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THE INVARIANT POLYNOMIAL ALGEBRAS FOR THE GROUPS $ISL(n)$ AND $ISp(n)$

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§ 0. Main theorems

This paper is a continuation to the previous one [3]. We shall show that, for the inhomogeneous linear group $ISL(n+1, \boldsymbol{R})$ (resp. $ISp(n, \boldsymbol{R})$), the coadjoint invariant polynomial algebra is generated by one (resp. n) algebraically independent element. We shall state our results more precisely.

(i) $ISL(n+1, \boldsymbol{R})$, $(n \geqq 1)$.

We can consider the following vector space $\mathfrak{H}_n$ to be a subspace of the dual space realized as in Section 1 of the Lie algebra of $ISL(n+1, \boldsymbol{R})$;

$$\mathfrak{H}_n = \left\{ \begin{pmatrix} 0 & & & 0 & 0 \\ & \ddots & 0 & \vdots & \vdots \\ y_{21} & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & 0 \\ & & y_{n+1,n} & 0 & y_{n+1} \end{pmatrix} \right\}.$$

Let $t = (\prod_{k=1}^{n} y_{k+1,k}^{k}) y_{n+1}^{n+1}$ be a polynomial function on $\mathfrak{H}_n$. Denote by $\mathscr{I}_n$ the C-algebra of the coadjoint invariant polynomial functions on the dual space of the Lie algebra of $ISL(n+1, \boldsymbol{R})$.

THEOREM 1. *The restriction map of $\mathscr{I}_n$ into the set of polynomials on $\mathfrak{H}_n$ is an injective algebra-homomorphism, whose image is $\boldsymbol{C}[t]$.*

(ii) $ISp(n, \boldsymbol{R})$ $(n \geqq 1)$.

In this case we can consider the following vector space $\mathfrak{H}_n$ to be a subspace of the dual space realized as in Section 2 of the Lie algebra of $ISp(n, \boldsymbol{R})$;

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$$\mathfrak{H}_n = \left\{ \begin{pmatrix} 0 & y_{12} & & & & 0 & 0 & 0 \\ y_{21} & 0 & \ddots & 0 & & \vdots & \vdots & \vdots \\ & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ 0 & & \ddots & 0 & y_{2n-3,2n-2} & \vdots & \vdots & \vdots \\ & & & y_{2n-2,2n-3} & 0 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & y_{2n-1,2n} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & y_{2n} \end{pmatrix} \right\}.$$

Let $s_i (0 \leqq i \leqq n-1)$ be the i-th fundamental symmetric polynomial in $y_{2k,2k-1} y_{2k-1,2k}$ $(1 \leqq k \leqq n=1)$ and set $t_i = s_i y_{2n-1,2n} y_{2n}^2$. It is not difficult to see that t_i $(0 \leqq i \leqq n-1)$ are algebraically independent over $\boldsymbol{C}$. Denote by $\mathscr{I}_n$ the algebra of coadjoint invariant polynomial functions on the dual space of the Lie algebra of $ISp(n, \boldsymbol{R})$.

THEOREM 2. *The restriction map of $\mathscr{I}_n$ into the set of the polynomials on $\mathfrak{H}_n$ is an injective algebra-homomorphism, whose image is $\boldsymbol{C}[t_0, \cdots, t_{n-1}]$.*

The proofs of Theorems 1 and 2 will be given in Section 1 and Section 2 respectively.

§ 1. The group $ISL(n+1, \boldsymbol{R})$

Let G_n and IG_n be the Lie groups $SL(n+1, \boldsymbol{R})$, and $ISL(n+1, \boldsymbol{R})$, respectively. Denote by $\mathfrak{g}_n$ and $I\mathfrak{g}_n$ their Lie algebras respectively. To be definite,

$IG_n = \left\{ \begin{pmatrix} u & a \\ o & 1 \end{pmatrix}; u \in G_n, a \in \boldsymbol{R}^{n+1} \right\}$, $I\mathfrak{g}_n = \left\{ \begin{pmatrix} X & x \\ o & 0 \end{pmatrix}; X \in \mathfrak{g}_n, x \in \boldsymbol{R}^{n+1} \right\}$. We can identify the dual space $I\mathfrak{g}_n^*$ of $I\mathfrak{g}_n$ with $\mathfrak{g}_n \times \boldsymbol{R}^{n+1}$ via the following bilinear form on $I\mathfrak{g}_n \times (\mathfrak{g}_n \times \boldsymbol{R}^{n+1})$;

$$\left\langle \begin{pmatrix} X & x \\ o & 0 \end{pmatrix}, (Y, y) \right\rangle = \langle X, Y \rangle_{s\ell(n+1)} + \langle x, y \rangle_{n+1},$$

where $\langle X, Y \rangle_{s\ell(n+1)} = 2(n+1) \operatorname{tr}(XY)$ i.e. the Killing form of the Lie algebra $\mathfrak{g}_n$ [4, p. 390] and $\langle x, y \rangle_{n+1} = {}^t x y$. Clearly the following e_i, e_{jk} and f_ℓ $(1 \leqq i \leqq n,\ 1 \leqq j, k, \ell \leqq n+1, j \neq k)$ form a basis of $I\mathfrak{g}_n$;

$$e_i = \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{bmatrix} ⓘ, \qquad e_{jk} = \begin{array}{c} \begin{pmatrix} & & & 0 & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \end{pmatrix} ⓙ \\ ⓚ \end{array}$$

and $f_\ell = {}^t(0 \cdots 0 \overset{\text{①}}{1} 0 \cdots 0)$. The dual basis is given by the following $\hat{e}_i$, $\hat{e}_{jk}$ and $\hat{f}_\ell$;

$$\hat{e}_i = -\frac{1}{2(n+1)^2}\begin{bmatrix} i-(n+1) & & & & \\ & \ddots & & & \\ & & i-(n+1) & & \\ & & & i & \\ & & & & \ddots \\ & & & & & i \end{bmatrix} \text{ⓘ}, \qquad \hat{e}_{jk} = \frac{1}{2(n+1)} e_{kj}$$

and $\hat{f}_\ell = f_\ell$. For a $g = \begin{pmatrix} u & a \\ o & 1 \end{pmatrix} \in IG_n$ we have

$$\left\langle g^{-1}\begin{pmatrix} X & x \\ o & 0 \end{pmatrix} g, (Y, y)\right\rangle = \langle X, uYu^{-1}\rangle_{s\ell(n+1)} + \langle Xa + x, {}^tu^{-1}y\rangle_{n+1}.$$

Consequently the coadjoint action $\mathrm{CoAd}(g)$ of g is given by

$$\mathrm{CoAd}(g)(Y, y) = (uYu^{-1} + A, {}^tu^{-1}y)$$

with $A = \sum \langle \omega a, {}^tu^{-1}y\rangle_{n+1}\hat{\omega}$, where ω ranges the above bases of $\mathfrak{g}_n$ (not of $I\mathfrak{g}_n$). In the sequel we shall use the notation $g\cdot(Y, y)$ for $\mathrm{CoAd}(g)(Y, y)$. Moreover, we identify G_n and $\boldsymbol{R}^{n+1}$ with the subgroups $\left\{\begin{pmatrix} u & o \\ o & 1 \end{pmatrix}; u \in G_n\right\}$ and $\left\{\begin{pmatrix} I_{n+1} & a \\ o & 1 \end{pmatrix}; a \in \boldsymbol{R}^{n+1}\right\}$ of IG_n respectively. Denote by $\mathscr{I}_n$ the algebra of IG_n-invariant polynomial functions on $I\mathfrak{g}_n^*$. $\mathfrak{H}_n$ stands for the same as in Section 0. The Theorem 1 is an easy consequence of the following three lemmas.

LEMMA 1.1. *The union of the orbits $\{g\cdot\mathfrak{H}_n; g \in IG_n\}$ is dense in $I\mathfrak{g}_n^* = \mathfrak{g}_n \times \boldsymbol{R}^{n+1}$. In particular the restriction map $F \to F|\mathfrak{H}_n$ of $\mathscr{I}_n$ into the set of polynomial functions on $\mathfrak{H}_n$ is an injective algebra-homomorphism.*

Proof. We can show that almost all $(Y, y) \in I\mathfrak{g}_n^*$ is conjugate to some element of $\mathfrak{H}_n$. Indeed, if $y_{n+1} \neq 0$, there exists a $u \in G_n$ such that $u\cdot(Y, y) = (uYu^{-1}, {}^tu^{-1}y)$ with $({}^tu^{-1}y) = {}^t(0, \cdots, 0, y_{n+1})$. Taking some $a \in \boldsymbol{R}^{n+1}$, we obtain

$$a\cdot u\cdot(Y, y) = \begin{bmatrix} & 0 & 0 \\ * & \vdots & \vdots \\ & 0 & 0 \\ & 0 & y_{n+1} \end{bmatrix}.$$

To complete the proof, it suffices to reverse the procedure in the proof of the following Lemma 1.3. The details, however, will be omitted.

LEMMA 1.2. *Let F be a homogeneous element of $\mathscr{I}_n$. Then the restriction $F|\mathfrak{H}_n$ takes the form $c((\prod_{k=1}^n y_{k+1,k}^k)y_{n+1}^{n+1})^m$ for some constant c and non-negative integer m.*

Proof. Let d be the degree of F. Then the restriction $F|\mathfrak{H}_n$ can be written as

$$\sum_{\alpha_1+\cdots+\alpha_{n+1}=d} a_{\alpha_1,\cdots,\alpha_{n+1}}\left(\prod_{k=1}^n y_{k+1,k}^{\alpha_k}\right)y_{n+1}^{\alpha_{n+1}}.$$

Since $F|\mathfrak{H}_n$ is invariant under the action of the diagonal matrix $[c_1,\cdots,c_{n+1}] \in G_n$, $F|\mathfrak{H}_n$ must be equal to

$$\sum_{\alpha_1+\cdots+\alpha_{n+1}=d} b_{\alpha_1,\cdots,\alpha_{n+1}}\left(\prod_{k=1}^n y_{k+1,k}^{\alpha_k}\right)y_{n+1}^{\alpha_{n+1}}$$

with $b_{\alpha_1,\cdots,\alpha_{n+1}} = a_{\alpha_1,\cdots,\alpha_{n+1}}(\prod_{k=1}^n c_{k+1}^{\alpha_1+\alpha_k-\alpha_{k+1}})$. It is now immediate that $a_{\alpha_1,\cdots,\alpha_{n+1}} = 0$ unless $\alpha_k = k\alpha_1$ for all k.

LEMMA 1.3. *There exists one and only one polynomial F in $\mathscr{I}_n$ such that the restriction $F|\mathfrak{H}_n$ takes the form $(\prod_{k=1}^n y_{k+1,k}^k)y_{n+1}^{n+1}$.*

Proof. Define subspaces $\mathscr{Y}_k$ of $I\mathfrak{g}_n^* = \mathfrak{g}_n \times \boldsymbol{R}^{n+1}$ and subgroups $G_{n,k}$ of G_n as follows $(1 \leqq k \leqq n+1)$.

$$\mathscr{Y}_k = \left\{\begin{bmatrix} y_{11} & \cdots & y_{1k} & & * & 0 \\ y_{k1} & \cdots & y_{kk} & & & \vdots \\ 0 \cdots 0 & & y_{k+1,k} & 0 & & \vdots \\ & 0 & \ddots & \ddots & \ddots & 0 \\ & & & 0 \quad y_{n+1,n} & 0 & y_{n+1} \end{bmatrix} + \tau\begin{pmatrix} -I_n & o & o \\ o & n & 0 \end{pmatrix};\ \tau \in \boldsymbol{R}\right\}$$

$$(1 \leqq k \leqq n)$$

$$\mathscr{Y}_{n+1} = \left\{\begin{bmatrix} Y & 0 \\ & \vdots \\ & 0 \\ & y_{n+1} \end{bmatrix} + \tau\begin{pmatrix} -I_n & o & o \\ o & n & 0 \end{pmatrix};\ y_{n+1,n+1} = 0,\ \tau \in \boldsymbol{R}\right\}$$

$$G_{n,k} = \left\{\begin{bmatrix} u & z & 0 \\ o & c_0 & \\ 0 & & C \end{bmatrix} \in G_n;\ z \in \boldsymbol{R}^{n-1},\ C = \operatorname{diag}[c_1, \cdots, c_{n-k+1}]\right\}$$

$$(1 \leqq k \leqq n+1).$$

Moreover, set $IG_{n,k} = \left\{\begin{pmatrix} u & a \\ o & 1 \end{pmatrix};\ u \in G_{n,k},\ a \in \boldsymbol{R}^{n+1}\right\}$ $(1 \leqq k \leqq n+1)$. Note that $IG_{n,k}$ leaves $\mathscr{Y}_k$ invariant $(1 \leqq k \leqq n+1)$. Let $Y_k(\tau)$ be a representative of $\mathscr{Y}_k$. Starting with a $G_{n,1}$-invariant polynomial function

$$F_1(Y_1(\tau)) = \left(\prod_{k=1}^{n} y_{k+1,k}^{k}\right) y_{n+1}^{n+1} \quad (Y_1(\tau) \in \mathscr{Y}_1)\,,$$

we shall define a polynomial function $F \in \mathscr{I}_n$ such that $F|\mathscr{Y}_1 = F_1$. For $Y_k(\tau) = Y_k + \tau\begin{pmatrix} -I_n & o & o \\ o & n & 0 \end{pmatrix} \in \mathscr{Y}_k (2 \leqq k \leqq n+1)$, put $z_k = (y_{k,1}, \cdots, y_{k,k-2})/y_{k,k-1}$ and

$$v_k = v(Y_k(\tau)) = \begin{bmatrix} 1 & & & 0 & \\ & \ddots & 0 & \vdots & 0 \\ 0 & & \ddots & 0 & \\ z_k & & 1 & y_{k,k}/y_{k,k-1} & \\ 0 & \cdots\cdots & 0 & 1 & \\ & 0 & & & I_{n-k+1} \end{bmatrix}.$$

By simple calculation we obtain

$$v_k \cdot Y_k(\tau) = \begin{bmatrix} Z_{k-1} & & & * & * & 0 \\ 0 \cdots 0 & y_{k,k-1} & & 0 & * & \vdots \\ \vdots & & \ddots & & \ddots & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 \cdots\cdots 0 & & y_{n+1,n} & & 0 & y_{n+1} \end{bmatrix} + \tau\begin{pmatrix} -I_n & o & o \\ o & n & 0 \end{pmatrix} \in \mathscr{Y}_{k-1}$$

with

$$Z_{k-1} = \begin{bmatrix} 1 & & \\ & \ddots & 0 \\ 0 & \ddots & \\ z_k & & 1 \end{bmatrix} \dot{Y}_{k-1} \begin{bmatrix} 1 & & \\ & \ddots & 0 \\ 0 & \ddots & \\ z_k & & 1 \end{bmatrix}^{-1} + y_{k,k} \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \\ & & & 1 \end{bmatrix},$$

where $\dot{Y}_{k-1}$ denotes the $(k-1) \times (k-1)$-matrix whose (i, j)-component $(1 \leqq i, j \leqq k-1)$ is the one of Y_k. Define functions F_k on $\mathscr{Y}_k$ inductively by $F_k(Y_k(\tau)) = F_{k-1}(v_k \cdot Y_k(\tau))$ $(2 \leqq k \leqq n+1)$. Note that F_k does not depend on $y_{1,j}, \cdots, y_{j-1,j}$ $(j \geqq k)$ nor τ. In particular F_k are invariant under the action of $a \in \boldsymbol{R}^{n+1}$. We shall show by induction on k that F_k are $G_{n,k}$-invariant polynomials. Elementary calculation reveals that $g' = v(g \cdot Y_k(\tau)) g v(Y_k(\tau))^{-1}$ takes the form

$$\begin{bmatrix} u' & & z' & * & \\ 0\cdots 0 & & c' & 0 & 0 \\ 0\cdots\cdots 0 & & & c_0 & \\ 0 & & & & C \end{bmatrix} \qquad \text{for } g = \begin{bmatrix} u & & z & \\ & & & 0 \\ 0\cdots 0 & & c_0 & \\ & 0 & & C \end{bmatrix} \in G_{n,k}\,.$$

We can verify easily that $F_{k-1}(g' \cdot Y_{k-1}(\tau)) = F_{k-1}(\tau)$ even though g' does not necessarily belong to $G_{n,k-1}$. Since $v(g\cdot Y_k(\tau))\cdot(g\cdot Y_k(\tau)) = g'\cdot(v(Y_k(\tau))\cdot Y_k(\tau))$, we have $F_k(g\cdot Y_k(\tau)) = F_k(Y_k(\tau))$. Note now that F_k is a polynomial in all variables except possibly for $y_{k,k-1}$. In case $k \geqq 3$, F_k is a polynomial function on $\mathscr{Y}_k$, because $F_k(g\cdot Y_k(\tau)) = F_k(Y_k(\tau))$ for

$$g = \begin{bmatrix} I_{k-3} & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & I_{n-k+2} \end{bmatrix} \in G_{n,k}\,.$$

In case $k \leqq 2$, $F_k = F_1$, which can be verified easily. To sum up, $F_k(1 \leqq k \leqq n+1)$ is an $IG_{n,k}$-invariant polynomial function on $\mathscr{Y}_k$. Now a function $F \in \mathscr{I}_n$ is to be defined. For $(Y, y) \in I\mathfrak{g}_n^*$, put

$$v = v(Y, y) = \begin{bmatrix} 1 & & \\ & \ddots & 0 \\ 0 & & \ddots \\ {}^t\tilde{y} & & 1 \end{bmatrix} \in G_n \qquad ({}^t\tilde{y} = (y_1, \cdots, y_n)/y_{n+1})\,.$$

Keeping in mind that $v\cdot(Y, y) = (vYv^{-1}, {}^tv^{-1}y) \in \mathscr{Y}_{n+1}$, we define F by $F(Y, y) = F_{n+1}(v\cdot(Y, y))$. Then F belongs to $\mathscr{I}_n$. To see this, firstly we shall show F to be G_n-invariant. By simple calculation we get for $u \in G_n$

$$v(u\cdot(Y, y))uv(Y, y)^{-1} = \begin{pmatrix} * & * \\ o & y_{n+1}/({}^tu^{-1}y)_{n+1} \end{pmatrix} \in G_{n,n+1}\,.$$

Since F_{n+1} is $G_{n,n+1}$-invariant, it follows that $F(u\cdot(Y, y)) = F(Y, y)$. The same argument as for F_k yields that F is a polynomial. Secondly, on account of the $IG_{n,n+1}$-invariance of F_{n+1}, we obtain for $a \in \mathbf{R}^{n+1}(\subset IG_n)$

$$\begin{aligned} F(a\cdot(Y, y)) &= F(vav^{-1}\cdot(v\cdot(Y, y))) = F_{n+1}(vav^{-1}\cdot(v\cdot(Y, y))) \\ &= F_{n+1}(v\cdot(Y, y)) = F(Y, y)\,, \end{aligned}$$

since $\mathbf{R}^{n+1}$ is a normal subgroup of IG_n. This completes the proof of Lemma 1.3.

§ 2. The group $ISp(n, R)$

Let now G_n and IG_n be the Lie groups $Sp(n, \boldsymbol{R})$ and $ISp(n, \boldsymbol{R})$ respectively. Namely,

$$G_n = \{u \in GL(2n, \boldsymbol{R});\ {}^t u J_n u = J_n\} \qquad \text{with } J_n = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

and $IG_n = \left\{\begin{pmatrix} u & a \\ o & 1 \end{pmatrix};\ u \in G_n,\ a \in \boldsymbol{R}^{2n}\right\}$. Denote by $\mathfrak{g}_n$ and $I\mathfrak{g}_n$ their Lie algebras respectively. We may assume $n \geqq 2$, since $G_1 \cong SL(2, \boldsymbol{R})$. We can identify the dual space $I\mathfrak{g}_n^*$ of $I\mathfrak{g}_n$ with $\mathfrak{g}_n \times \boldsymbol{R}^{2n}$ via the following bilinear form on $I\mathfrak{g}_n \times (\mathfrak{g}_n \times \boldsymbol{R}^{2n})$;

$$\left\langle \begin{pmatrix} X & x \\ o & 0 \end{pmatrix}, (Y, y) \right\rangle = \langle X, Y \rangle_{sp(n)} + \langle x, y \rangle_{2n}\,.$$

Here $\langle X, Y \rangle_{sp(n)} = 2(n+1)\operatorname{tr}(XY)$ i.e. the Killing form of $\mathfrak{g}_n$ and $\langle x, y \rangle_{2n} = {}^t x J_n y$. In the sequel we consider G_n and $\boldsymbol{R}^{2n}$ to be the subgroups $\left\{\begin{pmatrix} u & o \\ o & 1 \end{pmatrix};\ u \in G_n\right\}$ and $\left\{\begin{pmatrix} I_{2n} & a \\ o & 1 \end{pmatrix};\ a \in \boldsymbol{R}^{2n}\right\}$ of IG_n respectively. It is not difficult to see that the following e_i, $e_{2i-1,2i}$, $e_{2i,2i-1}$, $e_{j,2k-1}$, $e_{j,2k}$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq 2k-2,\ 2 \leqq k \leqq n)$ and f_ℓ $(1 \leqq \ell \leqq 2n)$ form a basis of $I\mathfrak{g}_n$;

$$e_i = \begin{array}{c} \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{bmatrix} \\ \text{(2i)} \end{array}, \qquad e_{2i-1,2i} = \begin{array}{c} \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 0 & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \\ \text{(2i)} \end{array},$$

$e_{2i,2i-1} = {}^t e_{2i-1,2i}$,

$$e_{j,2k-1} = \begin{bmatrix} O_{2k-2} & z_j & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & 0 \\ 0 \cdots\cdots 0 & & & \\ {}^t(J_{k-1} z_j) & & O_2 & \\ 0 & & & O_{2n-2k} \end{bmatrix}, \qquad \text{with } z_j = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{matrix} \\ \\ \\ \text{(j)} \\ \\ \\ \\ \end{matrix} \end{array} \in \boldsymbol{R}^{2k-2}.$$

$$e_{j,2k} = \begin{bmatrix} O_{2k-2} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & z_j & 0 \\ -{}^t(J_{k-1}z_j) & & O_2 & \\ 0 \cdots\cdots 0 & & & \\ 0 & & & O_{2n-2k} \end{bmatrix}, \quad f_\ell = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ①.$$

Elementary calculation shows that the following $\hat{e}_i$, $\hat{e}_{2i-1,2i}$, $\hat{e}_{2i,2i-1}$, $\hat{e}_{j,2k-1}$, $\hat{e}_{j,2k}$ and $\hat{f}_\ell$ form the dual basis of $I\mathfrak{g}_n^*$;

$$\hat{e}_i = e_i/4(n+1), \quad \hat{e}_{2i-1,2i} = e_{2i,2i-1}/2(n+1),$$
$$\hat{e}_{2i,2i-1} = e_{2i-1,2i}/2(n+1), \quad \hat{e}_{j,2k-1} = e_{j',2k}/2(-1)^j,$$
$$\hat{e}_{j,2k} = e_{j',2k-1}/2(-1)^j, \quad \hat{f}_\ell = -J_n f_\ell \quad (j' = j - (-1)^j).$$

Since

$$g^{-1}\begin{pmatrix} X & x \\ o & 0 \end{pmatrix} g = \begin{pmatrix} u^{-1}Xu & u^{-1}Xa + u^{-1}x \\ o & 0 \end{pmatrix} \quad \text{for } g = \begin{pmatrix} u & a \\ o & 1 \end{pmatrix} \in IG_n,$$

it follows that

$$\mathrm{CoAd}(g)(Y, y) = (uYu^{-1} + A, uy) \qquad \text{with } A = \sum \langle \omega a, uy \rangle_{2n} \hat{\omega},$$

where ω ranges the elements of the basis of $\mathfrak{g}_n$ (not of $I\mathfrak{g}_n$) given above. Simpler notation $g\cdot(Y, y)$ will be used for $\mathrm{CoAd}(g)(Y, y)$. $\mathfrak{H}_n$ and $\mathscr{I}_n$ stand for the same as in Section 0.

LEMMA 2.1. *The union of the orbits* $\{g\cdot\mathfrak{H}_n; g \in IG_n\}$ *contains an open set of* $I\mathfrak{g}_n^* = \mathfrak{g}_n \times \boldsymbol{R}^{2n}$. *In particular the restriction map* $F \to F|\mathfrak{H}_n$ *of* $\mathscr{I}_n$ *into the set of polynomial functions on* $\mathfrak{H}_n$ *is an injective algebra-homomorphism.*

Proof. Denote by $\tilde{\mathfrak{H}}_n$ the union $\bigcup g\cdot\mathfrak{H}_n$ $(g \in IG_n)$. Note that $\tilde{\mathfrak{H}}_n$ contains elements of the form

$$(\dot{Y}) = \begin{bmatrix} \dot{Y} & 0 & 0 \\ 0 & \begin{matrix} 0 & y_{2n-1,2n} \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ y_{2n} \end{matrix} \end{bmatrix} \quad (\dot{Y} \text{ belongs to an open set } \mathscr{O}_{n-1} \text{ of } \mathfrak{g}_{n-1}).$$

This follows from the Proposition 1.3.4.1 [5, p. 101] and the simple fact that the set consisting of the following elements $(u \in G_{n-1}, y_{ij} \in \boldsymbol{R})$

$$u\begin{bmatrix} 0 & y_{12} & & \\ y_{21} & 0 & \ddots & \\ & \ddots & \ddots & y_{2n-3,2n-2} \\ & & y_{2n-2,2n-3} & 0 \end{bmatrix}u^{-1}$$

contains a Cartan subalgebra $\{\lambda_1 e_1 + \cdots + \lambda_{n-1}e_{n-1};\ \lambda_i \in \boldsymbol{R}\}$ of $\mathfrak{g}_{n-1}$. Using the notation in the proof of Lemma 2.3, we have $v_{n+1}\cdot(Y, y, y_{2n}) \in \mathscr{Y}_n$ for $(Y, y, y_{2n}) \in \mathscr{Y}_{n+1}$. In other words, there exists a smooth map of $\mathscr{Y}_{n+1}\backslash\{y_{2n-2}y_{2n-1,2n} = 0\}$ into $\mathscr{Y}_n$, which contains the set $\{a\cdot(\dot{Y});\ \dot{Y} \in \mathscr{O}_{n-1},\ a \in \boldsymbol{R}^{2n}\}$ (recall that $\boldsymbol{R}^{2n}$ is regarded as a subgroup of IG_n). Thus there exists an open set $\mathscr{O}_{n+1}$ of $\mathscr{Y}_{n+1}$. Similar argument shows the existence of an open set $\mathscr{O}$ of $I\mathfrak{g}_n^*$ such that $\mathscr{O} \subset \tilde{\mathfrak{H}}_n$.

Let s_i $(0 \leqq i \leqq n-1)$ be the i-th fundamental symmetric polynomial in $y_{2k,2k-1}y_{2k-1,2k}$ $(1 \leqq k \leqq n-1)$ and set $t_i = s_i y_{2n-1}y_{2n}^2$.

LEMMA 2.2. *Let F be an element of $\mathscr{I}_n$. The restriction $F|\mathfrak{H}_n$ takes the form $\sum_{\alpha_i \geqq 0} a_{\alpha_0,\cdots,\alpha_{n-1}} t_0^{\alpha_0}\cdots t_{n-1}^{\alpha_{n-1}}$.*

Proof. Since $F|\mathfrak{H}_n$ is invariant under the action of the diagonal matrix $[1, \cdots, 1, c, c^{-1}] \in G_n$, it takes the form $\sum_{\beta \geqq 0} B_\beta(y_{2n-1,2n}y_{2n}^2)^\beta$, where B_β are polynomials in $y_{2k,2k-1}$, $y_{2k-1,2k}$ $(1 \leqq k \leqq n-1)$. Moreover, $F|\mathfrak{H}_n$ is invariant under any substitution $y_{2k,2k-1}$, $y_{2k-1,2k}$ for $-y_{2k,2k-1}$, $-y_{2k-1,2k}$ and the permutations of $y_{2k,2k-1}$, $y_{2k-1,2k}$ $(1 \leqq k \leqq n-1)$. Consequently B_β can be written as $\sum_{\alpha_k \geqq 0} b_{\alpha_1,\cdots,\alpha_{n-1},\beta}, s_1^{\alpha_1}\cdots s_{n-1}^{\alpha_{n-1}}$. It remains to prove that $\alpha_1 + \cdots + \alpha_{n-1} \leqq \beta$. By simple calculation we obtain

$$\left[\begin{array}{ccc} & 0 & 0 \\ I_{2n-2} & \vdots & \vdots \\ & 0 & \vdots \\ & \sigma & 0 \\ 0\cdots\cdots 0 & & \\ 0\cdots 0 \quad \sigma \quad 0 & & I_2 \end{array}\right]\cdot\left[\begin{array}{ccccccc} 0 & y_{12} & 0 & 0 & 0 & & 0 \\ y_{21} & 0 & \ddots & \vdots & \vdots & & \vdots \\ 0 & \ddots & y_{2n-3,2n-2} & \vdots & 0 & & \vdots \\ & y_{2n-2,2n} & 0 & 0 & y & & \vdots \\ 0\cdots 0 & -y & 0 & 0 & y_{2n-1,2n} & & 0 \\ 0\cdots\cdots & & 0 & 0 & 0 & & y_{2n} \end{array}\right]$$

$$= \left[\begin{array}{cccc} \left[\begin{array}{ccc} 0 & y_{12} & 0 \\ y_{21} & 0 & \ddots \\ 0 & \ddots & y_{2n-3,2n-2} \\ & y_{2n-3,2n-3} & 0 \end{array}\right] + \dfrac{y^2}{y_{2n-1,2n}}\left[\begin{array}{cccc} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & 0 & \vdots \\ 0 & \cdots 0 & 1 & 0 \end{array}\right] & \begin{array}{c} * \\ * \\ \\ * \end{array} & \begin{array}{c} 0 \\ \vdots \\ \vdots \\ 0 \end{array} & \begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \end{array} \\ 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad 0 & 0 & y_{2n-1,2n} & 0 \\ * \qquad\qquad\qquad * \qquad\qquad\qquad * & * & 0 & y_{2n} \end{array}\right]$$

for $\sigma = -y/y_{2n-1,2n}$. Since the value of F at this point must be represented as a polynomial in y, we conclude that $\alpha_1 + \cdots + \alpha_{n-1} \leqq \beta$ (note that the value of F at this point does not depend on the omitted components: cf. the proof of Lemma 2.1).

LEMMA 2.3. *There exists uniquely* $F^{(i)} \in \mathscr{I}_n$ *such that the restriction* $F^{(i)} | \mathfrak{H}_n = t_i$ $(0 \leqq i \leqq n-1)$.

Proof. Denote by $\mathscr{Y}_k$ (resp. $G_{n,k}$) $(k = n, n+1)$ the following subspaces (resp. subgroups) of $I\mathfrak{g}_n^*$ (resp. G_n);

$$\mathscr{Y}_n = \left\{\begin{bmatrix} \dot{Y} & * & o & \begin{matrix} 0 \\ \vdots \end{matrix} \\ o & * & y_{2n-1,2n} & 0 \\ * & * & * & y_{2n} \end{bmatrix}\right\}, \quad \mathscr{Y}_{n+1} = \left\{\begin{bmatrix} \dot{Y} & * & & \begin{matrix} 0 \\ \vdots \end{matrix} \\ -{}^t(J_{n-1}y) & * & y_{2n-1,2n} & 0 \\ * & * & * & y_{2n} \end{bmatrix}\right\},$$

$$G_{n,n} = \left\{\begin{bmatrix} u & 0 & 0 \\ 0 & * & 0 \\ & * & * \end{bmatrix};\ u \in G_{n-1}\right\}, \quad G_{n,n+1} = \left\{\begin{bmatrix} u & * & o \\ o & * & 0 \\ * & * & * \end{bmatrix};\ u \in G_{n-1}\right\}.$$

Set $IG_{n,k} = \left\{\begin{pmatrix} u & a \\ o & 1 \end{pmatrix};\ u \in G_{n,k},\ a \in \boldsymbol{R}^{2n}\right\}$ $(k = n, n+1)$. We shall define polynomial functions $F_k^{(i)}$ on $\mathscr{Y}_k$ $(k = n, n+1)$. The values of $F_n^{(i)}$ at

$$\begin{bmatrix} \dot{Y} & * & o & \begin{matrix} 0 \\ \vdots \end{matrix} \\ o & * & y_{2n-1,2n} & 0 \\ 0 & * & * & y_{2n} \end{bmatrix}$$

are defined by requiring that $\sum_{i=1}^{n-1} F_n^{(i)} T^{2(n-i-1)} = y_{2n-1,2n} y_{2n}^2 \times \det(T + \dot{Y})$, where T is an indeterminate. Note that the restriction $F_n^{(i)} | \mathscr{Y}_n$ is equal to t_i up to the signature. Keeping in mind that $F_n^{(i)}$ does not depend on the omitted components, we can easily verify that $F_n^{(i)}$ are $G_{n,n}$-invariant. For

$$(\dot{Y}, y, y_{2n}) = \begin{bmatrix} \dot{Y} & * & y & \begin{matrix} 0 \\ \vdots \end{matrix} \\ -{}^t(J_{n-1}y) & * & y_{2n-1,2n} & 0 \\ * & * & * & y_{2n} \end{bmatrix} \in \mathscr{Y}_{n+1} \quad (y \in \boldsymbol{R}^{2n}),$$

put

$$v_{n+1} = v(\dot{Y}, y, y_{2n}) = \begin{bmatrix} I_{2n-4} & o & -\tilde{y} & 0 & 0 \\ {}^t(J_{n-2}\tilde{y}) & 1 & -y_{2n-3}/y_{2n-2} & 0 & \vdots \\ o & 0 & 1 & \sigma & 0 \\ 0 \cdots\cdots & & 0 & & \\ & \sigma{}^t(J_{n-1}y)/y_{2n-2} & & & I_2 \end{bmatrix} \in G_n\,,$$

where ${}^t\tilde{y} = (y_1, \cdots, y_{2n-4})/y_{2n-2}$ and $\sigma = -y_{2n-2}/y_{2n-1,2n}$. Then we have

$$v_{n+1}\cdot(\dot{Y}, y, y_{2n}) = \begin{bmatrix} \left(u\dot{Y}u^{-1} + \begin{pmatrix} 0 & 0 & \\ & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} y^2_{2n-2}/y_{2n-1,2n} \right) & * & 0 & 0 \\ & * & \vdots & \vdots \\ & * & 0 & \vdots \\ 0 \cdots\cdots\cdots 0 & 0 & y_{2n-1,2n} & 0 \\ 0 \cdots\cdots\cdots * & * & 0 & y_{2n} \end{bmatrix}$$

where u is the first $(2n-2)\times(2n-2)$-block of the matrix v_{n+1}. Secondly, defining function $F^{(i)}_{n+1}$ on $\mathscr{Y}_{n+1}$ by $F^{(i)}_{n+1}(\dot{Y}, y, y_{2n}) = F^{(i)}_n(v_{n+1}\cdot(\dot{Y}, y, y_{2n}))$, we shall show that they are $IG_{n,n+1}$-invariant polynomial functions. An element g of $G_{n,n+1}$ can be represented as g_1g_2 for some

$$g_1 = \begin{pmatrix} u & 0 & \\ & c & 0 \\ 0 & b & c^{-1} \end{pmatrix} \in G_{n,n} \quad\text{and}\quad g_2 = \begin{pmatrix} I_{2n-2} & z & o \\ o & & \\ {}^t(J_{n-1}z) & & I_2 \end{pmatrix} \quad (z \in R^{2n-2})\,.$$

Clearly $g_j\cdot(\dot{Y}, y, y_{2n})$ $(j=1,2)$ are equal to

$$\begin{bmatrix} u\dot{Y}u^{-1} & * & c^{-1}uy & 0 \\ -c\,{}^t(J_{n-1}y)u^{-1} & * & c^2y_{2n-1,2n} & 0 \\ * & * & * & c^{-1}y_{2n} \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{Y} - z\,{}^t(J_{n-1}y) - (y + y_{2n-1,2n}z)\,{}^t(J_{n-1}z) & * & * & 0 \\ -{}^t(J_{n-1}y) & * & y_{2n-1,2n} & 0 \\ * & * & * & y_{2n} \end{bmatrix}$$

respectively. Elementary calculation yields

$$v(g_j\cdot(\dot{Y}, y, y_{2n}))g_jv(\dot{Y}, y, y_{2n})^{-1} \in G_{n,n} \quad (j=1,2)\,.$$

On account of $G_{n,n}$-invariance of $F^{(i)}_n$, it follows easily that $F^{(i)}_{n+1}$ are $G_{n,n+1}$-invariant. In particular, $F^{(i)}_{n+1}(g\cdot(\dot{Y}, y, y_{2n})) = F^{(i)}_{n+1}(\dot{Y}, y, y_{2n})$ for

$$g=\begin{pmatrix} I_{2n-4} & & \\ & J_1 & \\ & & I_2 \end{pmatrix}\in G_{n,n+1}\,.$$

This implies that $F^{(i)}_{n+1}$ are polynomials, since $F^{(i)}_{n+1}$ are polynomials in all variables except possibly for y_{2n-2}. Recalling that $F^{(i)}_{n+1}$ depend only on $\dot{Y}, y$ and y_{2n}, we conclude immediately that $F^{(i)}_{n+1}$ are invariant under the action of $a\in \boldsymbol{R}^{2n}$. Thus $F^{(i)}_{n+1}$ are $IG_{n,n+1}$-invariant polynomial functions. For $(Y,y)\in I\mathfrak{g}_n^*=\mathfrak{g}_n\times \boldsymbol{R}^{2n}$, let $v=v(Y,y)$ be the matrix

$$\begin{pmatrix} I_{2n-2} & o & -\tilde{y} \\ {}^t(J_{n-1}\tilde{y}) & 1 & -y_{2n-1}/y_{2n} \\ o & 0 & 1 \end{pmatrix}\qquad \text{with } {}^t\tilde{y}=(y,\cdots,y_{2n-2})/y_{2n}\,.$$

Then $v\cdot(Y,y)=(vYv^{-1},vy)$ belongs to $\mathscr{Y}_{n+1}$. To complete the proof of Lemma 2.3 we shall define functions $F^{(i)}$ on $I\mathfrak{g}_n^*$ by $F^{(i)}(Y,y)=F^{(i)}_{n+1}(v\cdot(Y,y))$ $(0\leqq i\leqq n-1)$ and show that $F^{(i)}$ are elements of $\mathscr{I}_n$. To being with, $F^{(i)}$ are G_n-invariant. Indeed, for $u\in G_n$, simple calculation reveals that

$$v(u\cdot(Y,y))uv(Y,y)^{-1}=\begin{bmatrix} * & * & * & \\ & & & o \\ * & * & * & \\ & o & 1 & 0 \\ * & * & * & * \end{bmatrix}\in G_{n,n+1}\,.$$

Since $F^{(i)}_{n+1}$ are $G_{n,n+1}$-invariant, it follows that $F^{(i)}$ are G_n-invariant. By the same argument as for $F^{(i)}_{n+1}$, we now conclude that $F^{(i)}$ are polynomials. Using the G_n-invariance of $F^{(i)}$ and $G_{n,n+1}$-invariance of $F^{(i)}_{n+1}$, we obtain for $a\in\boldsymbol{R}^{2n}$

$$\begin{aligned} F^{(i)}(a\cdot(Y,y)) &= F^{(i)}(v(Y,y)av(Y,y)^{-1}\cdot(v(Y,y)\cdot(Y,y))) \\ &= F^{(i)}_{n+1}(v(Y,y)av(Y,y)^{-1}\cdot(v(Y,y)\cdot(Y,y))) \\ &= F^{(i)}_{n+1}(v(Y,y)\cdot(Y,y))=F^{(i)}(Y,y)\,. \end{aligned}$$

IG_n being generated by G_n and $\boldsymbol{R}^{2n}$, $F^{(i)}$ are IG_n-invariant. The proof of Lemma 2.3 is complete.

Theorem 2 follows at once from Lemmas 2.1, 2.2 and 2.3.

Added in proof. After this paper had been accepted for publication, [6] appeared. [2] is now published (Comm. Math. Phy., 90 (1983), 353–372).

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