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ON A CERTAIN HOMOLOGY OF FINITE PROJECTIVE SPACES

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1. We denote by $P^n(q)$ the projective space of dimension n over a finite field $GF(q)$ with q elements, and we mean by an i-flat a linear subspace of dimension i in $P^n(q)$. Denoting

$$L_i = L_i(P^n(q)) = \{i\text{-flats in } P^n(q)\},$$
$$Z_{q+1} = Z/(q+1)Z,$$
$$C_{i+1}(P^n(q), Z_{q+1}) = \{\sum_{\sigma \in L_i} a_\sigma \sigma | a_\sigma \in Z_{q+1}\},$$
$$C_{i+1}(P^n(q), GF(l)) = \{\sum_{\sigma \in L_i} a_\sigma \sigma | a_\sigma \in GF(l)\}$$
$$C_0(P^n(q), Z_{q+1}) = Z_{q+1}, \quad C_0(P^n(q), GF(l)) = GF(l) \quad (l|q+1),$$

we have chain complexes

$$\begin{aligned} &C_{n+1}(P^n(q), Z_{q+1}) \xrightarrow{\partial_{n+1}} C_n(P^n(q), Z_{q+1}) \xrightarrow{\partial_n} \cdots \\ (1) \qquad &\xrightarrow{\partial_2} C_1(P^n(q), Z_{q+1}) \xrightarrow{\partial_1} Z_{q+1} \\ &= C_0(P^n(q), Z_{q+1}) \xrightarrow{\partial_0} \{0\} \end{aligned}$$

$$\begin{aligned} &C_{n+1}(P^n(q), GF(l)) \xrightarrow{\partial_{n+1}} C_n(P^n(q), GF(l)) \xrightarrow{\partial_n} \cdots \\ (2) \qquad &\xrightarrow{\partial_2} C_0(P^n(q), GF(l)) \xrightarrow{\partial_1} GF(l) \\ &= C_0(P^n(q), GF(l)) \xrightarrow{\partial_0} \{0\}, \end{aligned}$$

where the boundary operators ∂_i are defined as follows,

$$\begin{cases} \partial_{i+1}(\sum_{\sigma \in L_i} a_\sigma \sigma) = \sum_{\sigma \in L_i} a_\sigma (\sum_{\substack{\tau \in L_{i-1} \\ \tau \subset \sigma}} \tau) & (i \geq 1) \\ \partial_1(\sum_{\sigma \in L_0} a_\sigma \sigma) = \sum_{\sigma \in L_0} a_\sigma . \end{cases}$$

In fact $\partial_i \circ \partial_{i+1} = 0$, because for each pair $\sigma \supset \lambda (\sigma \in L_i, \lambda \in L_{i-2})$

$$|\{\tau \in L_{i-1} | \lambda \subset \tau \subset \sigma\}| = q+1$$

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and for each line $\sigma \in L_1$

$$|\{\tau \in L_0 | \tau \subset \sigma\}| = q + 1 .$$

In the present note we shall be concerned with homology groups of the chain complexes (1) and (2):

$$H_i(\boldsymbol{P}^n(q), Z_{q+1}) , \qquad H_i(\boldsymbol{P}^n(q), GF(l)) \qquad (l | q + 1) ,$$

which are $PGL_{n+1}(q)$-modules. Frobenius map also acts on these homology groups. Obviously $H_{n+1}(\boldsymbol{P}^n(q), \ ) = H_0(\boldsymbol{P}^n(q), \ ) = \{0\}$.

2. Using Gauss-polynomials in q

$$\binom{a}{b}_q = \frac{(q^a - 1)(q^{a-1} - 1)\cdots(q^{a-b+1} - 1)}{(q^b - 1)(q^{b-1} - 1)\cdots(q - 1)} , \tag{3}$$

we may express briefly

$$|\boldsymbol{P}^i(q)| = \binom{i+1}{1}_q = 1 + q + \cdots + q^i , \tag{4}$$

$$|L_i(\boldsymbol{P}^n(q))| = \binom{n+1}{i+1}_q = \frac{(q^{n+1} - 1)(q^n - 1)\cdots(q^{n-i+1} - 1)}{(q^{i+1} - 1)(q^i - 1)\cdots(q - 1)} . \tag{5}$$

Let us first calculate the 1-homologies:

PROPOSITION 1.

$$H_1(\boldsymbol{P}^1(q), Z_{q+1}) \cong Z_{q+1}^{q-1} , \quad H_1(\boldsymbol{P}^1(q), GF(l)) = GF(l)^{q-1} \quad (l | q + 1) , \tag{6}$$

$$H_1(\boldsymbol{P}^n(q), Z_{q+1}) = 0 , \quad H_1(\boldsymbol{P}^n(q), GF(l)) = 0 \quad (n \geq 2, l | q + 1) . \tag{7}$$

Proof. We denote $\omega_\infty = [0, 1]$, $\omega_\alpha = [1, \alpha]$ $(\alpha \in GF(q))$, then $\{\omega_\alpha - \omega_\infty | \alpha \in GF(q), \alpha \neq 0\}$ is a free base of $H_1(\boldsymbol{P}^1(q))$, because

$$(\omega_0 - \omega_\infty) + \sum_{\alpha \neq 0} (\omega_\alpha - \omega_\infty) - (\omega_\infty - \omega_\infty) = \partial_2 \boldsymbol{P}^1(q) - (q + 1)\omega_\infty \sim 0 ,$$

$$a_\infty \omega_\infty + a_0 \omega_0 + \sum_{\alpha \neq 0} a_\alpha \omega_\alpha \sim \sum_{\alpha \neq 0} (a_\alpha - a_0)(\omega_\alpha - \omega_\infty) + (a_\infty + a_0 + \sum_{\alpha \neq 0} a_\alpha)\omega_\infty .$$

This proves (6). Assume that $n \geq 2$. It is sufficient to show $\sigma_\infty \sim \tau_\infty$ $(\sigma_\infty, \tau_\infty \in L_0, \sigma_\infty \neq \tau_\infty)$. Let λ, λ', λ'' be three different lines such that

$$\lambda \wedge \lambda' = \sigma_\infty , \quad \lambda' \wedge \lambda'' = \tau_\infty , \quad \lambda \wedge \lambda'' = \sigma_0$$

with a point σ_0. Then there exist a system of points σ_i, τ_j, τ_{ij} $(1 \leq i, j \leq q - 1)$ such that

$$\partial_2 \lambda = \sigma_\infty + \sigma_0 + \sum_{i=1}^{q-1} \sigma_i , \qquad \partial_2 \lambda' = \tau_\infty + \sigma_0 + \sum_{j=1}^{q-1} \tau_j ,$$

$$\partial_2(\tau_\infty \vee \sigma_i) = \tau_\infty + \sigma_i + \sum_{j=1}^{q-1} \tau_{ij}\,,^{1)} \qquad \partial_2(\sigma_\infty \vee \tau_j) = \sigma_\infty + \tau_j + \sum_{i=1}^{q-1} \tau_{ij}\,.$$

Since $q = -1$ in $Z_{q+1}(GF(l))$, it follows

$$\begin{aligned}\sigma_\infty - \tau_\infty &\sim q\tau_\infty - q\sigma_\infty \\ &\sim (q-1)\tau_\infty + \tau_\infty - (q-1)\sigma_\infty - \sigma_\infty \\ &\sim \left(-\sum_{i=1}^{q-1}\sigma_i - \sum_{i,j=1}^{q-1}\tau_{ij} + \tau_\infty\right) + \left(\sum_{j=1}^{q-1}\tau_j + \sum_{i,j=1}^{q-1}\tau_{ij} - \sigma_\infty\right) \\ &\sim \left(-\sigma_0 - \sum_{i=1}^{q-1}\sigma_i - \sigma_\infty\right) + \left(\tau_\infty + \sigma_0 + \sum_{j=1}^{q-1}\tau_j\right) \sim 0\,.\end{aligned}$$

3. For each flat σ we denote

$$S(\sigma) = \sum_{\substack{\lambda \in L_0 \\ \lambda \not\subset \sigma}} \lambda \vee \sigma\,,^{2)} \tag{8}$$

LEMMA 1.

$$\partial_{i+2}S(\sigma) = \begin{cases} (-1)^i \sum\limits_{\substack{\tau \in L_{i-1} \\ \tau \subset \sigma}} S(\tau) & (odd\ n)\,, \\ \sigma + (-1)^i \sum\limits_{\substack{\tau \in L_{i-1} \\ \tau \subset \sigma}} S(\tau) & (even\ n) \end{cases} \tag{9}$$

$$(\sigma \in L_i(\boldsymbol{P}^n(q))\,.$$

Proof. Since an i-flat in $\lambda \vee \sigma$ is σ or $\mu \vee \tau$ with an $(i-1)$-flat τ in σ and $\mu \in L_0$ satisfying $\mu \vee \sigma = \lambda \vee \sigma$, hence it follows

$$\begin{aligned}\partial_{i+2}S(\sigma) &= \left[\binom{n+1}{1}_q - \binom{i+1}{1}_q\right]\sigma + \left[\binom{i+1}{1}_q - \binom{i}{1}_q\right]\sum_{\substack{\tau \in L_{i-1} \\ \tau \subset \sigma \\ \lambda \not\subset \sigma}} \lambda \vee \tau \\ &= (q^{i+1} + \cdots + q^n)\sigma + q^i \sum_{\substack{\tau \subset \sigma \\ \lambda \not\subset \sigma}} \lambda \wedge \tau\,.\end{aligned}$$

On the other hand

$$\lambda \vee \tau = \sigma \qquad (\lambda \in L_0, \tau \in L_{i-1}; \lambda \not\subset \tau; \lambda, \tau \subset \sigma)\,,$$

hence we have

$$\begin{aligned}\sum_{\substack{\tau \subset \sigma \\ \lambda \not\subset \sigma}} \lambda \vee \tau &= \sum_{\tau \subset \sigma}\sum_{\lambda \not\subset \tau} \lambda \vee \tau - \sum_{\substack{\lambda, \tau \subset \sigma \\ \lambda \not\subset \tau}} \lambda \vee \tau \\ &= \sum_{\tau \subset \sigma} S(\tau) - \left[\binom{i+1}{1}_q - \binom{i}{1}_q\right]\binom{i+1}{1}_q \sigma \\ &= \sum_{\tau \in \sigma} S(\tau) - q^i(1 + q + \cdots + q^i)\sigma\,.\end{aligned}$$

1) $\tau_\infty \vee \sigma_i$ means the join of τ_∞ and σ_i.

2) $\lambda \vee \sigma$ means the join of λ and σ.

From $q^2 = 1$ in $Z_{q+1}(GF(l))$, we have

$$\partial_{i+2}S(\sigma) = (1 + q + \cdots + q^n)\sigma + q^i \sum_{\tau\subset\sigma} S(\tau)$$
$$= \begin{cases} (-1)^i \sum\limits_{\substack{\tau\subset\sigma \\ \tau\in L_{i-1}}} S(\tau) & (\text{odd } n), \\ \sigma + (-1)^i \sum\limits_{\substack{\tau\subset\sigma \\ \tau\in L_{i-1}}} S(\tau) & (\text{even } n). \end{cases}$$

THEOREM 1.

$$H_{i+1}(P^{2n}(q), Z_{q+1}) = 0, \quad H_{i+1}(P^{2n}(q), GF(l)) = 0 \quad (l|q+1). \tag{10}$$

Proof. By virtue of (7), from the assumption

$$\partial_{i+1}(\sum_{\sigma\in L_i} a_\sigma\sigma) = \sum_{\tau}(\sum_{\sigma\supset\tau} a_\sigma)\tau = 0$$

it follows

$$\partial_{i+2}(\sum_{\sigma} a_\sigma S(\sigma)) = \sum_{\sigma} a_\sigma\sigma + \sum_{\substack{\tau \\ \tau\in L_{i-1}}} (\sum_{\sigma\supset\tau} a_\sigma) S(\tau) = \sum_{\sigma} a_\sigma\sigma .$$

When $i = 0$, we have already proved in Proposition 1.

PROPOSITION 2. *The Euler-characteristic is given by*

$$\sum_{i=0}^{n+1} (-1)^i \dim H_i(P^n(q), GF(l)) = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i}_q$$
$$= \begin{cases} (-1)^{(n+1)/2} \prod\limits_{i=1}^{(n+1)/2} (q^{2i-1} - 1) & (\textit{odd } n) \\ 0 & (\textit{even } n) \end{cases} \quad (l|q+1). \tag{11}$$

Proof. From the relations

$$\dim C_{n+1}(P^n(q)) = 1 = \dim \operatorname{Im} \partial_{n+1},$$
$$\dim C_{i+1}(P^n(q)) = \dim \operatorname{Im} \partial_{i+1} + \dim \operatorname{Ker} \partial_{i+1}$$
$$= \dim \operatorname{Im} \partial_{i+1} + \dim \operatorname{Im} \partial_{i+2} + \dim H_{i+1}(P^n(q)),$$
$$\dim C_1(P^n(q)) = 1 + \operatorname{Ker} \partial_1 = 1 + \operatorname{Im} \partial_2 + \dim H_1(P^n(q)),$$
$$\dim C_0(P^n(q)) = 1$$

it follows

$$\sum_{i=0}^{n+1} (-1)^i \dim H_i(P^n(q), GF(l)) = \sum_{i=0}^{n+1} (-1)^i \dim C_i(P^n(q)) + 1$$
$$= \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i}_q .$$

From the relations

$$\binom{n}{l}_q = \binom{n}{n-l}_q, \quad \binom{n}{l}_q = \binom{n-1}{l}_q + (q^{n-l}-1)\binom{n-1}{l-1}_q,$$

we have

$$\sum(-1)^l\binom{2m+1}{l}_q = 0$$

and

$$\begin{aligned}\sum(-1)^l\binom{2m}{l}_q &= \sum(-1)^l\binom{2m-1}{l}_q + \sum(-1)^l(q^{2m-l}-1)\binom{2m-1}{l-1}_q \\ &\quad + \sum(-1)^l\binom{2m-1}{l-1}_q \\ &= -\sum(-1)^{l-1}\frac{(q^{2m-1}-1)\cdots(q^{2m-l}-1)}{(q^{l-1}-1)\cdots(q-1)} \\ &= -(q^{2m}-1)\sum(-1)^l\binom{2(m-1)}{l}_q = (-1)^m\prod_{i=1}^{m}(q^{2i}-1).\end{aligned}$$

4. Using a system of homogeneous coordinates, we define the dual flat $\hat{\sigma} \in L_{n-i+1}$ of a flat $\sigma \in L_i$ as follows

$$\hat{\sigma} = \left\{[\hat{\alpha}_0, \cdots, \hat{\alpha}_n] \,\middle|\, \sum_{i=0}^{n} \hat{\alpha}_i\beta_i = 0 \ \ ([\beta_0, \cdots, \beta_n] \in \sigma)\right\} \qquad (2 \leq i \leq n+1)$$

and

$$\widehat{P^n(q)} = 1,$$

and we have the dual pairings;

$$\begin{gathered} C_{i+1}(P^n(q), Z_{q+1}) \times C_{n-i+2}(P^n(q), Z_{q+1}) \longrightarrow Z_{q+1}, \\ C_{i+1}(P^n(q), GF(l)) \times C_{n-i+2}(P^n(q), GF(l)) \longrightarrow GF(l), \end{gathered}$$

defined by

$$\text{(12)} \qquad \begin{cases} \langle\sigma, \hat{\tau}\rangle = \begin{cases} 1 & (\sigma = \tau) \\ 0 & (\sigma \neq \tau) \end{cases} & (\sigma \in L_i,\ i \leq n-1) \\ \langle 1, P^n(q)\rangle = 1 \end{cases}$$

LEMMA 2.

$$\text{(13)} \qquad \langle\partial_1(\sigma), P^n(q)\rangle = \langle\sigma, \partial_n P^n(q)\rangle = 1 \quad (\sigma \in L_0(P^n(q)),$$

$$\text{(14)} \qquad \langle\partial_{i+1}\sigma, \hat{\tau} = \langle\sigma, \partial_{n-i+1}\hat{\tau}\rangle \quad (1 \leq i \leq n, \sigma \in L_i, \hat{\tau} \in L_{n-i}).$$

Proof. The first equation (11) is a direct consequence of the definition $\langle\,,\rangle$. Since two conditions $\sigma \supset \lambda$ and $\hat{\sigma} \subset \hat{\lambda}$ are equivalent, it follows

$$\langle \partial_{i+1}\sigma, \hat{\tau}\rangle = \sum_{\tau \in L_{i-1}} \langle \lambda, \hat{\tau}\rangle = \begin{cases} 1 & (\tau \subset \sigma) \\ 0 & (\tau \not\subset \sigma) \end{cases},$$

$$\langle \sigma, \partial_{n-i+1}\hat{\tau}\rangle = \sum_{\substack{\hat{\lambda} \in L_{n-i-1} \\ \hat{\lambda} \subset \hat{\tau}}} \langle \sigma, \hat{\lambda}\rangle = \sum_{\substack{\lambda \in L_i \\ \lambda \supset \tau}} \langle \sigma, \hat{\lambda}\rangle = \begin{cases} 1 & (\tau \subset \sigma) \\ 0 & (\tau \not\subset \sigma) \end{cases}.$$

PROPOSITION 3.

$$H_{i+1}(\boldsymbol{P}^n(q), Z_{q+1}) \simeq H_{n-i}(\boldsymbol{P}^n(q), Z_{q+1}) \tag{15}$$

$$H_{i+1}(\boldsymbol{P}^n(q), GF(l)) \simeq H_{n-i}(\boldsymbol{P}^n(q), GF(l)) \quad (l|q+1). \tag{16}$$

Proof. By virtue of Lemma 2 we have

$$\operatorname{Ker} \partial_{i+1} = (\operatorname{Im} \partial_{n-i+1})^{\perp}$$
$$\operatorname{Im} \partial_{i+2} = (\operatorname{Ker} \partial_{n-i})^{\perp}$$
$$\operatorname{Ker} \partial_1 = (\operatorname{Im} \partial_{n+1})^{\perp},$$

where $(\ \ )^{\perp}$ means the orthogonal submodule with respect to the pairing. Proposition 3 is a direct consequence of the duality for modules with a pairing.

PROPOSITION 4.

$$\dim H_2(\boldsymbol{P}^3(q), GF(l)) = 1 - q - q^3 + q^4 \quad (l|q+1). \tag{17}$$

Proof. By virtue of Proposition 1, 2, 3 it follows

$$\dim H_2(\boldsymbol{P}^3(q), GF(l)) = \binom{4}{2}_q - 2\binom{4}{1}_q + 2 = 1 - q - q^3 + q^4.$$

EXAMPLES.

$$\dim H_2(\boldsymbol{P}^3(2), GF(3)) = 7,$$
$$\dim H_2(\boldsymbol{P}^3(3), GF(2)) = 52,$$
$$\dim H_2(\boldsymbol{P}^3(4), GF(5)) = 189,$$
$$\dim H_2(\boldsymbol{P}^3(5), GF(2)) = \dim H_2(\boldsymbol{P}^3(5), GF(3)) = 496.$$

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