

## ON EXISTENCE OF TOLERANCE STABLE DIFFEOMORPHISMS\*

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### § 1. Introduction

We consider a compact smooth manifold  $M$ .  $\text{Diff}^1(M)$  denotes the space of  $C^1$ -diffeomorphisms of  $M$  onto itself with the usual  $C^1$ -topology. In the research of the qualitative theory of dynamical systems there is a desire to find a concept of stability of geometric global structure of orbits such that this stable systems are dense in the space of dynamical systems on  $M$ . Structural stability does not satisfy the density condition in  $\text{Diff}^1(M)$ . Tolerance stability (see Section 2 for definition) is a candidate for the density property [7, p. 294]. Concerning tolerance stability there are researches as [6], [7], [8], and [2].

If  $f \in \text{Diff}^1(M)$  is structurally stable in strong sense,  $f$  is topologically stable in  $\text{Diff}^1(M)$  (see Section 2 for definition). Moreover, topological stability implies tolerance stability [A. Morimoto, 2]. The proof of this property will be introduced in Section 2.

The main result of this paper is to show the existence of diffeomorphisms on any compact manifold  $M$  which are tolerance stable but not topologically stable in  $\text{Diff}^1(M)$ , so that, not structurally stable in strong sense. This will be proved in Sections 3, 4 and 5.

### § 2. Definitions and statement of results

We denote by  $\text{Homeo}(M)$  the set of homeomorphisms of  $M$  onto itself; the topology on  $\text{Homeo}(M)$  is given by the neighborhood  $N_\varepsilon(f)$  of  $f \in \text{Homeo}(M)$

$$N_\varepsilon(f) = \{g; d(f, g) < \varepsilon\}, \quad \varepsilon > 0.$$

Here, for a Riemannian metric  $d$  on  $M$ ,  $d(f, g) < \varepsilon$  means

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$$d(f(x), g(x)) < \varepsilon \quad \text{for } x \in M .$$

To state the definition of tolerance stability, we need the following definition:

DEFINITION (2.1).  $f, g \in \text{Homeo}(M)$  are *orbit- $\varepsilon$ -equivalent*,  $\varepsilon > 0$ , if

1. for every  $f$ -orbit  $O_f$ , there is a  $g$ -orbit  $O_g$  such that

- (a)  $O_f \subset U_\varepsilon(O_g)$
- (b)  $O_g \subset U_\varepsilon(O_f)$ , and

2. for every  $g$ -orbit  $O_g$ , there is a  $f$ -orbit  $O_f$  such that

- (a)  $O_g \subset U_\varepsilon(O_f)$
- (b)  $O_f \subset U_\varepsilon(O_g)$ .

Here,  $U_\varepsilon(*)$  is the  $\varepsilon$ -neighborhood of  $*$ .

Suppose that a subset  $\mathcal{D}$  of  $\text{Homeo}(M)$  is given a topology not coarser than that of  $\text{Homeo}(M)$ .

DEFINITION (2.2). An element  $f \in \mathcal{D}$  is *tolerance-stable* in  $\mathcal{D}$  if for every  $\varepsilon > 0$  there is a neighborhood  $N$  of  $f$  in  $\mathcal{D}$  (with respect to the given topology on  $\mathcal{D}$ ) such that, for every  $g \in N$ ,  $f$  and  $g$  are orbit- $\varepsilon$ -equivalent.

DEFINITION (2.3). An element  $f \in \mathcal{D}$  is *topologically stable* in  $\mathcal{D}$ , if for any  $\varepsilon > 0$  there is a neighborhood  $N$  of  $f$  in  $\mathcal{D}$  such that for every  $g \in N$  there is a continuous map  $h: M \rightarrow M$  satisfying

- (a)  $d(h, i_M) < \varepsilon$ , where  $i_M$  is the identity map of  $M$ ,
- (b)  $hg = fh$ .

The following property is mentioned and proved by A. Morimoto in [2]. We introduce this:

PROPOSITION. *If  $M$  is a compact topological manifold and  $f \in \text{Homeo}(M)$  is topologically stable in  $\mathcal{D}$  then  $f$  is tolerance stable in  $\mathcal{D}$ , for any subset  $\mathcal{D} \subset \text{Homeo}(M)$ .*

*Proof.* For closed non-empty subsets  $A$  and  $B$  of  $M$ , let

$$\bar{d}(A, B) = \max \left\{ \max_{a \in A} d(a, B), \max_{b \in B} d(b, A) \right\},$$

where  $d(a, B) = \min_{b \in B} d(a, b)$ .  $O_f(x)$  denotes the  $f$ -orbit of  $x$ ;  $O_f(x) = \{f^i(x); i \in \mathbb{Z}\}$ . Put  $\bar{O}_f(x) = \text{Cl}(O_f(x))$ . By the assumption, for every  $\varepsilon > 0$ ,

there is a neighborhood  $N$  of  $f$  in  $\mathcal{D}$  such that for every  $g \in N$  there is  $h: M \rightarrow M$  satisfying (a) and (b) in Definition (2.3). By (b),  $h(O_g(x)) = O_f(h(x))$  for every  $x \in M$ . Hence,

$$\bar{d}(\bar{O}_g(x), \bar{O}_f(h(x))) = \bar{d}(\bar{O}_g(x), h(\bar{O}_g(x))) < \varepsilon .$$

Therefore, for any  $g$ -orbit  $O_g$  there is  $f$ -orbit  $O_f$  such that  $O_g \subset U_\varepsilon(O_f)$  and  $O_f \subset U_\varepsilon(O_g)$ . Since  $M$  is a compact manifold, we can prove that  $d(h, i_M) < \varepsilon$  implies that  $h: M \rightarrow M$  is a surjection if  $\varepsilon > 0$  is sufficiently small. We may assume that  $\varepsilon$  is taken so small that this property is satisfied. Hence for every  $x \in M$  there is  $y \in M$  such that  $h(y) = x$ . Then

$$\begin{aligned} \bar{d}(\bar{O}_f(x), \bar{O}_g(y)) &= \bar{d}(\bar{O}_f(h(y)), \bar{O}_g(y)) \\ &= \bar{d}(h(\bar{O}_g(y)), \bar{O}_g(y)) < \varepsilon . \end{aligned}$$

Hence, for any  $f$ -orbit  $O_f$  there is  $g$ -orbit  $O_g$  such that  $O_f \subset U_\varepsilon(O_g)$  and  $O_g \subset U_\varepsilon(O_f)$ . Therefore,  $f$  is tolerance stable in  $\mathcal{D}$ .

**DEFINITION (2.4).** Two elements  $f, g \in \text{Diff}^1(M)$  are *topologically  $\varepsilon$ -conjugate* if there is a homeomorphism  $h: M \rightarrow M$  such that  $hg = fh$  and  $d(h(x), x) < \varepsilon$  for every  $x \in M$ .  $f, g$  are *topologically conjugate* if there is a homeomorphism  $h$  such that  $hg = fh$ .

**DEFINITION (2.5).** An element  $f \in \text{Diff}^1(M)$  is *structurally stable in strong sense* if for every  $\varepsilon > 0$  there is a neighborhood  $N$  of  $f$  in  $\text{Diff}^1(M)$  such that every  $g \in N$  are topologically  $\varepsilon$ -conjugate to  $f$ .  $f$  is *structurally stable*, if there is  $N$  such that, for every  $g \in N$ ,  $f$  and  $g$  are topologically conjugate.

Structural stability in strong sense implies structural stability and topological stability in  $\text{Diff}^1(M)$ . If  $f \in \text{Diff}^1(M)$  satisfies Axiom A and strong transversality condition then  $f$  is structurally stable in strong sense [4].

**THEOREM.** *Let  $M$  be a compact differentiable manifold. There is a diffeomorphism  $f$ , in the boundary  $\partial\Sigma$  of the set  $\Sigma$  of all structurally stable elements in  $\text{Diff}^1(M)$ , such that*

- (a)  $f$  is tolerance-stable in  $\text{Diff}^1(M)$ ,      and
- (b)  $f$  is not topologically stable in  $\text{Diff}^1(M)$ , so that,  $f$  is not structurally stable in strong sense.

### §3. Construction of $f$

Theorem is proved in Sections 3, 4 and 5. In these sections  $M$  is assumed to have  $\dim M \geq 2$ . But to the readers of these sections the proof of Theorem in the case  $\dim M = 1$  will be obvious.

$f$  will be constructed as follows. If  $f_0$  is a diffeomorphism which is structurally stable in strong sense and has a periodic point  $p$  that is a sink or source, then  $f$  will be obtained from  $f_0$  by isotopically replacing  $f_0$  on a small neighborhood of  $p$ .

Let  $f_0$  be a time-one map of the flow of the vector field  $Y$  obtained by Theorem 2.1 of [5]. Then  $f_0$  is a Morse-Smale diffeomorphism having a fixed point  $p$  which is a sink. By [3],  $f_0$  is structurally stable in strong sense.

By replacing  $f_0$  by an isotopy on a small neighborhood  $U$  of  $p$  we obtain  $f_1$  such that

(i) every point in a small closed ball neighborhood  $B$  in  $U$ , with center  $p$ , is a fixed point of  $f_1$ , and

(ii) for every  $x$  in  $U - B$ ,  $\lim_{k \rightarrow \infty} f_1^k(x)$  exists in  $\partial B$ .

Let  $B_r$  be a closed ball in the euclidean space  $\mathbf{R}^m$  of the same dimension as  $M$ , centered on the origin with radius  $r$ . Let  $S_r = \partial B_r$ , a  $(m - 1)$ -sphere. After this, we regard  $B$  as a closed ball  $B_{r_0}$  in  $\mathbf{R}^m$ , and  $p$  as the origin of  $\mathbf{R}^m$ .

To construct  $f$  we will define a vector field  $V$  on  $B$ . On a neighborhood of  $p$ ,  $f$  will be the time-one map of the flow of  $V$ .

(1) Construction of  $V$ .

For this purpose we at first define a vector field  $X$ . Let

$$\varphi_0(r) = e^{-1/r^2} \sin \frac{1}{r}, \quad r > 0.$$

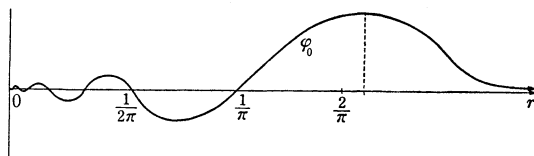


Fig. 1.

Take  $r_1 \in \mathbf{R}_+$  such that  $r_1 < r_0$ ,  $\varphi_0'(r_1) > 0$ , and

$$(3.1) \quad \frac{1}{2n\pi} < r_1 < \frac{1}{(2n-1)\pi} \quad \text{for a fixed } n \in \mathbf{Z}_+.$$

Let  $\alpha: [r_1, \infty) \rightarrow \mathbf{R}$  be a  $C^1$ -function such that  $\alpha(r) < 0$  and  $\alpha'(r) > 0$  for every  $r \in [r_1, \infty)$ , and that the function defined by

$$\varphi(r) = \begin{cases} 0 & \text{if } r = 0 \\ \varphi_0(r) & \text{if } 0 < r < r_1 \\ \alpha(r) & \text{if } r_1 \leq r \end{cases}$$

is  $C^1$ .

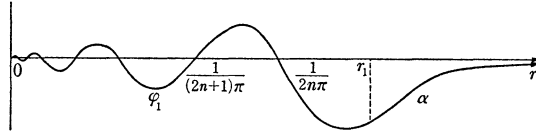


Fig. 2.

Define a vector field  $X$  on  $B$  by

$$X_x = \begin{cases} \varphi(\|x\|) \frac{x}{\|x\|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Here,  $\|\cdot\|$  is the euclidean norm on  $\mathbf{R}^m$ .

We show that  $X$  is  $C^1$ . Let  $x = {}^t(x_1, \dots, x_m) \in \mathbf{R}^m$  be a row vector, i.e. the transposition of  $(x_1, \dots, x_m)$ . If  $x \neq 0$

$$\begin{aligned} \frac{\partial}{\partial x_i} X_x &= \frac{\partial}{\partial x_i} \left( \frac{\varphi(\|x\|)}{\|x\|} \right) x + \frac{\varphi(\|x\|)}{\|x\|} \frac{\partial}{\partial x_i} x \\ &= \frac{\partial}{\partial x_i} \|x\| \frac{\varphi'(\|x\|)\|x\| - \varphi(\|x\|)}{\|x\|^2} x + \frac{\varphi(\|x\|)}{\|x\|} \frac{\partial}{\partial x_i} x \\ &= \frac{x_i}{\|x\|} \frac{\varphi'(\|x\|)\|x\| - \varphi(\|x\|)}{\|x\|^2} x + \frac{\varphi(\|x\|)}{\|x\|} \frac{\partial}{\partial x_i} x \\ &= x_i \left( \frac{\varphi'(\|x\|)}{\|x\|^2} - \frac{\varphi(\|x\|)}{\|x\|^3} \right) x + \frac{\varphi(\|x\|)}{\|x\|} \frac{\partial}{\partial x_i} x. \end{aligned}$$

Hence, for  $x \neq 0$

$$DX_x = \left( \frac{\varphi'(\|x\|)}{\|x\|^2} - \frac{\varphi(\|x\|)}{\|x\|^3} \right) x \cdot {}^t x + \frac{\varphi(\|x\|)}{\|x\|} I,$$

where  $DX_x$  is the Jacobian matrix, and  $I$  is the unit matrix. For a matrix  $A = (a_1, \dots, a_m)$  with row vectors  $a_1, \dots, a_m$ , we define the norm of  $A$  by

$$\|A\| = \max_j \|a_j\|.$$

Then,

$$\|DX_x\| \leq \left| \frac{\varphi'(\|x\|)}{\|x\|^2} - \frac{\varphi(\|x\|)}{\|x\|^3} \right| \cdot \|x\|^2 + \left| \frac{\varphi(\|x\|)}{\|x\|} \right|.$$

$DX_0 = 0$  since  $\varphi'(0) = 0$ . Therefore, since  $\varphi$  is  $C^1$ ,  $X$  is a  $C^1$ -vector field.

Next we define a vector field  $Y$  on  $B$ . Let  $\mu: [0, \infty) \rightarrow [0, \infty)$  be a  $C^1$ -function such that

$$\begin{cases} \mu \geq 0, & \text{and} \\ \mu(r) = 0 & \text{and } \mu'(r) = 0 \quad \text{if } r = 0 \text{ or } r \geq r_1. \end{cases}$$

Let  $C$  be a  $C^1$ -vector field, on the unit sphere  $S^{m-1}$ , such that  $C$  has two singular points  $p_+$  and  $p_-$ , where  $p_+$  is a source at the north pole and  $p_-$  is a sink at the south pole, and such that every other trajectory of  $C$  goes out of  $p_+$  and into  $p_-$ . Then  $Y$  is defined by

$$Y_x = \begin{cases} \mu(\|x\|)C_{x/\|x\|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

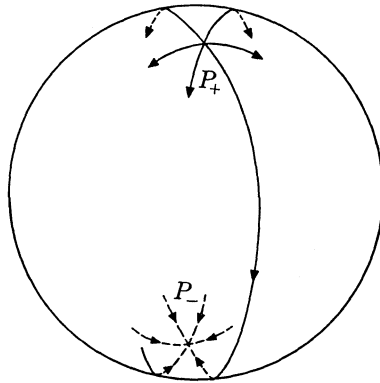


Fig. 3.

For the calculation of the derivative of  $Y_x$ , we take a  $C^1$ -extension  $\tilde{C}: U(S^{m-1}) \rightarrow \mathbf{R}^m$  of  $C: S^{m-1} \rightarrow \mathbf{R}^m$ , where  $U(S^{m-1})$  is a neighborhood of  $S^{m-1}$  in  $\mathbf{R}^m$ . Then, for  $x \neq 0$ , we have

$$\mu(\|x\|)C_{x/\|x\|} = \mu(\|x\|)\tilde{C}_{x/\|x\|}.$$

Let  $e_i$  be the  $i$ -th row vector of the unit matrix  $I$ . Let  $y = x/\|x\|$ , and let  $D$  be the notation of the derivative of variable  $x$ . Since

$$\frac{\partial}{\partial x_i} \frac{x}{\|x\|} = -\frac{x_i}{\|x\|^3}x + \frac{1}{\|x\|}e_i, \quad \text{and}$$

$$DY_x = D\mu(\|x\|) \cdot \tilde{C}_y + \mu(\|x\|) \cdot D\tilde{C}_y \cdot D\left(\frac{x}{\|x\|}\right),$$

we have

$$\begin{aligned} \frac{\partial}{\partial x_i} Y_x &= \frac{\partial}{\partial x_i} (\mu(\|x\|)) \tilde{C}_y + \mu(\|x\|) \cdot D\tilde{C}_y \cdot \frac{\partial}{\partial x_i} \frac{x}{\|x\|} \\ &= \frac{x_i}{\|x\|} \mu'(\|x\|) \tilde{C}_y + \mu(\|x\|) \cdot D\tilde{C}_y \cdot \left(-\frac{x_i}{\|x\|^3}x + \frac{1}{\|x\|}e_i\right). \end{aligned}$$

Consequently, if  $x \neq 0$  then

$$DY_x = \mu'(\|x\|) \tilde{C}_y \cdot t_y + \mu(\|x\|) \cdot D\tilde{C}_y \cdot \left(-\frac{1}{\|x\|^3}x \cdot t_x + \frac{1}{\|x\|}I\right).$$

Since  $\mu(0) = \mu'(0) = 0$  we have  $DY_0 = 0$ . Therefore  $Y$  is a  $C^1$ -vector field.

The  $C^1$ -vector field  $V$  on  $B$  is defined by

$$V_x = X_x + Y_x.$$

Fig. 4 shows the orbit structure of  $V$ . Here, we denote  $B(k) = B_{1/k\pi}$  and  $S(k) = \partial B(k)$ . Every singular point of  $V$  is hyperbolic except  $p$ .

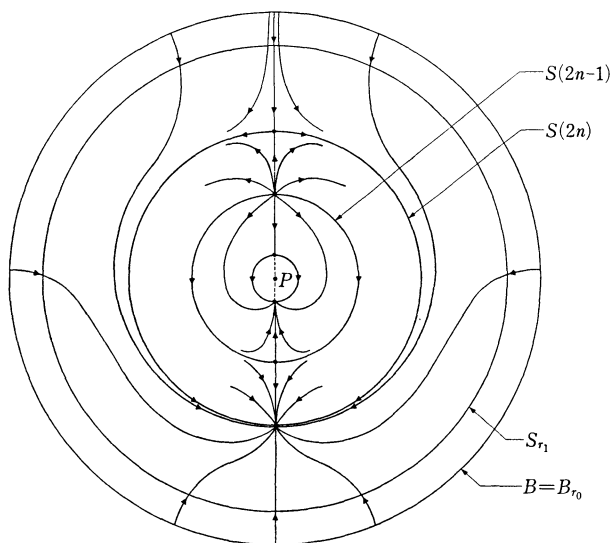


Fig. 4.

(2) Construction of  $f$ .

Let  $\Psi_1: B \rightarrow B$  be the time one map of the flow  $\Psi$  of  $V$ .  $\Psi_1$  is a  $C^1$ -diffeomorphism such that  $B - \Psi_1(B)$  is an annulus which is diffeomorphic to  $\partial B \times [0, 1)$ . Every fixed point of  $\Psi_1$  is hyperbolic except  $p$ . The property (ii) of  $f_1$  and the orbit structure of  $V$  enable us to obtain a diffeomorphism  $f: M \rightarrow M$  satisfying the following properties;

- (i)  $f|_B = \Psi_1$ ,
- (ii)  $f|(M - U) = f_1|(M - U)$ ,
- (iii) if  $x \in U - B$  then  $\lim_{k \rightarrow \infty} f^k(x)$  is the north pole or the south pole of  $S(2n)$ .

Moreover,  $f|(M - B)$  is obtained from  $f_1$  by an isotopy supported by  $U$ . Since  $\Psi_1$  is isotopic to  $i_B = f_1|_B$  by the isotopy  $\Psi_t$ ,  $t \in [0, 1]$ ,  $f$  is isotopic to  $f_1$  by an isotopy supported by  $U$ .

In Sections 4 and 5 it is proved that  $f$  possesses the desired properties (a), (b) of Theorem.

§ 4. Proof of tolerance-stability of  $f$  in  $\text{Diff}^1(M)$ 

Let sufficiently small  $\varepsilon > 0$  be given.

LEMMA. *There is a diffeomorphism  $h: M \rightarrow M$  such that (i)  $h = \text{identity}$  on  $M - B_{\varepsilon/4}$ , and (ii)  $f_\varepsilon = hf$  is structurally stable in strong sense.*

*Proof.* We may assume

$$(4.1) \quad \frac{\varepsilon}{3} < r_1.$$

Let  $\ell$  be a sufficiently large integer satisfying the following inequalities.

$$(4.2) \quad \frac{1}{2\ell\pi} + e^{-(\ell\pi)^2} < \frac{1}{(2\ell - 1)\pi} < \frac{\varepsilon}{4}.$$

Put  $1/2\ell\pi + e^{-(\ell\pi)^2} = r_2$ . Define a discontinuous function  $\eta_0: (0, r_2) \rightarrow \mathbf{R}_+$  by

$$\eta_0(r) = \begin{cases} r - e^{-(k\pi)^2} & \text{if } \frac{1}{(k+1)\pi} < r \leq \frac{1}{k\pi}, \\ r - e^{-4(2\ell\pi)^2} & \text{if } \frac{1}{2\ell\pi} < r \leq r_2, \end{cases}$$

where  $k = 2\ell, 2\ell + 1, 2\ell + 3, \dots$ . Let  $\eta: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  be a  $C^1$ -function satisfying



$$(4.3) \quad \left\{ \begin{array}{ll} 0 \leq \eta(r) \leq r, \\ \eta(r) = r & \text{if } r > \frac{1}{(2\ell - 1)\pi}, \\ \eta(r) < \eta_0(r) & \text{if } 0 < r \leq r_2, \\ \eta(0) = 0, \\ \eta'(r) > 0 & \text{for every } r \geq 0, \\ \eta'(0) < 1. \end{array} \right.$$

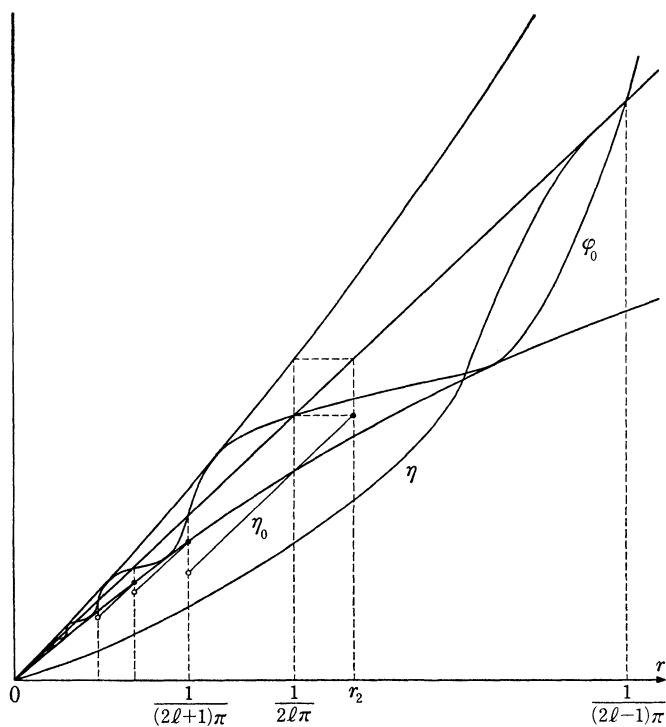


Fig. 5.

In fact,  $\eta$  exists. Especially we can find  $\eta$  such that  $0 < \eta'(0) < 1$ , since in a neighborhood of 0 the following properties hold.

$$(4.4) \quad \eta_0(r) > r - e^{-(1/r - \pi)^2},$$

$$(4.5) \quad \lim_{r \rightarrow 0} \frac{1}{r} (r - e^{-(1/r - \pi)^2}) = 1.$$

Define  $h: M \rightarrow M$  by

$$(4.6) \quad h(x) = \begin{cases} \eta(\|x\|) \frac{x}{\|x\|} & \text{if } x \in B \\ x & \text{if } x \notin B. \end{cases}$$

Since  $B = B_{r_0}$  and  $r_1 < r_0$ , the map  $h$  is well defined by (4.1), (4.2) and (4.3).  $h$  is a diffeomorphism. Define  $f_i$  by

$$f_i(x) = hf(x).$$

By (4.3)  $f_i(x) = f(x)$  if  $\|x\| \geq 1/(2\ell - 1)\pi$ .

Next, we show

$$(4.7) \quad \|f_i(x)\| < \|x\| \quad \text{if } \|x\| < \frac{1}{(2\ell - 1)\pi}.$$

Remember the definition of the vector field  $X$ , then we observe that  $\|f(x)\| \leq \|x\|$  when  $1/2k\pi < \|x\| < 1/(2\ell - 1)\pi$ . Since  $\eta(\|x\|) \leq \|x\|$ , it follows that

$$(4.8) \quad \|f_i(x)\| < \|x\| \quad \text{if } \frac{1}{2k\pi} < \|x\| < \frac{1}{(2k - 1)\pi}, \quad k \geq \ell.$$

Next, let  $1/(2k + 1)\pi < \|x\| < 1/2k\pi$ . Let  $\bar{\Psi}_t(x)$  be the flow of  $X$ , so that  $\bar{\Psi}_0(x) = x$ . Since  $V_x = X_x + Y_x$  and  $\|f(x)\| = \|\Psi_1(x)\| = \|\bar{\Psi}_1(x)\|$ , we have

$$(4.9) \quad \|f(x)\| = \|x\| + \int_0^1 \varphi(\|\bar{\Psi}_t(x)\|) dt,$$

where  $\varphi(r) = e^{-1/r^2} \sin 1/r$  as before.  $1/(2k + 1)\pi \leq \|x\| \leq 1/2k\pi$  implies  $0 \leq \sin(1/\|x\|) \leq 1$ .

Hence,

$$\varphi(\|x\|) \leq e^{-1/\|x\|^2} \leq e^{-(2k\pi)^2}.$$

Therefore, by (4.9),

$$\|f(x)\| \leq \|x\| + e^{-(2k\pi)^2}.$$

Using this and the definition of  $\eta_0$  we have

$$\begin{aligned} \|f_i(x)\| &= \|hf(x)\| = \eta(\|f(x)\|) \\ &\leq \eta(\|x\| + e^{-(2k\pi)^2}) < \eta_0(\|x\| + e^{-(2k\pi)^2}) \\ &< (\|x\| + e^{-(2k\pi)^2}) - e^{-(2k\pi)^2} = \|x\|. \end{aligned}$$

Hence,

$$(4.10) \quad \|f_\varepsilon(x)\| < \|x\| \quad \text{if} \quad \frac{1}{(2k+1)\pi} \leq \|x\| \leq \frac{1}{2k\pi}.$$

By (4.8) and (4.10) we have (4.7).

Hence  $f_\varepsilon$  contracts to  $p$  in  $\text{Int } B(2\ell - 1)$ . We have  $f_\varepsilon = f$  in  $M - B_{1/(2\ell-1)\pi}$  by (4.3). By the definition of  $f$ ,  $f|(M - B_{1/(2\ell-1)\pi})$  is Morse-Smale and  $\partial B_{1/(2\ell-1)\pi}$  is  $f$ -invariant. Therefore  $f_\varepsilon$  is Morse-Smale. Since a Morse-Smale diffeomorphism is structurally stable in strong sense by [3] this completes the proof of Lemma.

Since  $f_\varepsilon$  is structurally stable in strong sense there is a neighborhood  $N_0$  of  $f_\varepsilon$  in  $\text{Diff}^1(M)$  such that every element in  $N_0$  is topologically  $\varepsilon/24$ -conjugate to  $f_\varepsilon$ .

Since  $h$  is a  $C^1$ -diffeomorphism the map  $h_*: \text{Diff}^1(M) \rightarrow \text{Diff}^1(M)$  defined by  $h_*(g) = hg$  is continuous [1, p. 229, (B.8)]. Hence, for the neighborhood  $N_0$  of  $hf = f_\varepsilon$ , there is a neighborhood  $N$  of  $f$  in  $\text{Diff}^1(M)$  such that

$$g \in N \Rightarrow hg = g_\varepsilon \in N_0.$$

Hereafter, let  $g$  be included in  $N$ . Since  $h = \text{identity}$  on  $M - B_{\varepsilon/4}$  by (4.2), (4.3) and (4.6), we have

$$(4.11) \quad \begin{aligned} f_\varepsilon &= f \quad \text{and} \quad g_\varepsilon = g \quad \text{in} \quad M - B_{\varepsilon/4} \\ f_\varepsilon \quad \text{and} \quad g_\varepsilon &\text{ are topologically } \varepsilon/24\text{-conjugate:} \end{aligned}$$

there is a homeomorphism  $h_g: M \rightarrow M$  such that

$$(4.12) \quad h_g g_\varepsilon = f_\varepsilon h_g \quad \text{and} \quad d(h_g(x), x) < \varepsilon/24, \quad \text{for all } x.$$

We may assume that  $\varepsilon$  is small, so that there is an integer  $k$  satisfying  $3/\pi\varepsilon < k < 24/7\pi\varepsilon$ . Then we have

$$(4.13) \quad \frac{\varepsilon}{4} + \frac{\varepsilon}{24} < \frac{1}{k\pi} < \frac{\varepsilon}{3}.$$

(4.1), (4.13) and the definition of  $f$  imply that  $S_{1/k\pi}$  is  $f$ -invariant. Denote  $S_f = S_{1/k\pi}$ . Since  $S_f$  is contained in the complement of  $B_{\varepsilon/4}$ , (4.11) implies that  $S_f$  is also  $f_\varepsilon$ -invariant. Since  $f_\varepsilon$  and  $g_\varepsilon$  are topologically  $\varepsilon/24$ -conjugate, (4.11) and (4.13) imply that  $h_g(S_f)$  is contained in  $M - B_{\varepsilon/4}$  and is both  $g$  and  $g_\varepsilon$ -invariant. Denote  $h_g(S_f) = S_g$ ,  $B_{1/k\pi} = B_f$  and  $h_g(B_f) = B_g$ . Since  $\partial B_f = S_f$  and  $\partial B_g = S_g$  we have

$$(4.14) \quad \begin{cases} f_i = f & \text{in } M - B_f, \\ g_i = g & \text{in } M - B_g, \\ f|(M - B_f) \text{ and } g|(M - B_g) \end{cases} \text{ are topologically } \varepsilon/24\text{-conjugate.}$$

Precisely, the last part of (4.14) means that there is the commutative diagram

$$\begin{array}{ccc} (M - B_f) & \xrightarrow{f} & (M - B_f) \\ \downarrow h_g & & \downarrow h_g \\ (M - B_g) & \xrightarrow{g} & (M - B_g) \end{array}$$

and  $d(h_g(x), x) < \varepsilon/24$  for every  $x \in (M - B_f)$ . (4.14) implies

$$(4.15) \quad \begin{cases} B_f & \text{is } f\text{-invariant,} \\ B_g & \text{is } g\text{-invariant.} \end{cases}$$

For every  $g$  in  $N$ , we must show that  $f$  and  $g$  are orbit- $\varepsilon$ -equivalent. First, let  $O_f \subset M - B_f$ . Then,  $O_f$  is an  $f_i$ -orbit  $O_{f_i}$ . By (4.14),  $h_g(O_f) = O_{g_i}$  is contained in  $M - B_g$  and  $O_{g_i}$  is a  $g$ -orbit  $O_g$ . Since  $d(h_g, i_M) < \varepsilon/24$  then the conditions (a) and (b) of 1 in Definition (2.1) are satisfied in this case.

Next, let  $O_f \subset B_f$ . Take any orbit  $O_g$  in  $B_g$  (by using (4.15)). Then (a) and (b) of 1 in Definition (2.1) are satisfied. In fact, for any  $x \in B_f$  and  $y \in B_g$ , by (4.13) we have

$$\begin{aligned} \|x - y\| &\leq \|x\| + \|y\| \\ &\leq \frac{1}{k\pi} + \left( \frac{1}{k\pi} + \frac{\varepsilon}{24} \right) \\ &< \frac{\varepsilon}{3} + \left( \frac{\varepsilon}{3} + \frac{\varepsilon}{24} \right) < \varepsilon. \end{aligned}$$

Hence, the condition 1 in Definition (2.1) is satisfied. Similarly we can show the condition 2. Therefore  $f$  is tolerance-stable in  $\text{Diff}^1(M)$ .

## §5. Proof of topological unstability in $\text{Diff}^1(M)$

Suppose that  $f$  is topologically stable in  $\text{Diff}^1(M)$ . Then, for any  $\varepsilon_1 > 0$  there is a neighborhood  $N$  of  $f$  in  $\text{Diff}^1(M)$  such that for every  $g$  in  $N$  there is a continuous map  $\tau: M \rightarrow M$  satisfying

- (a)  $d(\tau, i_M) < \frac{\varepsilon_1}{2}$ ,
- (b)  $\tau g = f\tau$ .

For the fixed integer  $n$  in (3.1), let

$$\varepsilon_1 = \frac{1}{2n\pi} .$$

To introduce a contradiction, we take following  $g$ ;

$$g = hf ,$$

where  $h$  is a diffeomorphism defined by (4.6). But we must take  $g$  such that  $g \in N$ . By (4.4), (4.5) and the definition (4.3) of  $\eta$  we can choose  $\eta$ , by taking  $\ell$  sufficiently large, such that  $|\eta(r) - r|$  and  $|\eta'(r) - 1|$  are arbitrarily small. Hence we may assume that  $g \in N$  and  $1/(2\ell - 1)\pi < \varepsilon_1$ .

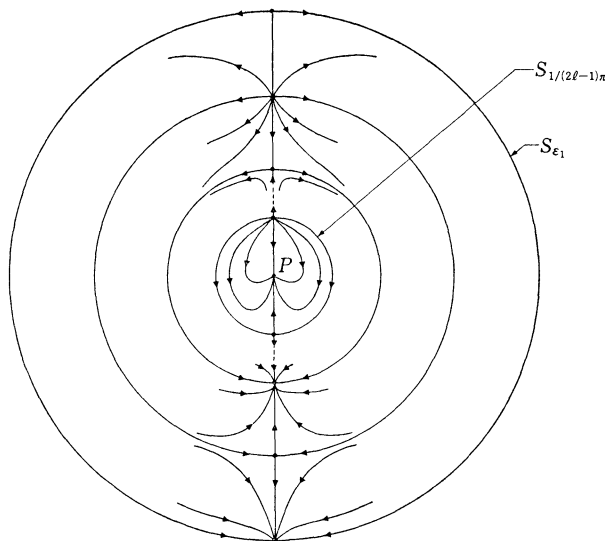


Fig. 6.

If  $y$  is a fixed point of  $f$  satisfying  $\|y\| < \varepsilon_1/2$  then  $\tau^{-1}(y)$  contains a fixed point of  $g$ . In fact, since  $\tau g = f\tau$ ,  $\tau^{-1}(y)$  is a  $g$ -invariant closed subset. By the condition (a) above, each  $x$  in  $\tau^{-1}(y)$  satisfies

$$\begin{aligned} \|x\| &\leq \|y\| + \|y - x\| \\ &= \|y\| + \|\tau x - x\| \\ &< \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1 . \end{aligned}$$

Any invariant closed subset of  $g$ , included in  $B_{\varepsilon_1}$ , contains at last one fixed point (see Fig. 6). Hence, for each fixed point  $y$  of  $f$  in  $B_{\varepsilon_1/2}$ , there

is a fixed point  $x$  of  $g$  such that  $\tau(x) = y$  and  $x \in B_{\varepsilon_1}$ . There are infinitely many fixed points of  $f$  in  $B_{\varepsilon_1/2}$ , but there are at most finite fixed points of  $g$  in  $B_{\varepsilon_1}$ . This is a contradiction. Therefore  $f$  is topologically unstable in  $\text{Diff}^1(M)$ .

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