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THE BOUNDARY BEHAVIOUR OF HADAMARD LACUNARY SERIES

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§ 1. Introduction

A convergent power series $f(z)$ in the open unit disk $\boldsymbol{D}$ is called Hadamard lacunary if it is expressed as follows:

$$(1) \qquad f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}, \quad n_{k+1}/n_k \geq q \ (k \geq 1) \qquad \text{for some } q > 1 .$$

We shall discuss the boundary behaviour of Hadamard lacunary series. For a subset X of $\boldsymbol{D}$, we put $b(X) = \bar{X} \cap \partial\boldsymbol{D}$, where $\bar{X}$ is the closure of X and $\partial\boldsymbol{D}$ the boundary of $\boldsymbol{D}$. We say that an analytic function $g(z)$ in $\boldsymbol{D}$ has an extended complex number ω as an asymptotic value if there exists a path $\gamma \subset \boldsymbol{D}$ with $b(\gamma) \neq \emptyset$ such that $\lim_{|z|\to 1, z\in\gamma} g(z) = \omega$. We say that $g(z)$ has an asymptotic value at $a \in \partial\boldsymbol{D}$ if there exists a path $\gamma \subset \boldsymbol{D}$ with $b(\gamma) = \{a\}$ such that $\lim_{z\to a, z\in\gamma} g(z)$ exists. The Maclane class $\mathscr{A}$ is the totality of analytic functions $g(z)$ in $\boldsymbol{D}$ such that $g(z)$ has asymptotic values at a dense subset of $\partial\boldsymbol{D}$.

In [5], G. R. Maclane proved that a power series $f(z)$ given by (1) with $q > 3$ belongs to $\mathscr{A}$. It is conjectured that Hadamard lacunary series belong to $\mathscr{A}$. In [1], J. M. Anderson noted that Maclane's result is deduced from a result of K. G. Binmore in [2]. In [3], K. G. Binmore and R. Hornblower gave an another partial answer to this question. We shall answer this question. The main purpose of this paper is to show

THEOREM. *Let $f(z)$ be an Hadamard lacunary series given by* (1) *with* $\limsup_{k\to\infty} |c_k| = \infty$. *Then $f(z)$ has an asymptotic value ∞ at every point of $\partial\boldsymbol{D}$.*

It is known that the Hadamard lacunary series in our theorem has no finite asymptotic value ([2]), and hence ∞ is a unique asymptotic value.

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If an Hadamard lacunary series $f(z)$ given by (1) satisfies $\limsup_{k\to\infty}|c_k| < \infty$, then Paley's theorem ([11]) yields $f \in \mathscr{A}$. Hence we have, by our theorem,

COROLLARY. *Hadamard lacunary series belong to* $\mathscr{A}$.

As application of our method, we shall note that Property (A) (which will be stated later) deduces Binmore's result in [2] and Sons's result on annular functions.

§ 2. Fundamental tools

LEMMA 1 ([4]). *Let p be a positive integer and $g(\zeta)$ an analytic function in $D(w, \rho) = \{\zeta; |\zeta - w| < \rho\}$ such that $|g^{(p)}(w)| \geq y_1$ and $|g^{(p)}(\zeta)| \leq y_2$ ($\zeta \in D(w, \rho)$). Then there exists $0 < \varepsilon < \rho$ such that*

$$|g(\zeta) - g(w)| \geq \eta(p)\rho^p y_1^{p+1} y_2^{-p}$$

for all $\zeta \in S(w, \varepsilon) = \{z; |z - w| = \varepsilon\}$, where $\eta(p)$ is a constant depending only on p.

In this lemma, we may assume that $\eta(1) \geq \eta(2) \geq \cdots$; consider $\min\{\eta(j);\ 1 \leq j \leq p\}$ $(p = 1, 2, \cdots)$ if necessary.

LEMMA 2 ([11]). *Given $q > 1$, there exist two constants $0 < A \leq 1$ and $B \geq 1$ depending only on q with the following property: For every lacunary polynomial $P(t) = \sum_{k=1}^{n} a_k e^{im_k t}$, $m_{k+1}/m_k \geq q$ and every interval I in $[0, 2\pi)$ of length $\geq B/m_1$, there exists $t_0 \in I$ such that $\operatorname{Re} P(t_0) \geq A \sum_{k=1}^{n} |a_k|$.*

LEMMA 3. *Let*

$$Q(\zeta) = \sum_{k=1}^{n} a_k \exp(m_k \zeta), \qquad m_{k+1}/m_k \geq q > 1\ (k \geq 1)\,. \tag{2}$$

Then, for every complex number w and $1 \leq d \leq n$, there exists an integer $\ell = \ell(Q, w, d)$ with $0 \leq \ell \leq n - 1$ such that

$$|Q^{(\ell)}(w)| \geq C m_d^{\ell} |a_d| \exp(m_d \operatorname{Re} w)\,, \tag{3}$$

where $C = 1/2 \cdot \prod_{k=1}^{\infty}\{(1 - q^{-k})/(1 + q^{-k})\}^2$.

Proof. This lemma is analogous to Lemma 8 in [6]. The following elegant proof was communicated by W. H. J. Fuchs. Without loss of generality, we may assume $a_d \neq 0$. Let us consider an equation:

$$(4)\qquad \begin{bmatrix} 1 & \cdots & 1 \\ m_1 & \cdots & m_n \\ & \cdots\cdots & \\ m_1^{n-1} & \cdots & m_n^{n-1} \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Then we have, with $\Delta = \prod_{k<j}(m_j - m_k)$,

$$|x_d| = \left|\det\begin{pmatrix} 1 & \cdots & 1 & y_1 & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_1^{n-1} & \cdots & m_{d-1}^{n-1} & y_n & m_{d+1}^{n-1} & \cdots & m_n^{n-1} \end{pmatrix}\right| \Big/ \Delta$$

$$\leq \sum_{\ell=1}^{n} \left|\det\left[\begin{array}{c|c} 1 & \cdots 1 \\ \vdots & \vdots \\ \hline \vdots & \vdots \\ m_1^{n-1} & \cdots m_n^{n-1} \end{array}\right]\right| |y_\ell|/\Delta$$

(omit the d^{th} column and the ℓ^{th} row from the determinant in (4))

$$= \sum_{\ell=1}^{n}\left\{\sigma_{d,\ell}\prod_{k<j;k,j\neq d}(m_j - m_k)\right\}|y_\ell|/\Delta = \prod_{k\neq d}|m_k - m_d|^{-1}\sum_{\ell=1}^{n}\sigma_{d,\ell}|y_\ell|,$$

where $\sigma_{d,\ell}$'s are defined by $\prod_{1\leq k\leq n;k\neq d}(x + m_k) = \sigma_{d,n}x^{n-1} + \sigma_{d,n-1}x^{n-2} + \cdots + \sigma_{d,1}$. If $|y_\ell| \leq Cm_d^{\ell-1}|x_d|$ ($\ell = 1, \cdots, n$), then

$$\begin{aligned} |x_d| &\leq \prod_{k\neq d}|m_k - m_d|^{-1}\sum_{\ell=1}^{n}\sigma_{d,\ell}m_d^{\ell-1}C|x_d| \\ &= \prod_{k\neq d}\{(m_k + m_d)/|m_k - m_d|\}C|x_d| \\ &\leq \prod_{k=1}^{\infty}\{(1 + q^{-k})/(1 - q^{-k})\}^2C|x_d| = |x_d|/2, \end{aligned}$$

and hence $x_d = 0$.

Now we put $y_\ell = Q^{(\ell-1)}(w)$ $(1 \leq \ell \leq n)$ in (4). Then $x_\ell = a_\ell \exp(m_\ell w)$ $(1 \leq \ell \leq n)$. If (3) does not hold for all ℓ with $0 \leq \ell \leq n-1$, then $x_d = 0$, that is, $a_d = 0$. This is a contradiction. Hence (3) holds for some ℓ with $0 \leq \ell \leq n-1$.

§3. Proof of Theorem

In this section, we shall show that our theorem follows from two properties, which will be stated later. Let $f(z)$ be an Hadamard lacunary series given by (1) with $\limsup_{k\to\infty}|c_k| = \infty$. Our purpose is to construct a path $\gamma \subset D$ with $b(\gamma) = \{a\}$ such that $f(z)$ has ∞ as an asymptotic value along with γ. Without loss of generality, we may assume $a = 1$. Adding terms with coefficient 0 if necessary, we may assume that $q \leq n_{k+1}/n_k \leq q^2$ $(k \geq 1)$.

To construct such an arc, we deal with an analytic function

$$F(\zeta) = f(e^\zeta) = \sum_{k=1}^{\infty} c_k \exp(n_k\zeta) \tag{5}$$

in a domain $U = \{\zeta; \operatorname{Re}\zeta < 0\}$ and shall construct a path $\Gamma \subset U$ with $b(\Gamma) = \{0\}$ such that $F(\zeta)$ has ∞ as an asymptotic value along with Γ.

Now we introduce some notation. Throughout the paper, A, B and C are the constants in Lemmas 2 and 3. We put $\theta = A/8$. For every $-1 \leq r < 0$, we put

$$\begin{cases} M_r = \max\{|c_k|\exp(n_k r); k \geq 1\} & \text{(the maximum term)} \\ \mu_r = \min\{k; |c_k|\exp(n_k r) = M_r\} & \text{(the smallest central index)} \\ \nu_r = \max\{k; |c_k|\exp(n_k r) = M_r\} & \text{(the largest central index)} \\ \alpha_r = r - \theta/n_{\mu_r} & \text{(the smallest dominant point)} \\ \beta_r = r - \theta/n_{\nu_r} & \text{(the largest dominant point)} \\ I(t, r) = \{x + it; \alpha_r \leq x \leq \beta_r\} & (|t| \leq \pi)\,. \end{cases} \tag{6}$$

Then $\lim_{r\to 0} M_r = \lim_{r\to 0}\mu_r = \lim_{r\to 0}\nu_r = \infty$ and $\lim_{r\to 0}\alpha_r = \lim_{r\to 0}\beta_r = 0$. We denote by $(\nu_m)_{m=1}^{\infty}$ $(\nu_{m+1} > \nu_m)$ the totality of the largest central indexes. Since ν_r is increasing and continuous on the right, we can find r_m, s_m such that $\cup\{r; \nu_r = \nu_m\} = [r_m, s_m)$ $(m \geq 1)$. We have $s_m = r_{m+1}$ $(m \geq 1)$.

Now we prove $\mu_{s_m} = \nu_m$. Since μ_r is continuous on the left, we have $\mu_{s_m} = \lim_{r\uparrow s_m}\mu_r \leq \lim_{r\uparrow s_m}\nu_r = \nu_m$. Let $\mathscr{R}$ be the (finite) set of all integers with $|c_k|\exp(n_k s_m) = M_{s_m}$ $(k \geq 1)$. Then the smallest integer in $\mathscr{R}$ is μ_{s_m}. We have

$$\psi'_{\mu^*}(s_m) < \psi'_k(s_m) \qquad (k \in \mathscr{R}, k \neq \mu^*)\,, \tag{7}$$

where $\mu^* = \mu_{s_m}$ and $\psi_k(r) = |c_k|\exp(n_k r)$. Hence $|c_{\mu^*}|\exp(n_{\mu^*}r) > |c_k|\exp(n_k r)$ $(\mu^* < k \leq \nu_{m+1})$ for all r $(r < s_m)$ sufficiently near to s_m. Since $\nu_r \leq \nu_{m+1}$ $(r \leq s_m)$, this signifies $\nu_r \leq \mu_{s_m}$ for all r $(r < s_m)$ sufficiently near to s_m. Thus $\nu_m = \lim_{r\uparrow s_m}\nu_r \leq \mu_{s_m} \leq \nu_m$. Consequently, $\mu_{s_m} = \mu_{r_{m+1}} = \nu_m$. By these facts, we have $\cup\{\beta_r; -1 \leq r < 0, \nu_r = \nu_m\} = [\beta_{r_m}, \alpha_{r_{m+1}})$ $(m \geq 1)$.

For every $-1 \leq r < 0$, we denote by ξ_r the largest integer in a set of m's $(m \geq 1)$ with $\sum_{k<m}|c_k| \leq A/2\cdot M_r$; if the set is empty, we put $\xi_r = 0$. Then $\lim_{r\to 0}\xi_r = \infty$. We need the following two properties.

(A) For every w with $\beta_{r_m} \leq \operatorname{Re} w \leq \alpha_{r_{m+1}}$ (for some m), there exists a positive number ε_w with $0 < \varepsilon_w \leq 1/n_{\nu_m}$ such that $|F(\zeta)| \geq DM_{r_m}$ ($\zeta \in S(w, \varepsilon_w)$), where D is a constant depending only on q.

(B) For every $m \geq 2$, there exist a point t_m with $|t_m| \leq 2B/n_\xi$ ($\xi = \xi_{r_m}$) and a corresponding Jordan curve Γ_m with $\operatorname{diam}(\Gamma_m) =$ (the diameter of Γ_m) $\leq 3/n_{\nu_{m-1}}$ such that $\langle \Gamma_m \rangle \supset I(t_m, r_m)$ and $|F(\zeta)| \geq EM_{r_m}$ ($\zeta \in \Gamma_m$), where $\langle \Gamma_m \rangle$ is the domain bounded by Γ_m and E a constant depending only on q.

We postpone the proof of (A) and (B) to the sections 4 and 5. From now, we construct a required path Γ assuming (A) and (B).

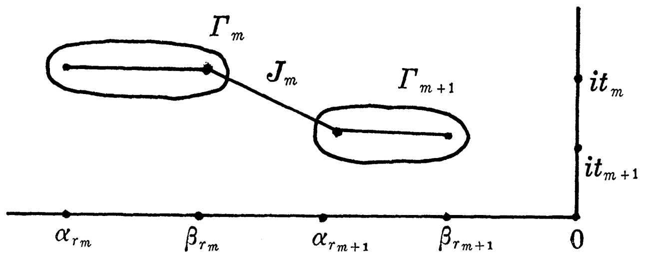

Fig.

Note that $[\beta_{r_1}, 0) = \bigcup_{m=1}^{\infty} [\beta_{r_m}, \alpha_{r_{m+1}}] \cup [\alpha_{r_{m+1}}, \beta_{r_{m+1}}]$. Let J_m be the segment which connects $\beta_{r_m} + it_m$ and $\alpha_{r_{m+1}} + it_{m+1}$ ($m \geq 1$). The property (A) shows that, for every $w \in J_m$, there exists $0 < \varepsilon_w \leq 1/n_{\nu_m}$ such that $|F(\zeta)| \geq DM_{r_m}$ ($\zeta \in S(w, \varepsilon_w)$). This shows that there exists a Jordan curve γ_m with $\kappa_m = \max\{$the distance of ζ and J_m; $\zeta \in \gamma_m\} \leq 1/n_{\nu_m}$ such that $\langle \gamma_m \rangle \supset J_m$ and $|F(\zeta)| \geq DM_{r_m}$ ($\zeta \in \gamma_m$). Put $\Gamma^* = \bigcup_{m=1}^{\infty} (\Gamma_m \cup \gamma_m)$. Since $\lim_{m \to \infty} t_m = 0$, we have $b(\Gamma^*) \ni 0$. Since $\sum_{j=m}^{\infty} \operatorname{diam}(\Gamma_j) + \sum_{j=m}^{\infty} \kappa_j = o(1)$ ($m \to \infty$), we have $b(\Gamma^*) = \{0\}$. Since Γ^* is arcwise connected, we can choose a path $\Gamma \subset \Gamma^*$ with $b(\Gamma) = \{0\}$. Then $F(\zeta)$ has ∞ as an asymptotic value along with Γ.

§ 4. Proof of (A)

In this section, we prove (A). Let w satisfy $\beta_{r_m} \leq \operatorname{Re} w \leq \alpha_{r_{m+1}}$. Put $r = \operatorname{Re} w + \theta/n_{\nu_m}$. Then $r_m \leq r \leq r_{m+1}$ and $M_r = |c_{\nu_m}| \exp(n_{\nu_m} r)$.

LEMMA 4. *There exists a positive integer $p = p(F, w)$ with $1 \leq p \leq N$ (N: a constant depending only on q) such that*

$$|F^{(p)}(w)| \geq C/2e \cdot M_r n_{\nu_m}^p \,. \tag{8}$$

Proof. Let us write

$$F'(\zeta)=\sum_{k=1}^{\infty} n_k c_k \exp(n_k\zeta)=\sum_{k<\nu_m-n}+\sum_{\nu_m-n\le k\le \nu_m+n}+\sum_{k>\nu_m+n}$$
$$=\phi(\zeta)+Q(\zeta)+\Phi(\zeta)\,,$$

where n is determined later. Lemma 3 shows that there exists $\ell=\ell(n)$ with $0\le \ell\le 2n$ such that

$$(9)\qquad |Q^{(\ell)}(w)|\ge Cn_{\nu_m}^{\ell+1}|c_{\nu_m}|\exp\{n_{\nu_m}(r-\theta/n_{\nu_m})\}\ge C/e\cdot M_r n_{\nu_m}^{\ell+1}\,.$$

We have

$$(10)\qquad |\phi^{(\ell)}(w)|\le M_r\sum_{k<\nu_m-n} n_k^{\ell+1}=M_r n_{\nu_m}^{\ell+1}\sum_{k<\nu_m-n}(n_k/n_{\nu_m})^{\ell+1}$$
$$\le M_r n_{\nu_m}^{\ell+1}\sum_{j=n+1}^{\infty} q^{-j(\ell+1)}\le\{1/q^n(q-1)\}M_r n_{\nu_m}^{\ell+1}\,.$$

Note that $x^{2n+1}e^{-\theta x}$ is decreasing in $[(2n+1)/\theta,\infty)$. We choose an integer $N_0=N_0(\theta)$ so that $q^j\ge(2j+1)/\theta$ $(j\ge N_0(\theta))$. Let $n\ge N_0$. Then

$$(11)\qquad \begin{aligned}|\Phi^{(\ell)}(w)|&\le\sum_{k>\nu_m+n} n_k^{\ell+1}|c_k|\exp\{n_k(r-\theta/n_{\nu_m})\}\\&\le M_r\sum_{k>\nu_m+n} n_k^{\ell+1}\exp(-\theta n_k/n_{\nu_m})\\&\le M_r n_{\nu_m}^{\ell+1}\sum_{k>\nu_m+n}(n_k/n_{\nu_m})^{2n+1}\exp(-\theta n_k/n_{\nu_m})\\&\le M_r n_{\nu_m}^{\ell+1}\sum_{j=n+1}^{\infty} q^{j(2n+1)}\exp(-\theta q^j)\,(=M_r n_{\nu_m}^{\ell+1}\tau_n(\theta),\text{ say})\,.\end{aligned}$$

Now we choose n $(\ge N_0)$ so that $1/q^n(q-1)\le C/4e$, $\tau_n(\theta)\le C/4e$ and put $p=p(F,w)=\ell(n)+1$, $N=2n+1$. Then (8) follows from (9), (10) and (11). Q.E.D.

LEMMA 5. *Let $p=p(F,w)$ be the integer in Lemma 4. Then, for any $\zeta\in D(w,1/2n_{\nu_m})$,*

$$(12)\qquad |F^{(p)}(\zeta)|\le D_0 M_r n_{\nu_m}^p\,,$$

where $D_0=\{1+(2/\theta)^{2N}(2N)!\}q/(q-1)$.

Proof. Note that $e^{-\theta x/2}\le(2/\theta)^{2p}(2p)!\,x^{-2p}$ $(x>0)$. Since $\mathrm{Re}\,\zeta\le r-\theta/2n_{\nu_m}$, we have

$$|F^{(p)}(\zeta)|\le\sum_{k=1}^{\infty} n_k^p|c_k|\exp\{n_k(r-\theta/2n_{\nu_m})\}$$
$$\le M_r\sum_{k=1}^{\infty} n_k^p\exp(-\theta n_k/2n_{\nu_m})=M_r\left\{\sum_{k=1}^{\nu_m}+\sum_{k=\nu_m+1}^{\infty}\right\}$$

$$\leq M_r n_{\nu_m}^p \left\{ \sum_{k=1}^{\nu m} (n_k/n_{\nu_m})^p + (2/\theta)^{2p}(2p)! \sum_{k=\nu m+1}^{\infty} (n_k/n_{\nu_m})^p (n_{\nu_m}/n_k)^{2p} \right\}$$

$$\leq M_r n_{\nu_m}^p \{1 + (2/\theta)^{2p}(2p)!\} \sum_{j=0}^{\infty} q^{-pj} \leq D_0 M_r n_{\nu_m}^p . \qquad \text{Q.E.D.}$$

Now we apply Lemma 1 to $g(\zeta) = F(\zeta)$ and $D(w, \theta/2n_{\nu_m})$. There exists $0 < \varepsilon \leq \theta/2n_{\nu_m}$ ($\leq 1/n_{\nu_m}$) such that, for any $\zeta \in S(w, \varepsilon)$,

$$\begin{aligned} |F(\zeta) - F(w)| &\geq \eta(p)(\theta/2n_{\nu_m})^p (C/2e \cdot M_r n_{\nu_m}^p)^{p+1} (D_0 M_r n_{\nu_m}^p)^{-p} \\ &= \{\eta(p)(\theta/2)^p (C/2e)^{p+1} D_0^{-p}\} M_r \\ &\geq \{\eta(N)(\theta/2)^N (C/2e)^{N+1} D_0^{-N}\} M_{r_m} \ (= 3DM_{r_m}, \text{ say}) . \end{aligned} \tag{13}$$

If $|F(w)| < 2DM_{r_m}$, then $|F(\zeta)| \geq DM_{r_m}$ ($\zeta \in S(w, \varepsilon)$). Hence $\varepsilon_w = \varepsilon$ is a required number. If $|F(w)| \geq 2DM_{r_m}$, we choose $0 < \varepsilon_w \leq 1/n_{\nu_m}$ so that $|F(\zeta)| \geq DM_{r_m}$ ($\zeta \in S(w, \varepsilon_w)$). This completes the proof of (A).

§ 5. Proof of (B)

In this section, we prove (B). For the sake of simplicity, we write, for a polynomial $P(t) = \sum_{k=1}^{n} a_k e^{i m_k t}$, $\|P\| = \sum_{k=1}^{n} |a_k|$, $\ell(P) =$ (the length of P) $= n$, s.e. $P =$ (the smallest exponent in P) $= m_1$, l.e. $P =$ (the largest exponent in P) $= m_n$.

Given $m \geq 2$, our purpose is to define a point t_m and a corresponding Jordan curve Γ_m having the required properties. We write simply $r = r_m$, $\xi = \xi_m$, $\mu = \mu_{r_m} (= \nu_{m-1})$, $\nu = \nu_m$. We need two constants λ, Λ depending only on q which are defined as follows.

Let λ be a positive integer such that $B/\{\theta q^{\lambda-1}(q-1)\} \leq A/32$ and Λ a positive integer such that $(A/2 + 1)/\Lambda \leq A/4$.

Using λ, Λ, we define polynomials $\bar{\Delta}_0$, $\underline{\Delta}_1$, $\bar{\Delta}_1$, $\underline{\Delta}_2$, $\bar{\Delta}_2$, $\cdots$ with $\ell(\bar{\Delta}_j) \leq 2\lambda(\Lambda - 1)$, $\ell(\underline{\Delta}_j) = \lambda$ ($j \geq 1$). Let $\Delta_0^*(t) = \sum_{k=\xi}^{\mu} c_k \exp\{n_k(r + it)\}$, $\Delta_\ell^*(t) = \sum_{\mu+\lambda(\ell-1) < k \leq \mu+\lambda\ell} c_k \exp\{n_k(r + it)\}$ ($\ell \geq 1$). Choosing a sequence $(\ell_j)_{j=1}^{\infty}$ of positive integers so that $\|\Delta_{\ell_j}^*\| = \min\{\|\Delta_\ell^*\|; \Lambda(j-1) < \ell \leq \Lambda j\}$, we put $\bar{\Delta}_0 = \Delta_0^* + \sum_{\ell<\ell_1} \Delta_\ell^*$, $\underline{\Delta}_j = \Delta_{\ell_j}^*$, $\bar{\Delta}_j = \sum_{\ell_j < \ell < \ell_{j+1}} \Delta_\ell^*$ ($j \geq 1$), where $\bar{\Delta}_j \equiv 0$ if $\ell_{j+1} = \ell_j + 1$. Thus the required polynomials are defined. We put $h_j =$ s.e. $\bar{\Delta}_j$ ($j \geq 1$), $H_j =$ l.e. $\bar{\Delta}_j$ ($j \geq 0$), where $h_j = H_j =$ l.e. $\underline{\Delta}_j$ if $\bar{\Delta}_j \equiv 0$. Denoting by σ the smallest non-negative integer such that $n_\nu \leq H_j$ ($j \geq 0$), we put $S_0 = \bar{\Delta}_0$, $S_j = \sum_{\ell=0}^{j} \bar{\Delta}_\ell + \sum_{\ell=1}^{j} \underline{\Delta}_\ell$ ($1 \leq j \leq \sigma$). Then s.e. $S_0 = n_\xi$, l.e. $S_j = H_j$ ($0 \leq j \leq \sigma$). The required point t_m is defined by

LEMMA 6. *There exists t_m with $|t_m| \leq 2B/n_\xi$ such that $|S_j(t_m)| \geq A/4 \cdot \|S_j\|$ ($0 \leq j \leq \sigma$).*

Proof. Using Lemma 2, we define inductively $\sigma+1$ points $(u_j)_{j=0}^{\sigma}$ in the following manner: Let u_0 be a point with $|u_0|\leq B/n_\xi$ such that $\operatorname{Re} S_0(u_0)\geq A\|S_0\|$ and u_j a point with $|u_j-u_{j-1}|\leq B/h_j$ such that $\operatorname{Re}\bar{\varDelta}_j(u_j)\geq A\|\bar{\varDelta}_j\|$ $(1\leq j\leq\sigma)$. We put $t_m=u_\sigma$ and prove that this is a required point.

We have

$$
(14)\quad \begin{aligned}|u_j-t_m|&\leq B\sum_{\ell>j}1/h_\ell\\ &=B/H_j\sum_{\ell>j}(H_j/h_\ell)\leq B/\{q^{\lambda-1}(q-1)H_j\}\leq A/(2H_j)\qquad(0\leq j\leq\sigma)\,.\end{aligned}
$$

In particular, $|t_m|\leq|u_0|+A/(2H_0)\leq B/n_\xi+A/(2n_\xi)\leq 2B/n_\xi$. By (14), we have

$$
(15)\quad \begin{aligned}\operatorname{Re}\bar{\varDelta}_j(t_m)&\geq\operatorname{Re}\bar{\varDelta}_j(u_j)-|u_j-t_m|\|\bar{\varDelta}_j'\|\\ &\geq A\|\bar{\varDelta}_j\|-(A/2H_j)H_j\|\bar{\varDelta}_j\|\geq A/2\cdot\|\bar{\varDelta}_j\|\qquad(1\leq j\leq\sigma)\end{aligned}
$$

and $\operatorname{Re} S_0(t_m)\geq\operatorname{Re} S_0(u_0)-|u_0-t_m|H_0\|S_0\|\geq A/2\cdot\|S_0\|$. Hence the required inequality holds for $j=0$. Let $1\leq j\leq\sigma$. Then (15) gives

$$
\begin{aligned}|S_j(t_m)|&\geq\operatorname{Re} S_j(t_m)\geq\operatorname{Re} S_0(t_m)+\sum_{\ell=1}^{j}\operatorname{Re}\bar{\varDelta}_\ell(t_m)-\sum_{\ell=1}^{j}\|\underline{\varDelta}_\ell\|\\ &\geq A/2\cdot\left(\|S_0\|+\sum_{\ell=1}^{j}\|\bar{\varDelta}_\ell\|\right)-\sum_{\ell=1}^{j}\|\underline{\varDelta}_\ell\|\\ &\geq\{A/2\cdot(1-1/\Lambda)-1/\Lambda\}\|S_j\|\geq A/4\cdot\|S_j\|\,.\end{aligned}\qquad\text{Q.E.D.}
$$

To define the required Jordan curve Γ_m, we assume, for a while, $\sigma\geq 2$ and consider intervals $[r-\theta/n_\mu, r-\theta/H_0]$, $[r-\theta/H_{j-1}, r-\theta/H_j]$ $(1\leq j\leq\sigma-1)$, $[r-\theta/H_{\sigma-1}, r-\theta/n_\nu]$. We prepare

LEMMA 7. *If* $x\in[r-\theta/H_{j-1}, r-\theta/H_j]$ *for some* $1\leq j\leq\sigma-1$, *then*

$$
(16)\qquad |F(x+it_m)|\geq A/16\cdot M_r-T_j\,,
$$

where $T_j=\|\underline{\varDelta}_j\|+\|\bar{\varDelta}_j\|+\|\underline{\varDelta}_{j+1}\|$.

Proof. Writing

$$
\begin{aligned}F(x+it_m)&=\sum_{k=1}^{\infty}c_k\exp\{n_k(x+it_m)\}\\ &=\sum_{k<\xi}+\sum_{n_\xi\leq n_k\leq H_{j-1}}+\sum_{H_{j-1}<n_k<h_{j+1}}+\sum_{n_k\geq h_{j+1}}\,,\end{aligned}
$$

we denote by $S_{j-1,x}$ the second term. Then

$$
\begin{aligned}|S_{j-1,x}|&\geq|S_{j-1}(t_m)|-|x-r|\|S_{j-1}'\|\\ &\geq A/4\cdot\|S_{j-1}\|-(\theta/H_{j-1})H_{j-1}\|S_{j-1}\|=A/8\cdot\|S_{j-1}\|\geq A/8\cdot M_r\,.\end{aligned}
$$

On the other hand, the sum of absolute values of other terms is dominated by

$$\sum_{k<\xi}|c_k| + \{\|\underline{\Delta}_j\| + \|\bar{\Delta}_j\| + \|\underline{\Delta}_{j+1}\|\} + M_r \sum_{n_k \ge h_{j+1}} \exp(-\theta n_k/H_j)$$
$$\le A/32\cdot M_r + T_j + 1/\{\theta q^{\lambda-1}(q-1)\}\cdot M_r \le A/16\cdot M_r + T_j\,.$$

Hence we have (16). Q.E.D.

For the definition of Γ_m, we choose, for every $x\in[\alpha_r,\beta_r]$, a number $0<\varepsilon_x\le 1/n_\mu$ such that $|F(\zeta)|\ge EM_r$ $(\zeta\in S(x+it_m,\varepsilon_x))$ (E: some constant). Let $x\in[r-\theta/H_{j-1}, r-\theta/H_j]$ $(1\le j\le\sigma-1)$. We must distinguish the following two cases:

(a) $T_j < A/32\cdot M_r$, (b) $T_j \ge A/32\cdot M_r$.

In the case (a), we have $|F(x+it_m)|\ge A/32\cdot M_r$ and hence we can choose $0<\varepsilon_x\le 1/n_\mu$ so that $|F(\zeta)|\ge E_1M_r$ $(\zeta\in S(x+it_m,\varepsilon_x))$ with $E_1=A/64$. In the case (b), the choice of ε_x will be analogous as in the proof of (A).

Since $T_j\ge A/32\cdot M_r$, $\ell(\underline{\Delta}_j+\bar{\Delta}_j+\underline{\Delta}_{j+1})\le 2\lambda\Lambda$, there exists d with $H_{j-1} < n_d \le h_{j+1}$ such that $|c_d|\exp(n_d r)\ge A/(64\lambda\Lambda)\cdot M_r$. Hence $|c_d|\exp(n_d x)\ge |c_d|\exp(n_d r-\theta n_d/H_{j-1})\ge A/\{64\lambda\Lambda\exp(\theta q^{4\lambda\Lambda})\}\cdot M_r$ $(=3E_2M_r$, say). First we prove that there exists a positive integer $p'=p'(F,x)$ with $1\le p'\le N'$ (N': a constant depending only on q) such that

$$(17)\qquad |F^{(p')}(x+it_m)|\ge CE_2M_rn_d^{p'}\,.$$

Let us write

$$F'(\zeta)=\sum_{k=1}^{\infty}n_kc_k\exp(n_k\zeta)=\sum_{k<d-n'}+\sum_{d-n'\le k\le d+n'}+\sum_{k>d+n'}$$
$$=\phi(\zeta)+Q(\zeta)+\Phi(\zeta)\,,$$

where n' will be determined later; we choose, for a while, so that $n'\ge N_0'$ ($N_0'=N_0(\theta q^{-4\lambda\Lambda})$: the function given in the proof of Lemma 4). Lemma 3 shows that there exists $\ell=\ell(n')$ with $0\le\ell\le 2n'$ such that

$$(18)\qquad |Q^{(\ell)}(x+it_m)|\ge 3CE_2M_rn_d^{\ell+1}\,.$$

We have

$$(19)\qquad |\phi^{(\ell)}(x+it_m)|\le\{1/q^{n'}(q-1)\}M_rn_d^{\ell+1}\qquad(\text{; see (10)}).$$

Since $x\le r-\theta/H_j=r-(\theta n_d/H_j)/n_d\le r-(\theta q^{-4\lambda\Lambda})/n_d$, we have

$$
(20)\qquad
\begin{aligned}
|\Phi^{(\ell)}(x+it_m)| &\le M_r \sum_{k>d+n'} n_k^{\ell+1} \exp(-\theta n_k/H_j) \\
&\le M_r n_d^{\ell+1} \sum_{k>d+n'} (n_k/n_d)^{2n'+1} \exp\{-(\theta q^{-4\lambda A})(n_k/n_d)\} \\
&\le M_r n_d^{\ell+1} \tau_{n'}(\theta q^{-4\lambda A}) \qquad (\,;\text{ see (11)})\,.
\end{aligned}
$$

Choosing $n'(\ge N_0')$ so that $1/q^{n'}(q-1) \le CE_2$, $\tau_{n'}(\theta q^{-4\lambda A}) \le CE_2$, we put $p' = p'(F, x) = \ell(n') + 1$, $N' = 2n' + 1$. Then (17) follows from (18), (19) and (20).

Next we prove

$$
(21)\qquad |F^{(p')}(\zeta)| \le E_3 M_r n_d^{p'} \qquad (\zeta \in D(x+it_m, \theta/2H_j))\,,
$$

where $E_3 = \{1 + (2/\theta)^{2N'}(2N')!\, q^{4N'\lambda A}\} q/(q-1)$.

Since $\operatorname{Re}\zeta \le x + \theta/2H_j \le r - \theta/2H_j$, we have

$$
\begin{aligned}
|F^{(p')}(\zeta)| &\le M_r \sum_{k=1}^{\infty} n_k^{p'} \exp(-\theta n_k/2H_j) = M_r \left\{ \sum_{k=1}^{d} + \sum_{k=d+1}^{\infty} \right\} \\
&\le M_r n_d^{p'} \left\{ \sum_{k=1}^{d} (n_k/n_d)^{p'} + (2/\theta)^{2p'}(2p')! \sum_{k=d+1}^{\infty} (n_k/n_d)^{p'} (H_j/n_k)^{2p'} \right\} \\
&\le M_r n_d^{p'} \left\{ q^{p'}/(q^{p'}-1) + (2/\theta)^{2p'}(2p')!\, (H_j/n_d)^{2p'} \sum_{k=d+1}^{\infty} (n_d/n_k)^{p'} \right\} \\
&\le M_r n_d^{p'} \{1 + (2/\theta)^{2p'}(2p')!\, q^{4p'\lambda A}\} q^{p'}/(q^{p'}-1) \le E_3 M_r n_d^{p'}\,.
\end{aligned}
$$

Now we apply Lemma 1 to $g(\zeta) = F(\zeta)$ and $D(x+it_m, \theta/2H_j)$. There exists $0 < \varepsilon \le \theta/2H_j$ such that, for any $\zeta \in S(x+it_m, \varepsilon)$,

$$
\begin{aligned}
|F(\zeta) - F(x+it_m)| &\ge \eta(p')(\theta/2H_j)^{p'} (CE_2 M_r n_d^{p'})^{p'+1} (E_3 M_r n_d^{p'})^{-p'} \\
&= \{\eta(p')(\theta/2)^{p'} (CE_2)^{p'+1} E_3^{-p'}\} M_r \\
&\ge \{\eta(N')(\theta/2)^{N'} (CE_2)^{N'+1} E_3^{-N'}\} M_r \ (= 3E_4 M_r, \text{ say})\,.
\end{aligned}
$$

If $|F(x+it_m)| \le 2E_4 M_r$, we put $\varepsilon_x = \varepsilon$. Then $0 < \varepsilon_x \le \theta/2H_j \le 1/n_\mu$ and $|F(\zeta)| \ge E_4 M_r$ $(\zeta \in S(x+it_m, \varepsilon_x))$. If $|F(x+it_m)| \ge 2E_4 M_r$, we choose $0 < \varepsilon_x \le 1/n_\mu$ so that $|F(\zeta)| \ge E_4 M_r$ $(\zeta \in S(x+it_m, \varepsilon_x))$.

Thus we have chosen, for every $x \in [r - \theta/H_0, r - \theta/H_{\sigma-1}]$, a number $0 < \varepsilon_x \le 1/n_\mu$ such that $|F(\zeta)| \ge \min\{E_1, E_4\} M_r$ $(\zeta \in S(x+it_m, \varepsilon_x))$.

If $x \in [r - \theta/H_{\sigma-1}, r - \theta/n_\nu]$, we can use the method given in (b), since $|c_\nu| \exp(n_\nu x) \ge M_r \exp(-\theta n_\nu/H_{\sigma-1}) \ge \exp(-\theta q^{4\lambda A}) M_r$. Analogously, we can use the method for $x \in [r - \theta/n_\mu, r - \theta/H_0]$. Consequently, in the case $\sigma \ge 2$, we can choose, for every $x \in [\alpha_r, \beta_r]$, a number $0 < \varepsilon_x \le 1/n_\mu$ satisfying the required inequality with some constant.

In the case $\sigma = 0, 1$ also, we can use the method given in (b). Thus in any case, we can choose, for every $x \in [\alpha_r, \beta_r]$, a number $0 < \varepsilon_x \leq 1/n_\mu$ such that $|F(\zeta)| \geq EM_r$ $(\zeta \in S(x + it_m, \varepsilon_x))$ (E: some constant).

Now we choose a finite covering $D(x_j + it_m, \varepsilon_{x_j})$ $(j = 1, \cdots, u)$ of $I(t_m, r)$ and put $\Gamma_m = \partial\{\bigcup_{j=1}^u D(x_j + it_m, \varepsilon_{x_j})\}$. Then we have $\operatorname{diam}(\Gamma_m) \leq 3/n_\mu = 3/n_{\nu_{m-1}}$, according to $\operatorname{diam}(I(t_m, r)) = \theta/n_\mu - \theta/n_\nu \leq 1/n_\mu$ and $0 < \varepsilon_{x_j} \leq 1/n_\mu$ $(j = 1, \cdots, u)$. We have also $|F(\zeta)| \geq EM_r$ $(\zeta \in \Gamma_m)$. This completes the proof of (B).

§ 6. Application

APPLICATION 8. In [2], K. G. Binmore showed that an Hadamard lacunary series $f(z)$ given by (1) has no finite asymptotic value if $\limsup_{k\to\infty} |c_k| > 0$. We note that the discussion in the proof of (A) (; in particular (13),) gives a new proof of this fact. For the sake of simplicity, we work only with $\limsup_{k\to\infty} |c_k| = \infty$.

Let $\tilde{\gamma}$ be a path in $\boldsymbol{D}$ with $b(\tilde{\gamma}) \neq \emptyset$. Without loss of generality, we may assume $b(\tilde{\gamma}) \ni 1$. Then there exists a path $\tilde{\Gamma}$ in U with $b(\tilde{\Gamma}) \ni 0$ and $\iota(\tilde{\Gamma}) = \tilde{\gamma}$, where ι is the mapping defined by $\iota(\zeta) = e^\zeta$. It is sufficient to prove that $F(\zeta)$ has no finite asymptotic value along with $\tilde{\Gamma}$. Let $(w_m)_{m=1}^\infty$ be a sequence in $\tilde{\Gamma}$ with $\operatorname{Re} w_m = \beta_{r_m}$ $(m \geq 1)$. Then (13) shows that

$$|F(\zeta) - F(w_m)| \geq 3DM_{r_m} \qquad (\zeta \in S(w_m, \varepsilon_{w_m})) \,. \tag{22}$$

Let w'_m be a point in $\tilde{\Gamma} \cap S(w_m, \varepsilon_{w_m})$ $(m \geq 1)$. Then (22) holds for $\zeta = w'_m$. Since $\lim_{m\to\infty} M_{r_m} = \infty$, $F(\zeta)$ has no finite asymptotic value along with $\tilde{\Gamma}$.

APPLICATION 9. We say that an analytic function $g(z)$ in $\boldsymbol{D}$ is annular if there exists a sequence $(\gamma_m^*)_{m=1}^\infty$ of Jordan curves in $\boldsymbol{D}$ such that $\langle\gamma_m^*\rangle \ni 0$ $(m \geq 1)$ and $\lim_{m\to\infty} \min\{|g(z)|; z \in \gamma_m^*\} = \infty$. We say that $g(z)$ is strongly annular if we can choose $(\gamma_m^*)_{m=1}^\infty$ so that γ_m^*'s are circles with center 0 in addition to the above conditions. L. R. Sons showed that an Hadamard lacunary series $f(z)$ given by (1) is annular if and only if $\limsup_{k\to\infty} |c_k| = \infty$. The "only if" part is immediately seen; if $\limsup_{k\to\infty} |c_k| < \infty$, then $f(z)$ is normal ([8]) and hence $f(z)$ is not annular ([9] p. 267). Let us show that the "if" part is deduced from (A). Put $I_m = \{\zeta; \operatorname{Re}\zeta = \beta_{r_m}, 0 \leq \operatorname{Im}\zeta \leq 2\pi\}$ $(m \geq 1)$. Given $m \geq 1$, (A) shows that, for every $w \in I_m$, there exists $0 < \varepsilon_w \leq 1/n_{\nu_m}$ such that $|F(\zeta)| \geq DM_{r_m}$ $(\zeta \in S(w, \varepsilon_w))$. We choose a finite covering $D(w_j, \varepsilon_{w_j})$ $(j = 1, \cdots, u)$ of I_m and put $V_m = \iota(\bigcup_{j=1}^u D(w_j, \varepsilon_{w_j}))$. Then $V_m \supset S(0, \beta_{r_m})$. Let $(\gamma_m^*)_{m=1}^\infty$ be the sequence defined by $\gamma_m^* = \partial V_m \cap$

$D(0, \beta_{r_m})$. Then $\langle \gamma_m^* \rangle \ni 0$ $(m \geq 1)$ and $\lim_{m\to\infty} \min \{|f(z)|; z \in \gamma_m^*\} = \infty$. Hence $f(z)$ is annular.

Let us remark that, in Sons's result, "annular" cannot be replaced by "strongly annular". This is a consequence of the following proposition: Let $\phi(z) = \sum_{k=1}^{\infty} b_k z^{\lambda_k}$ be an analytic function in D such that, with $s_m = (\sum_{k=1}^{m} |b_k|^2)^{1/2}$ $(m \geq 1)$, $\lim_{m\to\infty} b_m/s_m = 0$ and $\liminf_{k\to\infty} \log \lambda_{k+1}/\log \lambda_k > 1$. Then $\phi(z)$ is not strongly annular.

The proof is as follows. Nothing is to be proved if $\lim_{m\to\infty} s_m < \infty$. Let $\lim_{m\to\infty} s_m = \infty$. Then the method given in [7] (Lemma 38) yields meas $\{t; |\phi(\rho e^{it})| \leq 2\omega\} \geq \delta(\omega/d_\rho)^2$ $(\rho_0 \leq \rho < 1)$ for some $0 < \rho_0 < 1$, where "meas" signifies the 1-dimensional Lebesgue measure,

$$\omega = 4\pi \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} |b_k| \lambda_k/\lambda_\ell\,, \qquad d_\rho = \left(\sum_{k=1}^{\infty} |b_k|^2 \rho^{2\lambda_k}\right)^{1/2}$$

and δ an absolute constant. Thus $\min \{|\phi(z)|; |z| = \rho\} \leq 2\omega$ $(\rho_0 \leq \rho < 1)$, and hence $\limsup_{\rho\to 1} \min \{|\phi(z)|; |z| = \rho\} \leq 2\omega$. This shows that $\phi(z)$ is not strongly annular.

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