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ON THE BASE FIELD CHANGE OF P-RINGS AND P-2 RINGS

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One finds the following example in [3, (34, B)]:

Let k be a field of characteristic p and $\underline{X} = \{X_1, \cdots, X_n\}$ be n-variables over k. Then if $p>0$ and $[k : k^p] = \infty$, $k^p[[\underline{X}]][k]$ is an n-dimensional regular local ring but not a Nagata ring. In particular it is not an excellent ring.

On the other hand, according to [1, Corollary 4.3], $k[[\underline{X}]][l]$ is an excellent ring if l is a separably algebraic field extension of k.

In Section 1 we study when a property such as being excellent ascends by a base field extension.

Conversely in Section 2 we study when such a property as in Section 1 descends by a base field reduction.

§ 1. Notation and definitions

We use the following notation and definitions, following [6]:

Let $\boldsymbol{P}$ be a property meaningful for a noetherian ring and satisfying the following four axioms:

Axioms:
1. If A is regular, then A has $\boldsymbol{P}$.
2. $\boldsymbol{P}$ is a local property.
3. If A is a complete local ring, then $\boldsymbol{P}(A) = \boldsymbol{P}$-locus of A is Zariski open.
4. Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a faithfully flat local homomorphism. Then $\boldsymbol{P}$ descends from B to A; if both A and $B/\mathfrak{m}B$ have $\boldsymbol{P}$, then $\boldsymbol{P}$ ascends from A to B.

We say that a noetherian ring is a $\boldsymbol{P}$-ring if its formal fibers are geometrically $\boldsymbol{P}$. Then we have:

LEMMA 1. ([6, n. 3, Lemma]). *A noetherian local ring A is a $\boldsymbol{P}$-ring iff, for any finite A-algebra B which is a domain, and for any prime ideal Q of $\hat{B}$ with $Q \cap B = (0)$, the local ring $\hat{B}_Q$ is $\boldsymbol{P}$.*

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Moreover, imitating the proof of [3, Theorem 77], we have the following theorem (cf. [2, Theorem (7.4.4)]):

THEOREM 2. *Let A be a **P**-ring and B an A-algebra of finite type. Then B is a **P**-ring.*

We define ***P**-i* ($i = 0, 1, 2$) as follows:

DEFINITION 1. A ring A is **P**-0 iff **P**(A) contains a non-empty open set.

2. A is **P**-1 iff **P**(A) is an open set (maybe empty).
3. A is **P**-2 iff every A-algebra of finite type is **P**-1.

Next we define NC:

DEFINITION. A property **P** satisfying four axioms above has NC iff the following property holds for **P**; a noetherian ring A is **P**-1 if, for every $\mathfrak{p} \in \mathrm{Spec}(A)$, $A/\mathfrak{p}$ is **P**-0.

Remark. If **P** = regular, *CI*, Gorenstein or *CM*, then **P** has the four axioms and *NC*. ([6, n.2, Remark 1])

Then we have the following proposition:

PROPOSITION 3 ([6, n.2, Proposition 1]). *Let **P** satisfy NC. Then, for a noetherian ring A, the following are equivalent:*

1. *A is **P**-2;*
2. *any finite A-algebra is **P**-1;*
3. *for any $\mathfrak{p} \in \mathrm{Spec}(A)$, and for any finite radical extension L of the fraction field of $A/\mathfrak{p}$ there is a finite A-algebra B, containing $A/\mathfrak{p}$ and having L as fraction field, such that B is **P**-0.*

Besides the above, we freely use the notation and the definitions of [3].

§ 2. Base field extension

In this section we fix the notation as follows: A, k and l are noetherian rings, and A and l are k-algebras. We assume that there is a multiplicatively closed subset S of $A \otimes_k l$ such that $S^{-1}(A \otimes_k l)$ is a noetherian ring.

Our first result is about **P**-rings:

PROPOSITION 4. *If A is a **P**-ring, and k and l are fields such that l is separably generated over k, then $S^{-1}(A \otimes_k l)$ is a **P**-ring.*

Proof. Let B be an essentially finite $S^{-1}(A \otimes_k l)$-algebra which is a local domain. By Lemma 1, we have only to show that, for any $Q \in \operatorname{Spec}(\hat{B})$ such that $Q \cap B = (0)$, the local ring $\hat{B}_Q$ is $\boldsymbol{P}$.

We can put $B = (S^{-1}(A \otimes_k l)[\underline{X}])_M/\mathfrak{p}$, where $\underline{X} = \{X_1, \cdots, X_n\}$ are n-variables over $S^{-1}(A \otimes_k l)$ and $M \in \operatorname{Spec}(S^{-1}(A \otimes_k l)[\underline{X}])$, $\mathfrak{p} \in \operatorname{Spec}((S^{-1}(A \otimes_k l)[\underline{X}])_M)$. Since $\mathfrak{p}$ is finitely generated, there is an intermediate field K between k and l which satisfies the following three conditions:

(a) K is finitely generated over k;

(b) l is separably generated over K;

(c) if we put $\mathfrak{m} = M \cap (A \otimes_k K)[\underline{X}]$, $((A \otimes_k K)[\underline{X}])_{\mathfrak{m}}$ contains a certain system of generators of $\mathfrak{p}$.

Put $\mathfrak{q} = \mathfrak{p} \cap ((A \otimes_k K)[\underline{X}])_{\mathfrak{m}}$ and $C = ((A \otimes_k K)[\underline{X}])_{\mathfrak{m}}/\mathfrak{q}$, then we have the following commutative diagram:

$$\begin{array}{ccc} \hat{C} & \xrightarrow{\psi} & \hat{B} \\ \uparrow & & \uparrow \\ C & \xrightarrow{\varphi} & B \end{array}$$

where ψ is induced by φ. By (b) and (c), φ is formally smooth, hence so is ψ. Since $\hat{C}$ is excellent, ψ is regular by André's Theorem. Put $Q \cap \hat{C} = P$, then $P \cap C = (Q \cap B) \cap C = (0)$. Now C is an essentially finite type over A, therefore C is a $\boldsymbol{P}$-ring by Theorem 2. Therefore $\hat{C}_P$ is $\boldsymbol{P}$. Since ψ is regular, $\hat{B}_Q$ is $\boldsymbol{P}$. Q.E.D.

Now if $\boldsymbol{P} =$ regular, we have a stronger result as follows:

PROPOSITION 5. *Let A be a G-ring, and k and l be noetherian rings such that l is smooth over k. Then $S^{-1}(A \otimes_k l)$ is a G-ring.*

This proposition is a special case of the following theorem:

THEOREM ([7, Theorem 17] or [1]). *Let $u : A \to B$ and $v : B \to C$ be formally smooth homomorphisms of noetherian local rings. Suppose that A is a G-ring and $\Omega_{B/A} \otimes_B (C/Q)$ is a separated C/Q-module for any $Q \in \operatorname{Spec}(C)$. Then v is regular.*

In fact, let $P \in \operatorname{Spec}(S^{-1}(A \otimes_k l))$ and $P \cap A = \mathfrak{p}$. We put $A = A_{\mathfrak{p}}$, $B = S^{-1}(A \otimes_k l)_P$ and $C = \hat{B}$. Then, by our assumption, A is a G-ring, and $A \to B$ and $B \to C$ are formally smooth homomorphisms. Moreover $A \to B$ is smooth, so $\Omega_{B/A}$ is a projective B-module. Therefore $\Omega_{B/A} \otimes_B (C/Q)$ is a separated C/Q-module for any $Q \in \operatorname{Spec}(C)$. Thus the assumption of the above theorem is fulfilled. So $B \to C$ is regular, and $S^{-1}(A \otimes_k l)$ is a G-ring.

Remark. By Proposition 5, we have [1, Corollary 4.3]. We will generalize this result later.

Next proposition is about $\boldsymbol{P}$-2:

PROPOSITION 6. *Let $\boldsymbol{P}$ have NC. Then if A is $\boldsymbol{P}$-2 and k and l are fields such that l is separably generated over k, $S^{-1}(A \otimes_k l)$ is $\boldsymbol{P}$-2.*

Proof. Let B be a separating basis of l over k and $S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p}$ be a finite extension domain over $S^{-1}(A \otimes_k l)$, where $\underline{X} = \{X_1, \cdots, X_n\}$ are n-variables over $S^{-1}(A \otimes_k l)$ and $\mathfrak{p} \in \mathrm{Spec}(S^{-1}(A \otimes_k l)[\underline{X}])$. By Proposition 3, it is enough to prove that $S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p}$ is $\boldsymbol{P}$-0.

Since $\mathfrak{p}$ is finitely generated, there is an intermediate field K between k and l which satisfies the following three conditions:

(a) K is finitely generated over k;

(b) l is separably generated over K;

(c) $(A \otimes_k K)[\underline{X}]$ contains a system of generators of $\mathfrak{p}$.

Then $S^{-1}(A \otimes_k l)[\underline{X}]$ is smooth over $(A \otimes_k K)[\underline{X}]$. Put $\mathfrak{q} = \mathfrak{p} \cap (A \otimes_k K)[\underline{X}]$. Then from the definition of the field K, $\mathfrak{p} = \mathfrak{q}(S^{-1}(A \otimes_k l))[\underline{X}]$. So $S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p}$ is smooth over $(A \otimes_k K)[\underline{X}]/\mathfrak{q}$. Now since A is $\boldsymbol{P}$-2 and K is finitely generated over k, $(A \otimes_k K)[\underline{X}]/\mathfrak{q}$ is $\boldsymbol{P}$-2. Therefore $U = \boldsymbol{P}((A \otimes_k K)[\underline{X}]/\mathfrak{q})$ is a non-empty open set in $\mathrm{Spec}((A \otimes_k K)[\underline{X}]/\mathfrak{q})$. So the inverse image of U in $\mathrm{Spec}(S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p})$ is a non-empty open set contained in $\boldsymbol{P}(S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p})$. Thus $S^{-1}(A \otimes_k l)$ is $\boldsymbol{P}$-2. Q.E.D.

Remark. We have not yet succeeded in proving the proposition under the weaker assumption that l is separable over k.

PROPOSITION 7. *If A is a universally catenary noetherian ring and l and k are fields, then $S^{-1}(A \otimes_k l)$ is universally catenary.*

Proof. Similar to the proof of [2, Lemma (18.7.5.1)].

Summing up, if we assume that $\boldsymbol{P}$ = regular in particular, we get the following theorem:

THEOREM 8. *If A is a (quasi-)excellent ring and k and l are fields such that l is separably generated over k, then $S^{-1}(A \otimes_k l)$ is a (resp. quasi-) excellent ring.*

Next, we make examples which satisfy the assumptions of the above Theorem.

LEMMA 9. *Let A be a noetherian ring, k be a field contained in A, and l be a (not necessarily finite) algebraic field extension of k. Then if $A\otimes_k l$ is noetherian, $A[[\underline{X}]]\otimes_k l$ is noetherian where $\underline{X}=\{X_1,\cdots,X_n\}$ is n-variables over A.*

Proof. Similar to that of [4, (E3.1)].

PROPOSITION 10. *Let A be a noetherian ring, k be a field contained in A, and l be a field extension of k satisfying $\text{tr.deg}_k l<\infty$. Let B be a transcendence base of l over k. Then if $A\otimes_k l$ is noetherian, so is $S^{-1}(A[[\underline{X}]]\otimes_k l)$, where $\underline{X}=\{X_1,\cdots,X_n\}$ is n-variables and $S=1+\underline{X}(A[[\underline{X}]]\otimes_k k(B))$.*

Proof. Since $S^{-1}(A[[\underline{X}]]\otimes_k k(B))$ is a Zariski ring in the $(\underline{X})$-adic topology by the definition of S, and $(A\otimes_k k(B))[[\underline{X}]]$ is the $(\underline{X})$-adic completion of $S^{-1}(A[[\underline{X}]]\otimes_k k(B))$, the ring $(A\otimes_k k(B))[[\underline{X}]]$ is faithfully flat over $S^{-1}(A[[\underline{X}]]\otimes_k k(B))$. Therefore $(A\otimes_k k(B))[[\underline{X}]]\otimes_{k(B)} l$ is faithfully flat over $S^{-1}(A[[\underline{X}]]\otimes_k k(B))\otimes_{k(B)} l\cong S^{-1}(A[[\underline{X}]]\otimes_k l)$.

Now since $A\otimes_k l$ is noetherian and l is algebraic over $k(B)$, by Lemma 9, $(A\otimes_k k(B))[[\underline{X}]]\otimes_{k(B)} l$ is noetherian. Therefore $S^{-1}(A[[\underline{X}]]\otimes_k l)$ is noetherian. Q.E.D.

Remark. If $B=\phi$, each element of S is unit in $A[[\underline{X}]]\otimes_k l$. So $S^{-1}(A[[\underline{X}]]\otimes_k l)=A[[\underline{X}]]\otimes_k l$. This is the case of Lemma 9.

COROLLARY 11. *Let k be a field and l be a field separably generated over k such that $\text{tr.deg}_k l<\infty$. Denote a transcendence base of l over k by B. Put $S=1+\underline{Y}(k[\underline{X}][[\underline{Y}]]\otimes_k k(B))$ where $\underline{X}=\{X_1,\cdots,X_m\}$ ($\underline{Y}=\{Y_1,\cdots,Y_n\}$) are m-(resp. n-)variables over k. Then $S^{-1}(k[\underline{X}][[\underline{Y}]]\otimes_k l)$ is an excellent ring.*

Proof. By Proposition 6, $S^{-1}(k[\underline{X}][[\underline{Y}]]\otimes_k l)$ is noetherian. By [5], $k[\underline{X}][[\underline{Y}]]$ is an excellent ring. Thus by Theorem 8, $S^{-1}(k[\underline{X}][[\underline{Y}]]\otimes_k l)$ is an excellent ring. Q.E.D.

Remark. This is the generalization of [1, Corollary 4.3].

§ 3. Base field reduction

In this section we consider a base field reduction.

PROPOSITION 12. *Let A be a noetherian ring containing a field k, and l be a field separable over k. We assume that $\boldsymbol{P}$ satisfies the four axioms (but not NC). Then if $A\otimes_k l$ is a $\boldsymbol{P}$-ring, so is A.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of A. Since $A\otimes_k l$ is faithfully flat over A, there is a prime ideal M of $A\otimes_k l$ such that $M\cap A=\mathfrak{m}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \hat{A}_{\mathfrak{m}} & \xrightarrow{\psi} & (\widehat{A\otimes_k l})_M \\ \varphi\uparrow & & \uparrow v \\ A_{\mathfrak{m}} & \xrightarrow[u]{} & (A\otimes_k l)_M \end{array}$$

where u is the natural map and v, φ and ψ are induced by completions. Then since l is separable over k, u is regular. By the assumption, v is a $\boldsymbol{P}$-homomorphism. Therefore $v\circ u=\psi\circ\varphi$ is a $\boldsymbol{P}$-homomorphism. Now ψ is faithfully flat. Thus φ is a $\boldsymbol{P}$-homomorphism by [6, n.1, Remark 3].

Q.E.D.

PROPOSITION 13. *Let A be a noetherian ring containing a field k and l be a field extension of k. We assume that $\boldsymbol{P}$ satisfies NC. Then if $A\otimes_k l$ is $\boldsymbol{P}$-2, so is A.*

Proof. By Proposition 3, we assume that A is a domain, and we have only to show that $\boldsymbol{P}(A)$ contains a non-empty open set.

Let $\mathfrak{p}$ be an element of $\mathrm{Ass}(A\otimes_k l)$. Then $\mathfrak{p}\cap A=(0)$. Now $A\otimes_k l/\mathfrak{p}$ is $\boldsymbol{P}$-2 by the assumption, so there is an element $t\in A\otimes_k l$ such that $\boldsymbol{P}(A\otimes_k l/\mathfrak{p})\supseteq D(t)\neq\phi$. Since $\mathfrak{p}$ is finitely generated, there is an intermediate field K between k and l such that K is finitely generated over k and $A\otimes_k K$ contains the element t and a system of generators of $\mathfrak{p}$. Put $\mathfrak{q}=\mathfrak{p}\cap(A\otimes_k K)$. Then $A\otimes_k K/\mathfrak{q}\to A\otimes_k l/\mathfrak{p}$ is faithfully flat, therefore $\boldsymbol{P}(A\otimes_k K/\mathfrak{q})\supseteq D(t)\neq\phi$.

Now we denote a transcendence base of K over k by $\underline{X}=\{X_1,\cdots,X_n\}$. Since A is a domain, so is $A\otimes_k k(\underline{X})$. Hence $\mathfrak{q}\cap A\otimes_k k(\underline{X})=(0)$, and $A\otimes_k k(\underline{X})\to A\otimes_k K/\mathfrak{q}$ is a finite injective homomorphism. Therefore, by the generic flatness and the fact that a flat morphism of finite type is an open map, there is an element $s\in A\otimes_k k[\underline{X}]$ such that $\boldsymbol{P}(A\otimes_k k(\underline{X}))\supseteq D(s)\neq\phi$. We put $B=A\otimes_k k[\underline{X}]\cong A[\underline{X}]$ and $C=A\otimes_k k(\underline{X})$.

Now for any $P\in D(s)\subseteq\mathrm{Spec}(B)$, put $Q=P\cap A$. Then $QC\in\mathrm{Spec}(C)$. Since $QC\cap B=QB\subseteq P$, the prime ideal $QC\in D(s)\subseteq\mathrm{Spec}(C)$. So C_{QC} is

$\boldsymbol{P}$. Therefore A_Q is $\boldsymbol{P}$, because $A \to C$ is flat and $QC \cap A = Q$. Since $A_Q \to A_Q[\underline{X}]$ is regular, $A_Q[\underline{X}]$ is $\boldsymbol{P}$. So its localization $B_P \cong (A_Q[\underline{X}])_{PA_Q[X]}$ is also $\boldsymbol{P}$. Thus we have $\boldsymbol{P}(B) \supseteq D(s) \neq \phi$. Now $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open map, therefore $\boldsymbol{P}(A)$ contains a non-empty open set. This is what we want. Q.E.D.

References

[1] A. Brezuleanu and N. Radu, Excellent rings and good separation of the module of differentials, Rev. Roumaine Math. Pures Appl., **23** (1978), 1455–1470.
[2] A. Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique, Ch. IV, Première Partie, Publ. IHES, No. 24 (1965), No. 32 (1967).
[3] H. Matsumura, Commutative Algebra, Benjamin, New York (1970).
[4] M. Nagata, Local Rings, Interscience, New York (1962).
[5] P. Valabrega, On the excellent property for power series rings over polynomial rings, J. Math. Kyoto Univ., **15** (1975), 387–395.
[6] ——, Formal fibers and openness of loci, ibid., **18** (1978), 199–208.
[7] N. Radu, Sur les algèbres dont le module des différentielles est plat, Rev. Roumaine Math. Pures Appl., **21** (1976), 933–939.

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