_{i+1}$, of the primary ideals of $\mathfrak{p}_{i+1}A_{j+1}$, that is $A_j = \{x \in A_{j+1} \mid x \bmod (\mathfrak{p}_{i+1}A_{j+1}) \in \varphi(k(\mathfrak{p}_{i+1}))\}$, where $\varphi$ is the canonical map: $k(\mathfrak{p}_{i+1}) \hookrightarrow U^{-1}(A_{j+1}/\mathfrak{p}_{i+1}A_{j+1})$, $U = A/\mathfrak{p}_{i+1} - \{\bar{0}\}$.

Now, $U = k(\mathfrak{p}_{i+1}) - \{\bar{0}\} = k(\mathfrak{P}_j) - \{\bar{0}\}$ (for the assumptions) $= T$, so the hypothesis on $A_j$ can be written: $A_j = \{x \in A_{j+1} \mid x \bmod (\mathfrak{P}_j A_{j+1}) \in \varphi(k(\mathfrak{p}_{i+1}))\}$, and it is enough to prove: $\mathfrak{p}_{i+1}A_{j+1} = \mathfrak{P}_j A_{j+1}$.

Let $S = A - \mathfrak{p}_{i+1}$. As before seen, for $j \in \{j_i + 2, \cdots, j_{i+1}\}$, $S^{-1}A_j$ is local, with maximal ideal $\mathfrak{P}_j S^{-1}A_j$, and contains the ring $S^{-1}A_{j_i+2}$, obtained from $A_{\mathfrak{p}_{i+1}}$ by blowing-up $\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}$. Moreover, $e((A_j)_{\mathfrak{P}_j}) = e(A_{\mathfrak{p}_{i+1}})$, so $e(A_{\mathfrak{p}_{i+1}}) = e((S^{-1}A_j)_{\mathfrak{P}_j S^{-1}A_j})$; besides, $k(\mathfrak{P}_j S^{-1}A_j) = k(\mathfrak{P}_j) = k(\mathfrak{p}_{i+1}) = k(\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}})$. Then, owing to Proposition 2.4, we have: $\mathfrak{p}_{i+1}S^{-1}A_j = \mathfrak{P}_j S^{-1}A_j$, and this implies $\mathfrak{p}_{i+1}A_{j+1} = \mathfrak{P}_j A_{j+1}$, for the assumptions on $S$. So, the result follows for $j \in \{j_i + 2, \cdots, j_{i+1}\}$. As regards $A_{j_i+1}$, we know that $S^{-1}A_{j_i+1} = A_{\mathfrak{p}_{i+1}}$, so its maximal ideal $\mathfrak{P}_{j_i+1}S^{-1}A_{j_i+1}$ equals $\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}} = \mathfrak{p}_{i+1}S^{-1}A_{j_i+1}$; then, $\mathfrak{P}_{j_i+1}S^{-1}A_{j_i+2} = \mathfrak{p}_{i+1}S^{-1}A_{j_i+2}$. Then, the result follows for each $j \in \{j_i + 1, \cdots, j_{i+1}\}$.

Now, we show certain classes of rings, such that $(*)$ satisfies the two equivalent conditions of Theorem 2.6.

COROLLARY 2.7. *Under the same assumptions as in Theorem* 2.6, *a ring $A$ such that* ${}^{A}\sqrt{\mathfrak{b}} = \mathfrak{b}$, $\mathfrak{b} = A :_A \bar{A}$, *satisfies condition* 1) *of Theorem* 2.6.

*Proof.* We shall prove that $A$ satisfies 2) of Theorem 2.6; it is enough to show that this condition holds for each $i \in \{0, \cdots, k-1\}$, $j \in \{j_i + 1, \cdots, j_{i+1}\}$, if ${}^{A}\sqrt{\mathfrak{b}} = \mathfrak{b}$. So, let $i \in \{0, \cdots, k-1\}$, $S = A - \mathfrak{p}_{i+1}$. At the beginning of the proof of Theorem 2.6 we showed that the localization of $(*)'$ at $S$ is:

$$A_{\mathfrak{p}_{i+1}} = S^{-1}A_{j_i+1} \subset S^{-1}A_{j_i+2} \subset \cdots \subset S^{-1}A_{j_{i+1}} \subset \overline{A_{\mathfrak{p}_{i+1}}}\,.$$

In this particular case, we have: $A_{\mathfrak{p}_{i+1}} = S^{-1}A_{j_i+1} \subset S^{-1}A_{j_i+2} = \cdots = \overline{A_{\mathfrak{p}_{i+1}}}$, since (as we shall prove) the conductor of $S^{-1}A_{j_i+2}$ in $\overline{A_{\mathfrak{p}_{i+1}}}$ is not a proper ideal. Let $\mathfrak{b}_{j_i+2}$ be the conductor of $A_{j_i+2}$ in $\bar{A}$; then, the conductor of $S^{-1}A_{j_i+2}$ in $\overline{A_{\mathfrak{p}_{i+1}}}$ is $\mathfrak{b}_{j_i+2}S^{-1}A_{j_i+2}$. If this ideal is proper, it is the intersection of the prime ideals $\mathfrak{P}_{a_1}, \cdots, \mathfrak{P}_{a_r}$ of $S^{-1}A_{j_i+2}$ such that $\{\mathfrak{P}_{a_j} \cap A_{\mathfrak{p}_{i+1}}, j = 0, \cdots, r\} = \operatorname{Ass}(A_{\mathfrak{p}_{i+1}}/\mathfrak{b}A_{\mathfrak{p}_{i+1}}) - \{\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}\}$ (see Corollary 1.10); but $\mathfrak{b}A_{\mathfrak{p}_{i+1}} = \mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}$, since $\mathfrak{b} = {}^{A}\sqrt{\mathfrak{b}}$, so $\mathfrak{b}_{j_i+2}S^{-1}A_{j_i+2}$ is not proper.

So, it follows that in $(*)'$ the only "link" concerning blowing-up of prime ideals over $\mathfrak{p}_{i+1}$ is $A_{j_i+1} \subset A_{j_i+2}$; then, it is enough to show that $A_{j_i+1}$ satisfies 2) of Theorem 2.6. Indeed, we have: in $A_{j_i+1}$, $\mathfrak{P}_{j_i+1}$ is the only prime ideal over $\mathfrak{p}_{i+1}$, because $S^{-1}A_{j_i+1} = A_{\mathfrak{p}_{i+1}}$ is local, and its maximal ideal is $\mathfrak{P}_{j_i+1}S^{-1}A_{j_i+1}$. So, we have also: $(A_{j_i+1})_{\mathfrak{P}_{j_i+1}} = S^{-1}A_{j_i+1}$ ([1], p. 40) $= A_{\mathfrak{p}_{i+1}}$. Besides, since ${}^{A}\sqrt{\mathfrak{b}} = \mathfrak{b}$, the conductor of $A_{\mathfrak{p}_{i+1}}$ in $\overline{A_{\mathfrak{p}_{i+1}}}$ is $\mathfrak{p}_{i+1}A_{\mathfrak{p}_{i+1}}$, then, owing to the above facts, we have also: the conductor of $S^{-1}A_{j_i+1}$ in $S^{-1}A_{j_i+2}$ is $\mathfrak{p}_{i+1}S^{-1}A_{j_i+1}$, which equals $\mathfrak{P}_{j_i+1}S^{-1}A_{j_i+1}$. It follows (Lemma 2.1): $\operatorname{emdim}((S^{-1}A_{j_i+1})_{\mathfrak{P}_{j_i+1}S^{-1}A_{j_i+1}}) = e((S^{-1}A_{j_i+1})_{\mathfrak{P}_{j_i+1}S^{-1}A_{j_i+1}})$, i.e. $\operatorname{emdim}((A_{j_i+1})_{\mathfrak{P}_{j_i+1}}) = e((A_{j_i+1})_{\mathfrak{P}_{j_i+1}})$. So, $A_{j_i+1}$ is as required.

COROLLARY 2.8. *Under the same assumptions as in Theorem* 2.6, *if $A$ is seminormal, then $A$ satisfies condition* 1) *of Theorem* 2.6.

*Proof.* If $A$ is seminormal, then ${}^{A}\sqrt{\mathfrak{b}} = \mathfrak{b}$; so, we can apply Corollary 2.7.

COROLLARY 2.9. *Under the same assumptions as in Theorem* 2.6, *let $A$ be local, analytically irreducible and such that* $\operatorname{emdim}(A) = 2$. *Then, condition* 1) *of Theorem* 2.6 *holds if and only if* $e(A) = 2$.

*Proof.* If $A$ satisfies 1) of Theorem 2.6, in particular we have: $e(A) = \operatorname{emdim}(A) = 2$. (Theorem 2.6). On the contrary, suppose $e(A) = 2$. For each $A_j$ in $(*)'$, $A_j$ is a local ring (since $\overline{A}$ is a discrete valuation ring, see [3], p. 748), so it is enough to prove: $\operatorname{emdim}(A_j) = e(A_j) = e(A) = 2$. Let $\mathfrak{m}$ (resp.: $\mathfrak{P}_j$) *be* the maximal ideal of $A$ (resp.: of $A_j$). We have: $e(A_j) \leqslant e(A)$. In fact, $e(A) = 1_A(A/xA)$ (for a suitable regular $x$) $= 1_A(\overline{A}/\mathfrak{m}\overline{A})$ (see [4], Remark a), b) p. 657, Lemma 1.8), and also $e(A_j) = 1_{A_j}(\overline{A}_j/\mathfrak{P}_j\overline{A}_j)$ (see [4], as above). Now, $\overline{A}_j = \overline{A}$, so $e(A_j) = 1_{A_j}(\overline{A}/\mathfrak{P}_j\overline{A})$. Besides, owing to Lemma 2.3, putting $k = k(\mathfrak{m}) = k(\mathfrak{P}_j)$, we have: $1_A(\overline{A}/\mathfrak{m}\overline{A}) = 1_k(\overline{A}/\mathfrak{m}\overline{A})$, $1_{A_j}(\overline{A}/\mathfrak{P}_j\overline{A}) = 1_k(\overline{A}/\mathfrak{P}_j\overline{A})$. We have also: $\overline{A}/\mathfrak{P}_j\overline{A}$ is isomorphic to $(\overline{A}/\mathfrak{m}\overline{A})/(\mathfrak{P}_j\overline{A}/\mathfrak{m}\overline{A})$ as a $k$-vector space. So, $e(A_j) = 1_k(\overline{A}/\mathfrak{P}_j\overline{A}) = 1_k(\overline{A}/\mathfrak{m}\overline{A}) - 1_k(\mathfrak{P}_j\overline{A}/\mathfrak{m}\overline{A}) \leqslant 1_k(\overline{A}/\mathfrak{m}\overline{A}) = e(A)$.

Then, we have: $\operatorname{emdim}(A_j) \leqslant e(A_j)$ ([4], Corollary 1.10) $\leqslant e(A)$ (as before seen) $= 2$. On the other hand, $\operatorname{emdim}(A_j) \geqslant 2$, because $A_j$ is not regular. It follows: $\operatorname{emdim}(A_j) = e(A_j) = e(A) = 2$.

So, Corollary 2.9 shows that, if $C$ is an analytically irreducible plane curve with singular point $P$, the local ring of $C$ at $P$ satisfies condition 1) of Theorem 2.6 if and only if $P$ is a double point. Also for a larger class of analytically irreducible curves we can characterize the rings $A$ satisfying condition 1) of Theorem 2.6: see the next Corollary 2.10, which shows how Proposition 2.3 of [7] can be deduced from Theorem 2.6.

Let $A$ be the local ring of a monomial curve: $A = k[\![t^{n_1}, \cdots, t^{n_p}]\!]$, with $k$ algebrically closed. By $S = \langle n_1, \cdots, n_p \rangle$ we denote the semigroup generated by $n_1, \cdots, n_p$.

COROLLARY 2.10. *Let* $A = k[\![t^{n_1}, \cdots, t^{n_p}]\!]$, *where* $n_1 < \cdots < n_p$ *generate minimally* $S = \langle n_1, \cdots, n_p \rangle$. *Then, condition* 1) *of Theorem* 2.6 *holds if and only if* $n_2 \equiv 1 \bmod (n_1)$, $n_j = n_{j-1} + 1$ *for* $3 \leqslant j \leqslant p$.

*Proof.* Since each $A_j$ in $(*)$ is local, it is enough to prove: "emdim $(A_j)$ $= e(A_j) = e(A)$ for $j = 1, \cdots, n-1$" if and only if "$n_2 \equiv 1 \bmod (n_1)$, $n_h = n_{h-1} + 1$ for $3 \leqslant h \leqslant p$" (see Theorem 2.6). One has: the condition "$e(A_j)$ $= e(A)$ for $j = 1, \cdots, n-1$" is equivalent to "$n_2 \equiv 1 \bmod (n_1)$". In fact, if $e(A_j) = e(A)$, $j \in \{1, \cdots, n-1\}$ then $e(A_j) = n_1$; it implies that the remainder $r$ of the division of $n_2$ by $n_1$ is equal to 1, otherwise there is a $A_j$ such that $e(A_j) = r < n_1$, for a $j \in \{1, \cdots, n-1\}$ (see [7], Lemma 2.1). Contrariwise, if $n_2 \equiv 1 \bmod (n_1)$, owing to Lemma 2.1 of [7] we have: $e(A_j) = n_1$ for $j = 1, \cdots, n-1$, so $e(A_j) = e(A)$

Now, it is enough to apply Proposition 3.1 and Theorem 1.5 of [7], to complete the proof.

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*Istituto di Matematica*
*Università di Genova*
*Via L. B. Alberti 4*
*16132 Genova*
*Italy*