H. Kaneta
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# IRREDUCIBILITY OF SOME UNITARY REPRESENTATIONS OF THE POINCARÉ GROUP WITH RESPECT TO THE POINCARÉ SUBSEMIGROUP, II

HITOSHI KANETA

Let $P(3)$ and $P_+(3)$ be the 3-dimensional space-time Poincaré group and the Poincaré subsemigroup, that is, $P(3) = R^3 \times_s SU(1, 1)$ and $P_+(3) = V_+(3) \times_s SU(1, 1)$ where $V_+(3) = \{x_0^2 - x_1^2 - x_2^2 \geq 0, x_0 \geq 0\}$. The multiplication is defined by the formula $(x, g)(x', g') = (x + g^{*-1}x'g^{-1}, gg')$ for $x, x' \in R^3$ and $g, g' \in SU(1, 1)$. Here $x = (x_0, x_1, x_2)$ is identified with the matrix $\begin{pmatrix} x_0 & x_2 - ix_1 \\ x_2 + ix_1 & x_0 \end{pmatrix}$.

The purpose of this paper is to give an affirmative answer to the problem if there is any irreducible unitary representation of $P(3)$ such that its restriction to the semigroup $P_+(3)$ is reducible. To be more precise, we shall determine all $P_+(3)$-invariant, closed proper subspaces for the irreducible unitary representations $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$ $(\eta \in R, \varepsilon = 0, 1/2)$, which are associated with the one-sheeted hyperboloid $V_{iM}(3) = \{y_0^2 - y_1^2 - y_2^2 = -M^2\}$ $(M > 0)$. As for the other irreducible unitary representations of $P(3)$ it is easy to show that they are irreducible even when they are restricted to $P_+(3)$ (see [5], Theorem 5). Recall that all the irreducible unitary representations of the 2-dimensional space-time Poincaré group are irreducible even when they are restricted to the Poincaré subsemigroup ([5], Theorem 1). Using, among other things, the results in § 1, we shall show in the forthcoming Part III that the irreducible unitary representations of the 4-dimensional space-time Poincaré group whose irreducibility relative to the Poincaré subsemigroup remains unsettled in [5] are reducible as the representations of the semigroup.

The basic tools of our approach are i) the eigenfunction expansions for second order ordinary differential operators $\mathscr{L}_{k,\eta}$ (see (1.1)), which are connected with the Laplacian of $SU(1, 1)$, and ii) rephrased versions of the Hilbert transform and the Frobenius method for ordinary differential

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equations with a regular singularity.

This paper consists of two sections and an appendix. In § 1 we enumerate closed proper subspaces of $L^2(R)$ left invariant under the self-adjoint operator $\mathscr{L}_{k,\eta}$ and a semigroup $T_t = \exp(it \operatorname{sh} \tau)$ $(t \geq 0)$ of multiplication operators (Theorems 1.1–1.3). Toward the end of § 1 we shall determine nontrivial sequences $\{D_k\}_{k \in Z+\varepsilon}$ $(\varepsilon = 0, 1/2)$ of subspaces such that i) $D_k$ is a closed, proper subspace of $L^2(R)$ left invariant under $\mathscr{L}_{k,\eta}$ and $T_t (t \geq 0)$, ii) $F_{\pm,k,\eta} D_k \subset D_{k\pm 1}$, where $F_{\pm,k,\eta} = -d/d\tau + (\pm k + 1/2) \operatorname{th} \tau \pm \eta/\operatorname{ch} \tau$ with domain $H_2(R)$, the Sobolev space of order 2 (Theorem 1.4). In § 2 we firstly define the representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$ $(\eta \in R, \varepsilon = 0, 1/2)$ of the group $P(3)$, and then describe all the $P_+(3)$-invariant, closed proper subspaces $\mathscr{D}^{\eta,\varepsilon}_{\pm}$ in $\mathfrak{H}^{\eta,\varepsilon}$ and $\mathscr{D}^{0,0}_{\pm 1}$ in $\mathfrak{H}^{0,0}$. Namely, there are four such subspaces in $\mathfrak{H}^{0,0}$ in the special case $(\eta, \varepsilon) = (0, 0)$. It should be noted that Corollary 2.3 plays an important role in verifying that $SU(1, 1)$ leaves $\mathscr{D}^{\eta,\varepsilon}_{\pm}$ in $\mathfrak{H}^{\eta,\varepsilon}$ as well as $\mathscr{D}^{0,0}_{\pm 1}$ in $\mathfrak{H}^{0,0}$. The appendix is devoted to a quick review of Frobenius method in our context.

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Notation and terminology

$Z$ is the set of integers and $Z_+ = \{n \in Z; n \geq 0\}$.

$R$ is the set of real numbers, $R_+ = \{\lambda \in R; \lambda > 0$ and $R^* = R \backslash \{0\}$.

$C$ is the set of complex numbers, $C^* = C \backslash \{0\}$ and $T = \{z \in C, |z| = 1\}$.

More subsets of $C$ is to be defined. $D_\tau = \{z \in C; |\operatorname{Im} z| < \pi/2\}$, $\bar{D}_\tau = \{z \in C; |\operatorname{Im} z| \leq \pi/2\}$ and $\dot{D}_\tau = \bar{D}_\tau \backslash \{\pm i\pi/2\}$. An element of these three sets will be denoted by $\tau$. Throughout the paper $\sigma = \tau - i\pi/2$. A polynomial in $\log \sigma$ with holomorphic coefficients will be denoted by $h(\sigma, \log \sigma)$, that is, $h(\sigma, \log \sigma) = \sum_{n=0}^{m} (\log \sigma)^n h_n(\sigma)$, where $h_n(\sigma)$ are holomorphic around $\sigma = 0$. For a function $f(\sigma)$ we denote by $Rf(\sigma)$ the function $f(-\sigma)$. An integral $\int_R f(\tau)\, d\tau$ will be abbreviated to $\int f\, d\tau$ or $\langle f \rangle$. The relation $a \propto b$ for two elements $a$ and $b$ in a linear space means $a = cb$ for some $c$ in $C^*$.

$M_{m,n}$, $m, n \in Z_+ + 1$, is the set of complex $m \times n$-matrices and $M_n = M_{n,n}$. $M_n^+$ (resp. $M_n^{++}$) stands for the set of non-negative (resp. positive) definite $n \times n$-matrices. $I_n$ means the unit matrix in $M_n$. For a matrix $A = (a_{jk})$ in $M_{m,n}$, we set $\bar{A} = (\bar{a}_{jk})$, ${}^t A =$ the transpose of $A$, $A^* = {}^t \bar{A}$ and $A = \max_k \sum_{j=1}^{m} |a_{jk}|$.

$C^r(S)^n$ $(r = 0, 1, \cdots, \infty; n \in Z_+ + 1)$ for a $C^\infty$-manifold $S$ is the totality of $C^n$-valued $C^r$-functions on $S$. $C_0^r(S)^n = \{f \in C^r(S)^n;\ f$ is compactly supported$\}$. $C_0(S)^n = C_0^0(S)^n$. $H_r(R)$, $r \in Z_+$, is the Sobolev space of order $r$ on $R$. $H_r(R)^n$ means the direct sum $\sum_{j=1}^n \oplus H_r(R)$. Of course $H_0(R) = L^2(R)$, the Hilbert space consisting of $C$-valued square integrable functions on $R$. Let $(B, \Sigma)$ be a measurable space, where $B$ is a Borel set of $R^n$ and $\Sigma$ is the set of all Borel sets in $B$. $L^2(B, \mu)$ is the usual $L^2$-space defined in terms of a measure $\mu$ on $(B, \Sigma)$. Let $\rho(x)$ be a $M_m^{++}$-valued measurable functions on a Borel set $B$ of $R^n$. Then $L^2(B, \rho)$ denotes the Hilbert space consisting of $C^m$-valued measurable functions $f$ on $B$ such that $\int_B f^*(x)\rho(x)f(x)\,dx$ is finite. Here $dx$ is the Lebesgue measure.

Let $L$ be a linear operator from $H_1$ to $H_2$. When both $H_j$, $1 \leq j \leq 2$, are Hilbert spaces, $L^*$ means the (formal) adjoint of $L$. In this paper a Hilbert space is assumed to be separable. $LH_1$ is the range of $L$, namely, $LH_1 = \{Lh;\ h$ in $H_1$ belongs to the domain of $L\}$. For a subspace $H_0$ of $H_1$, $L|H_0$ denotes the restriction of $L$ to the subspace $H_0$. Let $D$ be a subset of a Hilbert space. Then $D^\perp$ is the set of all elements which are orthogonal to $D$. $\| \ \|$ and $\langle\ ,\ \rangle$ denote the norm and the inner product in a Hilbert space ($C^n$, $L^2(B, \mu)$, etc.) respectively. However, $\langle x, y\rangle = x_0y_0 - x_1y_1 - x_2y_2$ for $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2)$ in $R^3$. Recall that $\langle f\rangle$ is an abbreviation to the integral $\int_R f(\tau)\,d\tau$. A closed subspace $D$ of a Hilbert space is said to be invariant under a selfadjoint operator $L$ if $P_D L = LP_D$, where $P_D$ denotes the orthogonal projection $H \to D$. As is well-known, $D$ is invariant under $L$ iff the one-parameter unitary group $\exp(itL)$ leaves $D$ invariant.

$T_t = \exp(it \operatorname{sh} \tau)$ $(t \geq 0)$ is a continuous semigroup in $L^2(R)$ such that $T_t f(\tau) = \exp(it \operatorname{sh} \tau) f(\tau)$. $G_\alpha = (\alpha - i \operatorname{sh} \tau)^{-1}$ $(\operatorname{Re} \alpha > 0)$ are resolvent operators for the semigroup. By abuse of notation $G_\alpha$ also means the function $(\alpha - i \operatorname{sh} \tau)^{-1}$ of $\tau$. Finally, $f'$ means the derivative for either an absolutely continuous function $f$ on $R$ or a holomorphic function $f$.

## §1. Invariant subspaces common to $\mathscr{L}_{k,\eta}$ and $T_t(t \geq 0)$

The purpose of this section is to determine all closed proper subspaces in $L^2(R)$ which stay invariant under the selfadjoint operator $\mathscr{L}_{k,\eta}$ with domain $H_2(R)$ and the semigroup $T_t(t \geq 0)$ on $L^2(R)$;

$$\mathscr{L}_{k,\eta} = -\,d^2/d\tau^2 + (1/4 - k^2 + \eta^2 + 2k\eta \operatorname{sh} \tau)/\operatorname{ch}^2\tau \qquad (k \in Z/2,\ \eta \in R)\,, \tag{1.1}$$

(1.2) $$T_t = e^{it\,\mathrm{sh}\,\tau}.$$

To this end, first the case $k = 0$ or $1/2$ will be discussed. Then the general case can be dealt with by the aid of the following differential operator

(1.3) $$F_{\pm,k,\eta} = -\,d/d\tau + (\pm\, k + 1/2)\,\mathrm{th}\,\tau \pm \eta/\mathrm{ch}\,\tau\,.$$

Throughout the rest of this section the suffix $\eta$ will frequently be omitted.

In case $(k, \eta) = (0, \eta)$ or $(1/2, 0)$ clearly $\mathscr{L}_k$ reduces to an operator of the following form.

(1.4) $$\mathscr{N}_\kappa = -\,d^2/d\tau^2 + \kappa/\mathrm{ch}^2\tau\,, \qquad \kappa \geq 0\,.$$

We shall search for closed proper invariant subspaces common to $\mathscr{N}_\kappa$ and $T_t$ $(t \geq 0)$. To begin with, denote by $\Phi = (\phi_1, \phi_2)$ the solution of an ordinary differential equation $(\mathscr{N}_\kappa - \lambda)\Phi = 0$ with initial value ${}^t({}^t\Phi, {}^t\Phi')_{\tau=0} = I_2$, the unit matrix. Since $\kappa/\mathrm{ch}^2\tau$ is integrable and $\mathscr{N}_\kappa$ is positive definite, there exists a so-called spectral density $\rho$ on $R_+$ satisfying the following conditions i)～iii) [4].

i) $\rho$ is an $M_2^{++}$-valued continuous function on $R_+$.

ii) The operator $\mathscr{F} : L^2(R) \to L^2(R_+, \rho)$ (refer to the Notation) defined by

(1.5) $$\mathscr{F}f(\lambda) = \lim_{N\to\infty}\int_{|\tau|<N} {}^t\Phi(\tau, \lambda) f(\tau) d\tau$$

is an onto isometry, whose inverse $\mathscr{F}^{-1}$ is given by

(1.6) $$\mathscr{F}^{-1}g(\tau) = \lim_{N\to\infty}\int_{0<\lambda<N} \Phi(\tau, \lambda)\,\rho(\lambda)\,g(\lambda)\,d\lambda\,.$$

iii) $\mathscr{F}\mathscr{N}_\kappa\mathscr{F}^{-1}g(\lambda) = \lambda g(\lambda)$ if $\lambda g(\lambda)$ lies in $L^2(R_+, \rho)$.

On the other hand the equation $(\mathscr{N}_\kappa - \lambda)\zeta = 0$ has a regular singularity at $\tau = i\pi/2$, that is, $\sigma = 0$. The Frobenius method yields linearly independent solutions $\zeta_\pm(\tau, \lambda)$ which, being holomorphic in $\dot{D}_\tau \times C$, admit the following expansions around $\tau = i\pi/2$;

(1.7)
$$\zeta_\pm = \sigma^{\alpha_\pm}\Big(\sum_{n=0}^{\infty} z_{\pm,n}\sigma^n\Big) \qquad \text{if } \kappa \neq 1/4\,,$$
$$\zeta_+ = \sigma^{1/2}\Big(\sum_{n=0}^{\infty} z_{+\,n}\sigma^n\Big)$$
$$\zeta_- = \zeta_+ \log\sigma + \sigma^{1/2}\Big(\sum_{n=1}^{\infty} z_{-,n}\sigma^n\Big) \qquad \text{if } \kappa = 1/4\,,$$

where $\alpha_\pm = (1 \pm \sqrt{1 - 4\kappa})/2$ and $z_{\pm,0} = 1$. Set $\zeta = (\zeta_-, \zeta_+)$, and define $X(\lambda)$

$\in M_2$ and $s_\pm(\lambda)$, $r_\pm(\lambda) \in M_{2,1}$ as follows.

$$\zeta = \Phi X\,, \quad s_\pm = X^t v_\pm\,, \quad r_\pm = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} s_\pm\,, \tag{1.8}$$

where $v_\pm = {}^t(1 \pm 1, 1 \mp 1)$ or ${}^t(0, 2)$ according as $\kappa \neq 1/4$ or not. Now we are in a position to introduce invariant subspaces

$$D_\pm^\kappa = \mathscr{F}^{-1}\{g \in L^2(R_+, \rho);\ {}^t s_\pm(\lambda) g(\lambda) = 0 \text{ a.e.}\}\,. \tag{1.9}$$

Notice that $\mathscr{F} D_\pm^\kappa = \{r_\pm h \in L^2(R_+, \rho);\ h \in L^2(R_+, r^* \rho r)\}$. This is because ${}^t s_\pm r_\pm = 0$.

THEOREM 1.1. *Let $D$ be a closed proper subspace of $L^2(R)$. Then $D$ is invariant under the selfadjoint operator $\mathscr{N}_\kappa$ and the semigroup $T_t = e^{it\,\mathrm{sh}\,\tau}$ $(t \geq 0)$ iff it coincides with one of $D_\pm^\kappa$.*

For the proof we prepare two lemmas and two propositions.

LEMMA 1.1. (i) *The domain $D_\tau = \{|\mathrm{Im}\,\tau| < \pi/2\}$ is holomorphically isomorphic to a domain $\{\mathrm{Im}\, z \neq 0$ or $z \in (0, 1)\}$ via the map $z = (1 + i\,\mathrm{sh}\,\tau)/2$.*

(ii) *Let $f(\tau)$ be holomorphic in $\dot{D}_\tau$. Then $f(\tau)/\sqrt{z(1-z)}$ is holomorphic in $\{\mathrm{Re}\, z < 1\}$ iff $f(\tau)$ can be expanded as $\sum_{n=0}^\infty c_n \sigma^{2n+1}$ near $\tau = i\pi/2$, where $\sigma = \tau - i\pi/2$.*

*Proof.* It is easy to see that $z$ is a univalent function sending $D_\tau$ onto $\{\mathrm{Im}\, z \neq 0$ or $z \in (0, 1)\}$. Since the derivative $z'$ does not vanish on $D_\tau$, (i) follows. To verify (ii), assume that $f(\tau)/\sqrt{z(1-z)}$ is holomorphic in a neighborhood of $z = 0$. Then $f(\tau)/\sqrt{z}$ is holomorphic too. Since $\sqrt{z}$ is a holomorphic odd function of $\sigma$ in a vicinity of $\sigma = 0$, $f(\tau)$ has the desired expansion. Conversely, assume that $f$ satisfies the condition. Then $F(z)$ $f(\tau)/\sqrt{z(1-z)}$ is holomorphic in $\{\mathrm{Re}\, z < 1\} \backslash \{z \leq 0\}$. Notice that $F$ admits an analytic continuation across the line $\{z < 0\}$, for $z = (1 + i\,\mathrm{sh}\,\tau)/2$ is a local isomorphism of $C \backslash \{i\pi n/2;\ n \in Z\}$. By the condition on $f$ we see that $F(x + i0) = F(x - i0)$ for any negative $x > -\varepsilon$ $(\varepsilon > 0)$. Therefore $F(z)$ is holomorphic in $\{\mathrm{Re}\, z < 1\} \backslash \{0\}$. Since $F(z)$ is bounded in a punctured disc $\{0 < |z| < \varepsilon\}$, $z = 0$ is a removable singularity. This completes the proof of (ii). Q.E.D.

The next proposition is concerned with the Hilbert transform.

PROPOSITION 1.2. (i) *Assume that $F(z)$ is holomorphic in $\{\mathrm{Re}\, z < 1\}$. If the integral $\int |F(x + iy)|^p dy$ $(p > 1)$ is bounded on $x < 1 - \varepsilon$, $\varepsilon > 0$, then*

$$\int_{a-i\infty}^{a+i\infty} \frac{F(z)}{z-\alpha} dz = 0 \qquad \text{for } a < \min\{\operatorname{Re}\alpha, 1-\varepsilon\}.$$

(ii) *Assume that $F(z)$ is holomorphic in a strip $1/2 - 2\varepsilon < \operatorname{Re} z < 1/2 + 2\varepsilon$, $\varepsilon > 0$. If the integral $\int |F(x+iy)|^2 dy$ is bounded on $[1/2-\varepsilon, 1/2+\varepsilon]$, then $F(z)$ has the following integral representation in $1/2 - \varepsilon < \operatorname{Re} z < 1/2 + \varepsilon$.*

$$F(z) = \frac{1}{2\pi i}\left(-\int_{1/2-\varepsilon-i\infty}^{1/2-\varepsilon+i\infty} + \int_{1/2+\varepsilon-i\infty}^{1/2+\varepsilon+i\infty}\right)\frac{F(\zeta)}{\zeta - z} d\zeta .$$

*Proof.* To prove (i), we apply a lemma [9, p. 125] to $F$ to show that the integral in question is independent of $a$. On the other hand Hölder's inequality implies that the integral tends to zero as $a \to -\infty$. Now (i) follows. The statement (ii) is well-known [9, p. 130]. Q.E.D.

As to an estimate of the solution $\Phi(\tau, \lambda)$ we have the following

LEMMA 1.3. *Let $\Psi(\tau, \lambda) \in M_{1,2}$ be a solution of the following equation with initial value ${}^t({}^t\Psi, {}^t\Psi')_{\tau=0} = I_2$;*

$$\{-d^2/d\tau^2 + (a + b\operatorname{sh}\tau)/\operatorname{ch}^2\tau - \lambda\}\Psi(\tau, \lambda) = 0, \qquad a, b \in C.$$

*Fix $\lambda_0 \in R_+$. Then for any $\varepsilon > 0$ there exist positive $K$ and $\delta$ such that*

i) $|\Psi(\tau, \lambda_0)| + |\Psi'(\tau, \lambda_0)| < K \qquad$ *on* $\bar{D}_\tau \cap \{|\operatorname{Re}\tau| \geq 1\}$,

ii) $|\Psi(\tau, \lambda)| + |\Psi'(\tau, \lambda)| < Ke^{\varepsilon|\tau|} \qquad$ *on* $R \times \{|\lambda - \lambda_0| < \delta\}$.

*Proof.* We shall prove the existence of $K$ satisfying only i), for we can argue similarly to show the existence of $K$ and $\delta$ satisfying the condition ii). Put $S = \begin{pmatrix} 1 & 1 \\ \sqrt{-\lambda} & -\sqrt{\lambda} \end{pmatrix}$, and define $\chi$ by the relation ${}^t({}^t\Psi, {}^t\Psi') = S\left\{\exp\begin{pmatrix} \sqrt{-\lambda} & 0 \\ 0 & -\sqrt{-\lambda} \end{pmatrix}\tau\right\}\chi$. Then we note that $\chi(\tau, \lambda_0)$ is bounded on $\bar{D}_\tau \cap \{|\operatorname{Re}\tau| = 1\}$ and that $\chi' = V(\tau)\chi$, where $|V(\tau)|$ is bounded by a function $v(\operatorname{Re}\tau)$ on $\bar{D}_\tau \cap \{|\operatorname{Re}\tau| \geq 1\}$. Here $v$ is integrable on $I = (-\infty, -1] \cup [1, \infty)$. Consequently the integral $\int_I |V(\tau + i\varepsilon)| d\tau$ is bounded on $|\varepsilon| \leq \pi/2$. Hence $\chi(\tau, \lambda_0)$ is bounded on $\bar{D}_\tau \setminus \{|\operatorname{Re}\tau| < 1\}$ (see Problem 1 [1, p. 97]), from which follows that $|\Psi(\tau, \lambda_0)| + |\Psi'(\tau, \lambda_0)|$ is bounded there. Q.E.D.

Let $\delta$ be an atomic measure on a finite subset $\Lambda$ of $R$ such that $\delta(\{\lambda\}) = 1$ for each $\lambda \in \Lambda$, $\rho_2$ be an $M_2^{++}$-valued Borel measurable function on a Borel set $B$ of $R$. Set $H_p = L^2(\Lambda, \delta)$, $H_{ac} = L^2(B, \rho_2)$ and $H = H_p \oplus H_{ac}$. We denote by $e^{it\lambda}$, $t \in R$, the one-parameter unitary group acting on $H$ as multiplication.

Then we have

PROPOSITION 1.4. *A closed subspace $D$ of $H$ is invariant under the one-parameter group $e^{it\lambda}$ iff there exist a subset $\Lambda_0$ of $\Lambda$, disjoint Borel subsets $B_1$, $B_2$ of $B$ ($\Lambda_0$ and $B_j$ may be a null set) and a Borel measurable function $s$ on $B_1$ with values in $M_{2,1}\backslash\{0\}$ almost everywhere such that $D$ coincides with*

$$(1.10)\quad \begin{aligned} &L^2(\Lambda_0, \delta)\oplus\{{}^t(g_1, g_2) \in H_{ac}; (g_1, g_2)s = 0 \text{ a.e. on } B_1,\ (g_1, g_2) = 0 \\ &\text{a.e. outside } B_1\}\oplus\{{}^t(g_1, g_2) \in H_{ac}; (g_1, g_2) = 0 \text{ a.e. outside } B_2\}. \end{aligned}$$

*Proof.* It suffices to show that the conditions are necessary. We regard $e^{it\lambda}$ as a representation of $R$ in $H$, and apply Theorem 8.6.6 [2] to this representation. Then there exist a subset $\Lambda_0$ of $\Lambda$ and disjoint Borel sets of B such that the representation in $D$ is unitarily equivalent to the following representation

$$\int_{\Lambda_0}^{\oplus} e^{it\lambda}d\delta(\lambda)\oplus\int_{B_1}^{\oplus} e^{it\lambda}d\lambda\oplus[2]\int_{B_2}^{\oplus} e^{it\lambda}d\lambda$$

in $\tilde{H} = L^2(\Lambda_0, \delta) \oplus L^2(B_1) \oplus [2]L^2(B_2)$. Let $U: \tilde{H} \to D$ be an onto isometry ensuring the equivalence. By Proposition 8.4.6 [2] $U$ sends $L^2(\Lambda_0, \delta)$ in $\tilde{H}$ onto $L^2(\Lambda_0, \delta)$ in $H_p$ while $L^2(B_1) \oplus [2]L^2(B_2)$ in $\tilde{H}$ into $H_{ac}$. Choose $f_i \in L^2(B_i)$, $i = 1, 2$, such that $f_j \neq 0$ a.e. on $B_i$, and denote by $D_1$, $D_{21}$ and $D_{22}$ the closed subspaces of $H_{ac}$ cyclically generated by the vectors ${}^t(h_1, h_2) = U(0, f_1, 0, 0)$, ${}^t(h_{11}, h_{12}) = U(0, 0, f_2, 0)$ and ${}^t(h_{21}, h_{22}) = U(0, 0, 0, f_2)$ respectively. For the sake of simplicity assume that both $B_1$ and $B_2$ are non-null sets. In case either $B_1$ or $B_2$ is a null set, we can argue similarly. Note that $(h_1, h_2)$ and $(h_{i1}, h_{i2})$ do not vanish a.e. on $B_1$ and $B_2$ respectively. Moreover, $\det(h_{ij}) \neq 0$ a.e. on $B_2$, for if it happened to vanish on a set of positive measure, the representation in $D_{21} \oplus D_{22}$ contains a subrepresentation of the multiplicity one, which contradicts Theorem 8.6.6 [2]. Since the Fourier transform for $L^1(R)$ is injective, it is not hard to see that $D_{21} \oplus D_{22}$ constitutes the third component of (1.10). Finally $D_1 = \{\mathrm{rh} \in H_{ac}; h \in L^2(B_1, r^*\rho_2 r)\}$ coincides with the second component of (1.10) with $s=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}{}^t(h_1, h_2)$.

Q.E.D.

We are ready for the

*Proof of Theorem* 1.1. 1) We shall prove the sufficiency of the condition. To begin with, we note that $D^{\kappa}_{\pm}$ are closed proper subspaces variant under $\mathscr{N}_{\kappa}$. Indeed $\mathscr{F}\exp(it\mathscr{N}_{\kappa})\mathscr{F}^{-1}$, $t \in R$, is the multiplication operator $e^{it\lambda}$ in $L^2(R_+, \rho)$. In order to see that $T_t$ $(t \geq 0)$ leaves $D^\epsilon_\pm$ invariant, it suffices to show that the resolvent $G_\alpha$ $(\mathrm{Re}\,\alpha > 0)$ of the semigroup sends a dense subspace $\mathscr{F}^{-1}\{r_\pm h;\ h \in C_0(R_+)^1\}$ in $D^\epsilon_\pm$ into $D^\epsilon_\pm$, that is,

$$(1.11)\qquad {}^t s_\pm(\lambda)[\mathscr{F} G_\alpha \mathscr{F}^{-1} r_\pm h](\lambda) = 0, \qquad h \in C_0(R_+)^1 .$$

To verify (1.11) we shall show that

$$(1.12)\qquad \int {}^t s_\pm(\lambda)\, {}^t\Phi(\tau, \lambda)\, G_\alpha \Phi(\tau, \xi) \rho(\xi) r_\pm(\xi)\, d\tau = 0 .$$

Note that (1.11) follows from (1.12) immediately by integrating the both sides of (1.12) with respect to a signed measure $h(\xi)d\xi$ (we can safely change the order of integration on account of Lemma 1.3). To show (1.12), put, for positives $\lambda$ and $\xi$,

$$I_{\alpha,\lambda,\xi} = \int {}^t\zeta(\tau, \lambda)\, G_\alpha \zeta(\tau, \xi)\, d\tau, \qquad \tilde\rho = X^{-1} \rho\, {}^t X^{-1} = (\tilde\rho_{ij}),$$
$$J_{\alpha,\lambda,\xi} = I_{\alpha,\lambda,\xi} \tilde\rho(\xi) .$$

Then, using the relation $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\, {}^t Y^{-1} \det Y$, the left side of (1.12) can be written as

$$(1.13)\qquad v_\pm J_{\alpha,\lambda,\xi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v_\pm \det X(\xi) .$$

See (1.8) for the definition of $v_\pm, \zeta$ and $X$. We shall show that

$$(1.14)\qquad I_{\alpha,\lambda,\xi} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \text{ if } \kappa \neq 1/4, \qquad \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \text{ if } \kappa = 1/4,$$

$$(1.15)\qquad \tilde\rho = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \text{ if } \kappa \neq 1/4, \qquad \begin{pmatrix} 0 & \tilde\rho_{12} \\ \tilde\rho_{12} & \tilde\rho_{22} \end{pmatrix} \text{ if } \kappa = 1/4,$$

to the effect that $J_{\alpha,\lambda,\xi}$ is diagonal or of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ according as $\kappa \neq 1/4$ or not, which proves (1.12) since (1.13) turns out to vanish. To see (1.14), let $R$ be an operator assigning a function $f(\sigma)$ to $f(-\sigma)$ and $\mathscr{N}_\kappa(\sigma)$ be the differential operator $\mathscr{N}_\kappa$ expressed in terms of $\sigma = \tau - i\pi/2$. Then $R\mathscr{N}_\kappa(\sigma)R = \mathscr{N}_\kappa(\sigma)$. This relation gives rise to a symmetry of coefficients $z_{\pm,n}$ in (1.7). That is,

$$(1.16)\qquad z_{\pm,n}(-1)^n = z_{\pm,n} \text{ if } \kappa \neq 1/4, \quad z_{+,n}(-1)^n = z_{+,n} \text{ if } \kappa = 1/4 .$$

In particular ${}^t\zeta_\pm \zeta_\mp$ (resp. ${}^t\zeta_+ \zeta_+$) can be expanded as $\sum_{n=0}^\infty c_n \sigma^{2n+1}$ near $\sigma = 0$ in the case $\kappa \neq 1/4$ (resp. $\kappa = 1/4$). Since $I_{\alpha,\lambda,\xi}$ is equal to

$$(1.17)\qquad \int_{1/2-i\infty}^{1/2+i\infty} \frac{{}^t\zeta(\tau,\lambda)\zeta(\tau,\xi)\{z(1-z)\}^{-1/2}}{z-\alpha}\,dz\,, \qquad z=(1+i\,\mathrm{sh}\,\tau)/2\,,$$

(1.14) follows from Proposition 1.2 (i) in view of Lemmas 1.1 and 1.3. Finally, to see (1.15), let $g$ belong to $C_0(R_+)^2$. Since $\alpha G_\alpha$ converges to the identity operator as $\alpha \to \infty$, there is a sequence $\alpha_n$ tending to $\infty$ such that $\alpha_n \mathscr{F} G_{\alpha_n} \mathscr{F}^{-1} g$ converge to $g$ a.e. In other words

$$(1.18)\qquad \alpha_n \int_{R_+} I_{\alpha_n,\lambda,\xi}\,\tilde{\rho}(\xi)\,{}^tX(\xi)g(\xi)\,d\xi \longrightarrow {}^tX(\lambda)g(\lambda) \qquad \text{a.e. as } n\to\infty\,.$$

Set ${}^tXg = {}^t(a, b)$. Then, if $\kappa \neq 1/4$, the first (resp. second) component of the left side of (1.18) does not depend on $b$ (resp. $a$), while the right side of (1.18) is equal to ${}^t(\tilde{\rho}_{22}a - \tilde{\rho}_{12}b,\ -\tilde{\rho}_{21}a + \tilde{\rho}_{11}b)$. Thus $\tilde{\rho}_{12} = \tilde{\rho}_{21} = 0$ if $\kappa \neq 1/4$. Similar argument, together with the fact that $\rho$ is diagonal, yields $\tilde{\rho}_{11} = 0$ if $\kappa = 1/4$. This completes the proof of (1.15). 2) We shall show that the condition is necessary. Applying Proposition 1.4 to the one-parameter group $e^{it\lambda}$ on $L^2(R_+, \rho)$, we define Borel sets $B_1$, $B_2$ of $R_+$ and a Borel measurable function $s$ with values in $M_{2,1}\backslash\{0\}$ a.e. on $B_1$. Since the image $G_\alpha D$ is dense in $D$, $\det(\mathscr{F}G_\alpha f_1, \mathscr{F}G_\alpha f_2) \neq 0$ a.e. on $B_2$ for some $f_1$, $f_2 \in D$. If $B_2$ is not a null set, the determinant does not vanish a.e. on $R_+$, for it is holomorphic in a neighborhood of $R_+$. Therefore, if $B_2$ is not a null set, $D = L^2(R)$, which is a contradiction. Thus we may assume that $B_2 = \phi$ and $B_1 = R_+$ on account of the analyticity of $\mathscr{F}G_\alpha f(\lambda)$, $f \in D$. Set $r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} s$. Then $\mathscr{F}D = \{rh \in L^2(R_+, \rho);\ h \in L^2(R_+, r^*\rho r)\}$. Consequently we can replace $r$ and $s$ by real analytic functions $\mathscr{F}G_{\alpha_0}f$, $f \in D\backslash\{0\}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} r$ respectively. Since $rh$, $h \in C_0(R_+)^1$, belongs to $\mathscr{F}D$, we have ${}^ts(\lambda)[\mathscr{F}G_\alpha\mathscr{F}^{-1}rh](\lambda) = 0$ on $R_+$. Letting $h$ converge to the Dirac measure supported at $\xi \in R_+$, we obtain $\langle {}^ts(\lambda)\Phi(\tau, \lambda)G_\alpha\Phi(\tau, \xi)\rho(\xi)r(\xi)\rangle = 0$. Namely,

$$(1.19)\qquad {}^t(X^{-1}s)(\lambda)J_{\alpha,\lambda,\xi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(X^{-1}s)(\xi) = 0\,, \qquad \mathrm{Re}\,\alpha > 0\,.$$

Put $X^{-1}s = {}^t(a, b)$. Then (1.19) implies, by Proposition 1.2 (ii), that the following function of $z = (1 + i\,\mathrm{sh}\,\tau)/2$

$$\begin{aligned} &(a\,\zeta_-\zeta_-\rho_{11}b - b\,\zeta_+\zeta_+\rho_{22}a)/\sqrt{z(1-z)}\,, &&\kappa \neq 1/4\,, \\ &a(\zeta_-\zeta_-\rho_{12} + \zeta_-\zeta_+\rho_{22})a/\sqrt{z(1-z)}\,, &&\kappa = 1/4\,, \end{aligned}$$

is holomorphic at $z = 0$, from which it is immediate that

$$a(\lambda)\,b(\xi) = b(\lambda)\,a(\xi) = 0 \text{ for } \kappa \neq 1/4, \text{ while } a(\lambda)\,a(\xi) = 0 \text{ for } \kappa = 1/4\,.$$

Since $a$ as well as $b$ is real analytic, either $a$ or $b$ must vanish identically if $\kappa \neq 1/4$, and $a = 0$ if $\kappa = 1/4$. Thus there exists a Borel measurable function $c_\pm$ with values in $C^*$ such that $s = c_\pm s_\pm$ a.e. Q.E.D.

We return to the study of invariant closed subspaces common to $\mathscr{L}_0$ and $T_t$ $(t \geq 0)$. In case $\alpha_\pm = 1/2 \pm i\eta$, denote by $\zeta_{0,\pm}$, $\zeta_0$, $X_0$, $s_{0,\pm}$ and $r_{0,\pm}$, respectively, $\zeta_\pm$, $\zeta$, $X$, $s_\pm$ and $r_\pm$ in (1.8). Then we define subspaces $D^\eta_{0,\pm}$ of $L^2(R)$ by

$$D^\eta_{0,\pm} = \mathscr{F}_0^{-1}\{g \in L^2(R_+;\rho_0);\ {}^t s_{0,\pm}(\lambda) g(\lambda) = 0 \text{ a.e.}\}, \tag{1.20}$$

where $\rho_0$ is the spectral density for $\mathscr{L}_0$ with respect to $\Phi_0$ and $\mathscr{F}_0$ stands for the isometry associated with the eigenfunction expansion. Here, $\Phi_k$, $k \in Z/2$, is the solution of the following ordinary differential equation;

$$(\mathscr{L}_k - \lambda)\Phi_k(\tau, \lambda) = 0, \qquad {}^t({}^t\Phi_k, {}^t\Phi_k')_{\tau=0} = I_2. \tag{1.21}$$

Thanks to Theorem 1.1 $D^\eta_{0,\pm}$ are invariant, closed proper subspaces for $\mathscr{L}_0$ and $T_t$ $(t \geq 0)$, and there are no other closed proper subspaces with the invariant property.

We proceed to the study of invariant closed subspaces common to $\mathscr{L}_{1/2}$ and $T_t$ $(t \geq 0)$.

LEMMA 1.5. *The selfadjoint operator $\mathscr{L}_{1/2,\eta}$, $\eta \in R$, has no eigenvalues.*

*Proof.* Consider a selfadjoint operator $M_{1/2,\eta} = i\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} d\Big/d\tau + i\eta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Big/\operatorname{ch}\tau$ with domain $H_1(R)^2$ [6, p. 287]. We note that $(UM_{1/2,\eta}U^*)^2 = \mathscr{L}_{1/2,\eta} \oplus \mathscr{L}_{1/2,-\eta}$ for a unitary matrix $U = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\Big/\sqrt{2}$. This relation implies that an eigenvalue of $\mathscr{L}_{1/2,\pm\eta}$, if any, is equal to zero, because $\mathscr{L}_{1/2,\pm\eta}$ has no positive eigenvalues in virtue of Theorem 4 [4]. Now assume that $f$ is an eigenvector corresponding to the eigenvalue zero, say, of $\mathscr{L}_{1/2,\eta}$. Then $(UM_{1/2,\eta}U^*)^2\, {}^t(f, 0) = 0$. This contradicts the fact that $M_{1/2,\eta}$ has no eigenvalues by Theorem 2 [4] Q.E.D.

Since the function $(1/4 - k^2 + \eta^2 + 2k\eta \operatorname{sh}\tau)/\operatorname{ch}^2\tau$ is integrable, the spectral matrix for $\mathscr{L}_k$ relative to $\Phi_k$ has an $M_2^{++}$-valued continuous density $\rho_k$ on $R_+$ due to Theorem 4 [4]. On account of Lemma 1.5 we can define an onto isometry $\mathscr{F}_{1/2}\colon L^2(R) \to L^2(R_+, \rho_{1/2})$ and its inverse $\mathscr{F}_{1/2}^{-1}$ in a similar way as (1.5) and (1.6) respectively. To define invariant subspaces $D^\eta_{1/2,\pm}$ we first note that the equation (1.21) has a regular singularity at $\tau = i\pi/2$, the indicial roots at which are $1/2 \pm (i\eta - k)$. Therefore, the equation (1.21) for $k = 1/2$ has linearly independent solutions $\zeta_{1/2,\pm}(\tau, \lambda)$ which, being holomorphic in $\dot{D}_\tau \times C$, admit the following expansion near $\sigma = 0$.

$$\zeta_{k,\pm} = \sigma^{1/2\pm(i\eta-k)}\left(\sum_{n=0}^{\infty} z_{k,\pm,n}\sigma^n\right), \qquad z_{k,\pm,0} = 1, \tag{1.22}$$

where $k = 1/2$. It should be noted that $(\zeta_{1/2,-}, \zeta_{1/2,+}) = \Phi_{1/2}$ if $\eta = 0$. Let us define $X_k(\lambda) \in M_2$, $s_{k,\pm}(\lambda)$, $r_{k,\pm}(\lambda) \in M_{2,1}$ in terms of $\Phi_k$ and $\zeta_{k,\pm}$ as in (1.8), and set, for $k = 1/2$,

$$D^{\eta}_{k,\pm} = \mathscr{F}_k^{-1}\{g \in L^2(R_+, \rho_k);\, {}^t s_{k,\pm}(\lambda)g(\lambda) = 0 \text{ a.e.}\}. \tag{1.23}$$

Then, repeating the argument in the proof of Theorem 1.1, we get the next theorem.

THEOREM 1.2. *Let $D$ be a closed proper subspace of $L^2(R)$. Then the selfadjoint operator $\mathscr{L}_{1/2,\eta}$ and the semigroup $T_t$ $(t \geq 0)$ leave $D$ invariant iff $D$ coincides with one of $D^{\eta}_{1/2,\pm}$.*

From now on we shall be concerned with a general $\mathscr{L}_k$. The following lemma shows close relations among the operators $\mathscr{L}_k$ and $F_{\pm,k}$ (see (1.3)).

LEMMA 1.6. *Let $F_{\pm,k}$ and $\mathscr{L}_k$ be the differential operators on $C^\infty(R)$.*

(i) $F_{\mp,k\pm1}F_{\pm,k} = -\mathscr{L}_k - (k \pm 1/2)^2$.

(ii) $\mathscr{L}_{k\pm1}F_{\pm,k} = F_{\pm,k}\mathscr{L}_k$.

(iii) $F^*_{\pm,k} = -F_{\mp,k\pm1}$, $\qquad F^*_{\pm,k}F_{\pm,k} = \mathscr{L}_k + (k \pm 1/2)^2$.

(iv) *If $f$ satisfies $(\mathscr{L}_k - \lambda)f = 0$, then $(\mathscr{L}_{k\pm1} - \lambda)F_{\pm,k}f = 0$. In particular $F_{\pm,k}\Phi_k = \Phi_{k\pm1}X_{\pm,k}$, where*

$$X_{\pm,k} = \begin{pmatrix} \pm\eta & -1 \\ \lambda + (k \pm 1/2)^2 - \eta^2 & \pm\eta \end{pmatrix}.$$

*Proof.* Simple calculation is enough to verify (i)~(iii). The statement (iv) follows from (ii). Q.E.D.

As to eigenfunctions for $\mathscr{L}_k$ we assert

LEMMA 1.7. *Let $f_{\pm k,\pm k}$, $k > 1/2$, be an absolute continuous function on $R$ such that $F_{\mp,\pm k}f_{\pm k,\pm k} = 0$. Set $f_{\pm k\pm m,\pm k} = F_{\pm,\pm k\pm m\mp1}\cdots F_{\pm,\pm k}f_{\pm k,\pm k}$, $m \in Z_+$.*

(i) *$f_{\pm k\pm m,\pm k}$ lies in $H_2(R)$, satisfies the equation*

$$\{\mathscr{L}_{\pm k\pm m} + (k \mp 1/2)^2\}f_{\pm k\pm m,\pm k} = 0, \tag{1.24}$$

*and takes the following form near $\sigma = 0$.*

(1.25) $$\sigma^{1/2\pm i\eta-k-m}\Big(\sum_{n=0}^{\infty} z_n\sigma^n\Big), \quad z_0 \neq 0, \quad (-1)^n z_n = z_n .$$

(ii) *$f_{\pm k\pm m,\pm k}(\tau)$, as a function of $z = (1 + i\,\mathrm{sh}\,\tau)/2$, is bounded on $\{|z| \geq 2\}$.*

*Proof.* The function $f_{\pm k,\pm k}$ is clearly a constant multiple of the function $(\mathrm{ch}\,\tau)^{\mp k+1/2}\exp\Big(\pm\eta\int_0^\tau 1/\mathrm{ch}\,t\,dt\Big)$ which lies in $L^2(R)$ as well as its derivative. By Lemma 1.6 (i) we note that $f_{\pm k,\pm k}$ is an eigenfunction of $\mathscr{L}_{\pm k}$ corresponding to the eigenvalue $-(k \mp 1/2)^2$. Since $1/2 \pm (i\eta - k')$ and $1/2 \pm (i\eta - k')$ are indicial roots at $\sigma = 0$ for the equations $F_{\mp,k'}\, f = 0$ and $(\mathscr{L}_{k'} - \lambda)\, f = 0$ respectively, $f_{\pm k,\pm k}$ can be expanded as (1.25) for $m = 0$. From now on only $f_{k,k}$ will be discussed. By Frobenius method, together with what we have proved, it can be easily seen that the equation (1.24) for $m = 0$ has linearly independent solutions $\zeta_\pm$ such that

(1.26) $$\zeta_\pm = \sigma^{1/2\pm(i\eta-k)}\Big(\sum_{n=0}^{\infty} z_{\pm,n}\sigma^n\Big), \qquad z_{\pm,0} \neq 0 ,$$

where $\zeta_+ \propto f_{k,k}$. Let $\mathscr{L}_k(\sigma)$ stand for $\mathscr{L}_k$ represented in terms of the variable $\sigma$. Using the relation $R\mathscr{L}_k(\sigma)R = \mathscr{L}_k(\sigma)$, we can show that $(-1)^n z_{\pm,n} = z_{\pm,n}$. It is now immediate that $(-1)^n z_n = z_n$ when $m = 0$. This proves (i) for $m = 0$. To show (i) for any $m$, we can proceed by induction on $m$, keeping in mind that $F_{+,k+m-1}\cdots F_{+,k}\zeta_+$ takes the form $\sigma^{1/2+i\eta-k-m}(\sum_{n=0}^\infty z_n\sigma^n)$, $z_0 \neq 0$. To prove the statement (ii) we note that the equation (1.21) can be written as

(1.27) $$\Big\{\frac{d^2}{dz^2} + \frac{2z-1}{2(z^2-1)}\frac{d}{dz} + \frac{1/4-k^2+\eta^2-i(2z-1)}{4(z^2-1)^2} + \frac{\lambda}{4(z^2-1)}\Big\}\Psi_k = 0 ,$$

where $z = (1 + i\,\mathrm{sh}\,\tau)/2$ and $\Psi_k(z, \lambda) = \Phi_k(\tau, \lambda)$. The indicial equation at $z = \infty$ for the above equation is $\alpha^2 + \lambda = 0$. Since $f_{\pm k\pm m,\pm k}$ satisfies (1.24), it assumes the form $z^{-k+1/2}(\sum_{n=0}^\infty y_n z^{-n})$, $y_0 \neq 0$, near $z = \infty$. This is because $f_{\pm k\pm m,\pm k}(\tau)$ in $H_2(R)$ tends to zero as $\tau \to \pm\infty$ (i.e. $z \to 1/2 \pm i\infty$). Q.E.D.

DEFINITION. Let notation be as in Lemma 1.7. We denote by $e_{\pm k\pm m,\pm k}$, $m \in Z_+$, the normalized eigenvector $f_{\pm k\pm m,\pm k}/\|f_{\pm k\pm m,\pm k}\|$ of $\mathscr{L}_{\pm k\pm m}$ corresponding to the eigenvalue $-(k \mp 1/2)^2$. Let $\Lambda_k$ be the set of eigenvalues of $\mathscr{L}_k$ and $\tilde{E}_k$ be the Hilbert space $L^2(\Lambda_k, \delta_k)$, where $\delta_k$ is an atomic measure on $\Lambda_k$ such that $\delta_k(\{\lambda\}) = 1$ for each $\lambda \in \Lambda_k$.

We already know that $\Lambda_k = \phi$ if $|k| \leq 1/2$. It will be proved in the following proposition that

$$\Lambda_k = \{-(j+1/2)^2; j = k, k+1, \cdots < -1/2\} \text{ if } k < -1/2\,,$$
$$= \{-(j-1/2)^2; j = k, k-1, \cdots < 1/2\} \text{ if } k > 1/2\,.$$

According to the eigenfunction expansion theorem for $\mathscr{L}_k$ (see [1, p. 251]) we can define an onto isometry $\mathscr{F}_k : L^2(R) \to L^2(R_+, \rho_k) \oplus \tilde{E}_k$ and its inverse $\mathscr{F}_k^{-1}$ as follows.

$$\mathscr{F}_k f(\lambda) = \lim_{N\to\infty} \int_{|\tau|<N} {}^t\Phi_k(\tau, \lambda) f(\tau)\, d\tau \qquad \text{in } L^2(R_+, \rho_k)\,, \tag{1.28}$$
$$\mathscr{F}_k f(\lambda) = \langle e_{k,j}, f\rangle \text{ for } \lambda = -\{j - (\operatorname{sign} k)1/2\}^2 \in \Lambda_k\,.$$

$$\mathscr{F}_k^{-1} g(\tau) = \lim_{N\to\infty} \int_{0<|\tau|<N} \Phi_k(\tau, \lambda)\, \rho_k(\lambda)\, g(\lambda)\, d\lambda \tag{1.29}$$
$$\oplus\, \Sigma_j\, g(-\{j - (\operatorname{sign} k)1/2\}^2) e_{k,j}\,.$$

Here $\rho_k$ is the spectral density for $\mathscr{L}_k$ relative to $\Phi_k$. The next Proposition is concerned with the spectral property of $\mathscr{L}_k$.

PROPOSITION 1.8.

( i ) *The set of eigenvalues* $\Lambda_k$, $|k| > 1/2$, *is given as above.*

(ii) $\rho_{k+1}(\lambda) = -X_{+,k}(\lambda)\rho_k\, {}^tX^{-1}_{-,k+1}(\lambda)$, $\lambda \in R_+$,

*where* $X_{\pm,k}$ *stands for the same as in Lemma* 1.6.

*Proof.* We shall prove the assertion (i) only for $k > 1/2$. Assume that an $f$ in $H_2(R)\backslash\{0\}$ satisfies $(\mathscr{L}_k - \lambda)f = 0$ for $k = 1$ or $3/2$. Then $(\mathscr{L}_{k-1} - \lambda)F_{-,k}f = 0$ by Lemma 1.6 (ii). Particularly $F_{-,k}f$ belongs to $H_2(R)$. Since $\mathscr{L}_{k-1}$ has no eigenvalues, we conclude that $F_{-,k}f = 0$. Consequently a possible eigenvalue for $\mathscr{L}_k$ is $-(k-1/2)^2$ by Lemma 1.7. Conversely, the same lemma implies that $-(k-1/2)^2$ is really an eigenvalue. Recalling the well-known fact that the multiplicity of an eigenvalue for $\mathscr{L}_k$ is one, (i) has been proved in this case. Working by induction on $k$, we can complete the proof of (i). If $g$ belongs to $C_0(R_+)^2$, $f = \mathscr{F}_k^{-1}g$ lies in the domain of $\mathscr{L}_k$ and tends to zero as $|\tau| \to \infty$. Integration by parts, together with Lemma 1.6 (iv), yields $\mathscr{F}_{k+1}F_{+,k}f = X^{*-1}_{-,k}g$. Therefore we can represent $F_{+,k}f$ in two ways;

$$\int_{R_+} \Phi_{k+1} X_{+,k}\, \rho_k\, g\, d\lambda = \int_{R_+} \Phi_{k+1}\, \rho_{k+1}\, X^{*-1}_{-,k+1} g\, d\lambda\,,$$

which results in (ii), for $X_{-,k}$ is a real matrix. Q.E.D.

We are in a position to define invariant closed subspaces $D^{\eta}_{k,\pm}$ in $L^2(R)$. Since $s_{k,\pm}$ and $r_{k,\pm}$ for $k = 0$, $1/2$ are defined in connection with

$D^{\eta}_{k,\pm}$, $k=0$, $1/2$, the following definition makes sense.

$$s_{k,\pm}=X_{+,k-1}s_{k-1,\pm},\qquad r_{k,\pm}={}^{t}X_{-,k}r_{k-1,\pm}\,. \tag{1.30}$$

$$D^{\eta}_{k,\pm}=\mathscr{F}_k^{-1}\{g\in L^2(R_+,\rho_k);\ {}^{t}s_{k,\pm}(\lambda)g(\lambda)=0 \text{ a.e.}\}\oplus\mathscr{F}_k^{-1}\tilde{E}_{k,\pm}\,, \tag{1.31}$$

where $\tilde{E}_{k,\pm}=\tilde{E}_k$ if $\pm k>0$, while $\{0\}$ if $\pm k<0$. The following is one of the main theorems in this section.

THEOREM 1.3. *Let $D$ be a closed proper subspace of $L^2(R)$. Then the selfadjoint operator $\mathscr{L}_{k,\eta}$ and the semigroup $T_t$ $(t\geq 0)$ leave $D$ invariant iff $D$ coincides with one of $D^{\eta}_{k,\pm}$.*

To prove the theorem we need a lemma.

LEMMA 1.9. *Let $\lambda$ be positive.*

(i) ${}^{t}s_{k,\pm}(\lambda)\,r_{k,\pm}(\lambda)=0.$

(ii) *If either $\eta\in R^*$ or $k\in Z+1/2$, then*

$$\Phi_k(\tau,\lambda)s_{k,\pm}(\lambda)=O(\sigma^{1/2\pm(-i\eta+k)})\,,$$
$$\Phi_k(\tau,\lambda)\rho_k(\lambda)r_{k,\pm}(\lambda)=O(\sigma^{1/2\pm(i\eta-k)})\,.$$

*If $\eta=0$ and $k\in Z$, then*

$$\Phi_k(\tau,\lambda)s_{k,\pm}(\lambda),\ \Phi_k(\tau,\lambda)\rho_k(\lambda)r_{k,\pm}(\lambda)=O(\sigma^{1/2+|k|})\,.$$

*In the above $O(\sigma^{\alpha})$ denotes a holomorphic function on $\dot{D}_{\tau}$ which assumes the form $\sigma^{\alpha}(\sum_{n=0}^{\infty}c_n\sigma^{2n})$, $c_0\neq 0$, near $\sigma=0$.*

*Proof.* The relation (i) holds for $k=0$, $1/2$. Since $X_{-,k}(\lambda)X_{+,k-1}(\lambda)=-\lambda-(k-1/2)^2$, (i) follows from the definition of $s_{k,\pm}$ and $r_{k,\pm}$. As to the statement (ii) only the functions $\Phi_k s_{k,\pm}$ will be examined. We recall that

$$\Phi_k s_{k,\pm}=2\zeta_{k,+}\text{ if }(k,\eta)=(0,0)\text{ while }2\zeta_{k,\pm}\text{ if }k=1/2\text{ or }k=0,\ \eta\in R^*\,.$$

Therefore (ii) is valid for $k=0$, $1/2$. Assume that (ii) holds down to $k\leq 0$. To proceed by induction on $k$, we note that

$$F_{\pm,k}\Big(\sum_{n=0}^{\infty}c_n\sigma^{\alpha+2n}\Big)=\{1/2\pm(-i\eta+k)-\alpha\}c_0\sigma^{\alpha-1}+\sum_{n=1}^{\infty}d_n\sigma^{\alpha+2n-1}\,,$$
$$F_{-,k}\Phi_k(\tau,\lambda)s_{k,\pm}(\lambda)=-\{\lambda+(k-1/2)^2\}\Phi_{k-1}(\tau,\lambda)s_{k-1,\pm}(\lambda)\,.$$

Let $\Phi_k s_{k,\pm}$ take the form $\sum_{n=0}^{\infty}c_n\sigma^{\alpha+2n}$, $c_0\neq 0$. Then it can be easily seen that if $1/2-(-i\eta+k)-\alpha$ vanishes, $d_n$ is equal to zero unless $\mathrm{Re}\,(\alpha+2n-1)\geq\mathrm{Re}\,\{1/2-(k-1)+i\eta\}$. This is due to the fact that $F_{-,k}\Phi_k s_{k,\pm}$

is a nonzero solution of the equation $(\mathscr{L}_{k-1}-\lambda)f=0$ whose indicial roots at $\sigma=0$ are $1/2\pm(k-1-i\eta)$. This proves (ii) for $k<0$. In case $k>0$, we can argue similarly, using the equality $F_{+,k}\Phi_k s_{k,\pm}=\Phi_{k+1}s_{k+1,\pm}$. Q.E.D.

*Proof of Theorem* 1.3. The proof is much like that of Theorem 1.1. We may assume that $k\neq 0$, 1/2, and shall prove the theorem in the case $k>0$. On account of Lemmas 1.1, 1.3, 1.7 and 1.9, Proposition 1.2 (i) yields the following equalities.

$$\begin{aligned}&\int {}^t s_{k,+}(\lambda)\,{}^t\Phi_k(\tau,\lambda)\,G_\alpha\zeta_k(\tau,\xi)\,\sigma_k(\xi)\,r_{k,+}(\xi)\,d\tau=0\,,\\&\int {}^t s_{k,+}(\lambda)\,{}^t\Phi_k(\tau,\lambda)\,G_\alpha e_{k,j}(\tau)d\tau=0\,,\\&\int {}^t s_{k,-}(\lambda)\,{}^t\Phi_k(\tau,\lambda)\,G_\alpha\Phi_k(\tau,\xi)\,\rho_k(\xi)\,r_{k,-}(\xi)\,d\tau=0\,,\\&\int e_{k,j}(\tau)G_\alpha\Phi_k(\tau,\xi)\,\rho_k(\xi)\,r_{k,-}(\xi)\,d\tau=0\,,\end{aligned}$$

where $\lambda$ and $\xi$ are positive. We can show, as in the proof of Theorem 1.1, that the first two and last two equalities imply the invariance of $D^\eta_{k,+}$ and $D^\eta_{k,-}$ under the semigroup $T_t$ $(t\geq 0)$ respectively. Here we used the fact that $\bar{e}_{k,j}=ce_{k,j}$ for some constant $c$, $|c|=1$. On the other hand, $\mathscr{L}_k$ clearly leaves $D^\eta_{k,\pm}$ invariant. Conversely, let $D$ be a proper closed subspace with the desired invariant property. Arguing as in the proof of Theorem 1.1, we see that

$$D=\sum_{j\in I}\oplus\{e_{k,j}\}\oplus\mathscr{F}_k^{-1}\{g\in L^2(R_+,\rho_k);\ {}^t s(\lambda)g(\lambda)=0\ \text{a.e.}\}$$

for some subset $I$ of $\{k, k-1,\cdots,1\text{ or }3/2\}$ and a real analytic function $s$ on $R_+$ with values in $M_{2,1}\backslash\{0\}$ a.e. Denote by $\zeta_{k,\pm}(\tau,\lambda)$ linearly independent solutions of the equation $(\mathscr{L}_k-\lambda)\zeta=0$ such that they are holomorphic in $\dot{D}_\tau\times C$ and have the following expansion near $\sigma=0$.

$$\begin{aligned}\zeta_{k,\pm}&=\sigma^{1/2\pm(i\eta-k)}\Big(1+\sum_{n=1}^{\infty}z_{k,\pm,2n}\sigma^{2n}\Big)\,,\ \text{if }\eta\in R^*\text{ or }k\in Z+1/2\,,\\ \zeta_{k,+}&=\sigma^{1/2+|k|}\Big(1+\sum_{n=1}^{\infty}z_{k,+,2n}\,\sigma^{2n}\Big)\,,\ \text{if }\eta=0\text{ and }k\in Z\,.\\ \zeta_{k,-}&=(F_{+,k-1}\cdots F_{+,0}\zeta_{0,+})\log\sigma+\sigma^{1/2-|k|}\Big(\sum_{n=0}^{\infty}z_{k,-,n}\,\sigma^n\Big)\,,\ z_{k,0,-}\neq 0\,.\end{aligned}$$

Set $\zeta_k=(\zeta_{k,-},\zeta_{k,+})$, and define $X_k$ by $\zeta_k=\Phi_kX_k$. Then, it can be shown, as in the proof of Theorem 1.1, that the symmetric matrix $X_k^{-1}\rho_k\,{}^tX_k^{-1}$ is diagonal in the case either $\eta \in R^*$ or $k \in Z + 1/2$ while the matrix assumes the form $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$ in the case $\eta = 0$ and $k \in Z$. It is not hard to see that in the former case one of the components of $X_k^{-1}s$ must vanish identically while in the latter case the first component of $X^{-1}s$ must vanish (see the proof of Theorem 1.1). This means that there are, at most, two possibilities for $s$. Therefore, since $D^{\eta}_{k,\pm}$ possess the invariant property, there exists a $C^*$-valued measurable function $c_+$ or $c_-$ such that $s = c_+ s_{k,+}$ or $c_- s_{k,-}$ a.e. on $R_+$. Suppose $s = c_+ s_{k,+}$. We must show that $I = \{k, k-1, \cdots, 1 \text{ or } 3/2\}$, provided $\eta \in R^*$ or $k + 1/2 \in Z$ (recall that $s_{k,+} = s_{k,-}$ in the case when $\eta = 0$ and $k \in Z$). On account of Lemmas 1.1, 1.3, 1.7 and 1.9, using Proposition 1.2 (ii), we can show that for any eigenvector $e_{k,j}$, there exists an $\alpha'$, $\operatorname{Re} \alpha' > 0$, satisfying

$$\langle e_{k,j}(\tau) G_{\alpha'} \Phi_k(\tau, \xi) \rho_k(\xi) r_{k,+}(\xi) \rangle \neq 0$$

so that $\langle e_{k,j}(\tau) G_{\alpha'} \mathscr{F}_k^{-1} r_{k,+} h \rangle \neq 0$ for some $h \in C_0(R_+)^1$. This means $D = D^{\eta}_{k,+}$, that is, $I = \{k, k-1, \cdots, 1 \text{ or } 3/2\}$, for $D$ is $\mathscr{L}_k$-invariant. Next, assume $s = c_- s_{k,-}$. We must show that $I = \phi$, provided $\eta \in R^*$ or $k \in Z + 1/2$. To this end, we note that for any eigenvector $e_{k,j}$ and positive $\lambda$, there is an $\alpha'$, $\operatorname{Re} \alpha' > 0$, such that

$${}^t s_{k,-}(\lambda) \langle {}^t \Phi_k(\tau, \lambda) G_{\alpha'} e_{k,j}(\tau) \rangle \neq 0$$

on the same basis as above. This implies that $I = \phi$, since ${}^t s(\lambda)[\mathscr{F}_k G_\alpha f](\lambda) = 0$ a.e. for any $f \in D$. Finally, we note that for any eigenvectors $e_{k,i}$ and $e_{k,j}$, there exists an $\alpha'$, $\operatorname{Re} \alpha' > 0$, such that $\langle e_{k,i}, G_{\alpha'} e_{k,j} \rangle \neq 0$. This means $I = \phi$ or $\{k, k-1, \cdots, 1 \text{ or } 3/2\}$. Since $s_{k,-} = s_{k,+}$ in the case $\eta = 0$ and $k \in Z$, Theorem 1.3 has been shown for $k > 0$. In case $k < 0$, we can argue similarly. Q.E.D.

We set $W_k = L^2(R)$ for $k \in Z/2$ and regard $\mathscr{L}_k$ as a selfadjoint operator in $W_k$ and $F_{\pm,k}$ as an operator sending $W_k$ into $W_{k\pm 1}$. It is the next theorem that will be used in § 2.

THEOREM 1.4. *Let $\{D_k\}_{k \in Z+\varepsilon}$, $\varepsilon = 0, 1/2$, be a nontrivial sequence of closed subspaces of $W_k$. Then the sequence $\{D_k\}$ fulfils the following two conditions iff it coincides with one of*

$\{D^{\eta}_{k,-}\}$, $\{D^{\eta}_{k,+}\}$ *if* $\eta \in R^*$ *or* $1/2$,

$\{D^0_{k,\operatorname{sign}(-k+1/2)}\}$, $\{D^0_{k,-}\}$, $\{D^0_{k,+}\}$ *and* $\{D^0_{k,\operatorname{sign}(k+1/2)}\}$ *if* $\eta = \varepsilon = 0$.

i) $D_k$ *is invariant under the selfadjoint operator* $\mathscr{L}_{k,\eta}$ *and the semigroup* $T_t$ $(t \geq 0)$.

ii) $F_{\pm,k,\eta}D_k \subset D_{k\pm1}$, *where the domains of* $F_{\pm,k,\eta}$ *are* $H_2(R)$.

*Proof.* We shall first show the sufficiency of the condition. Assume that an $f$ in $H_2(R)$ satisfies $\mathscr{F}_k f = r_{k,\pm}h$, $h \in L^2(R_+, r^*_{k,\pm}\rho_k r_{k,\pm})$. Then integration by parts yields

$$(1.32) \qquad \mathscr{F}_{k+1}F_{+,k}f = -r_{k+1,\pm}h,\ \mathscr{F}_{k-1}F_{-,k}f = \{\lambda + (k-1/2)^2\}r_{k-1,\pm}h\,.$$

Making use of Lemma 1.6, we can verify easily that for $k$, $|k| > 1/2$,

$$(1.33) \qquad F_{\pm,k}e_{k,j} = \pm(\operatorname{sign} k)\sqrt{(k\pm 1/2)^2 - \{j-(\operatorname{sign} k)1/2\}^2}\, e_{k\pm1,j}\,.$$

By (1.32) and (1.33) the sequences mentioned in the theorem satisfy the conditions i) and ii). Conversely, let $\{D_k\}$ be a nontrivial sequence satisfying i) and ii). In view of Theorem 1.3 and the relations (1.32) and (1.33), $\{D_k\}$ must coincide with one of the aforementioned sequences, provided some $D_k$ is a proper subspace. Therefore it remains to show that all $D_k$ are proper subspaces. To this end, suppose $D_k = L^2(R)$ for some $k$. Let us show that $D_{k\pm1} = L^2(R)$. In fact, on account of the equality $G_\alpha F_{\pm,k} = F_{\pm,k}G_\alpha + G'_\alpha$ it is not hard to see that if an $f$ in $(D_{k\pm1})^\perp$ is orthogonal to the image $G_\alpha F_{\pm,k}C_0^\infty(R)$, then $f=0$. Assume now that $D_{k-1} = \{0\}$ and $D_k \neq \{0\}$ for some $k$. This contradicts Theorem 1.3 and (1.32). Thus each $D_k$ must be proper for the sequence $\{D_k\}$ to be nontrivial. Q.E.D.

Before concluding this section we shall rewrite the relation (1.32) in a more convenient manner. For this purpose, introduce Hilbert spaces $\tilde{D}^\eta_{k,\pm}$, $\hat{D}^\eta_{k,\pm}$ and an onto isometry $I^{\eta,\varepsilon}_{\pm,k} : \tilde{D}^\eta_{k,\pm} \to \hat{D}^\eta_{k,\pm}$, $k \in Z + \varepsilon$, as follows.

$$(1.34) \qquad \begin{aligned} &\tilde{D}^\eta_{k,\pm} = \{r_{k,\pm}h \in L^2(R_+, \rho_k);\ h \in L^2(R_+, r^*_{k,\pm}\rho_k r_{k,\pm})\} \oplus \tilde{E}_{k,\pm}\,. \\ &\hat{D}^\eta_{k,\pm} = L^2(R_+) \oplus \tilde{E}_{k,\pm}\,. \\ &(I^{\eta,\varepsilon}_{\pm,k} r_{k,\pm}h)(\lambda) = \langle r_{k,\pm}(\lambda), \rho_k(\lambda) r_{k,\pm}(\lambda)\rangle^{1/2} h(\lambda)\,, \qquad \lambda > 0\,, \\ &I^{\eta,\varepsilon}_{\pm,k}|\tilde{E}_{k,\pm} = \text{the identity operator.} \end{aligned}$$

Furthermore, for $F_{\pm,k}$ with domain $H_1(R)$, set

$$\hat{F}_{+,k,\pm} = I^{\eta,\varepsilon}_{\pm,k+1}\mathscr{F}_{k+1}F_{+,k}(I^{\eta,\varepsilon}_{\pm,k}\mathscr{F}_k)^{-1}\,,$$
$$\hat{F}_{-,k,\pm} = I^{\eta,\varepsilon}_{\pm,k-1}\mathscr{F}_{k-1}F_{-,k}(I^{\eta,\varepsilon}_{\pm,k}\mathscr{F}_k)^{-1}\,.$$

Then (1.32) yields

$$(1.35) \qquad \hat{F}_{\pm,k,s}h(\lambda) = \mp\sqrt{\lambda + (k\pm 1/2)^2}\, h(\lambda)\,, \quad h \in C_0(R_+)^1\,, \quad s = + \text{ or } -\,.$$

This is because $\langle r_{k,\pm}(\lambda), \rho_k(\lambda) r_{k,\pm}(\lambda)\rangle = \{\lambda + (k-1/2)^2\}\langle r_{k-1,\pm}(\lambda), \rho_{k-1}(\lambda) r_{k-1,\pm}(\lambda)\rangle$ by virtue of the definition of $r_{k,\pm}$ and Proposition 1.8 (ii).

## § 2. $P_+(3)$-invariant subspaces for the representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$

We begin by defining the representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$ of the group $P(3)$ (see the introduction for the definition of $P(3)$) associated with the one-sheeted hyperboloid $V_{iM}(3) = \{y_0^2 - y_1^2 - y_2^2 = -M^2\}$, $M > 0$, after Mackey [7]. Let $G$ be $SU(1,1)$, and $\omega_j$, $1 \le j \le 3$, be one-parameter subgroup of $G$;

$$\omega_1(t) = \begin{pmatrix} \mathrm{ch}\, t/2 & \mathrm{sh}\, t/2 \\ \mathrm{sh}\, t/2 & \mathrm{ch}\, t/2 \end{pmatrix}, \qquad \omega_2(t) = \begin{pmatrix} \mathrm{ch}\, t/2 & i\,\mathrm{sh}\, t/2 \\ -i\,\mathrm{sh}\, t/2 & \mathrm{ch}\, t/2 \end{pmatrix},$$

$$\omega_3(t) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}.$$

$G$ acts on $R^3$ as $y \cdot g = g^* y g$, where $y = (y_0, y_1, y_2)$ is identified with a matrix $\begin{pmatrix} y_0 & y_2 - iy_1 \\ y_2 + iy_1 & y_0 \end{pmatrix}$. It can be easily seen that the orbit of $\hat{y} = M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $V_{iM}(3)$ and that the isotropy group at $\hat{y}$ is $G_0 = \{\pm\, \omega_2(t);\ t \in R\}$. Let $\pi_{\eta,\varepsilon}$, $\eta \in R$, $\varepsilon = 0, 1/2$, be an irreducible unitary representation of $G_0$ such that $\pi_{\eta,\varepsilon}(\pm\, \omega_2(t)) = (\pm 1)^{2\varepsilon} \exp i\eta t$. We can identify the factor space $G_0\backslash G \simeq (R^3 \times_s G_0)\backslash(R^3 \times_s G)$ with $V_{iM}(3)$ via a projection $p$ of $G$ onto $V_{iM}(3)$ defined by $p(g) = g^* \hat{y} g$. As is well known, the measure $d\bar{y} = dy_1 dy_2 / M|y_0|$ on $V_{iM}(3)$ is $G$-invariant. Let $\tilde{\mathfrak{H}}^{\eta,\varepsilon}$ be the set of $C$-valued measurable functions on $P(3)$ such that

$$f((x', g_0)(x, g)) = e^{i\langle x', \hat{y}\rangle} \pi_{\eta,\varepsilon}(g_0) f(x, g), \qquad g_0 \in G_0,$$

and that $|f(x, g)|^2$, which is a function on $V_{iM}(3)$, is integrable relative to the measure $d\bar{y}$. Then $\tilde{\mathfrak{H}}^{\eta,\varepsilon}$ equipped with the inner product $\langle f, h\rangle = \int \bar{f} h \, d\bar{y}$ give rise to a Hilbert space, which we denote by $\tilde{\mathfrak{H}}^{\eta,\varepsilon}$ again. Let $U^{\eta,\varepsilon}(x, g)$, $(x, g) \in P(3)$, be a linear operator on $\tilde{\mathfrak{H}}^{\eta,\varepsilon}$ defined by

$$[U^{\eta,\varepsilon}(x, g) f](x', g') = f((x', g')(x, g)).$$

It is well-known that $(U^{\eta,\varepsilon}, \tilde{\mathfrak{H}}^{\eta,\varepsilon})$ is an irreducible unitary representation of $P(3)$ associated with $V_{iM}(3)$ and $\pi_{\eta,\varepsilon}$. We prefer to realize this representation in $L^2(V_{iM}(3), d\bar{y})$. For this purpose, note that a map $p(\omega_1(\tau)\omega_3(\theta))$ of $R \times (0, 2\pi)$ into $V_{iM}(3)$ is a diffeomorphism onto an open dense set of $V_{iM}(3)$, and fix a Borel measurable section $s_e$ of $V_{iM}(3)$ into $G$ such that $s_e \circ p(\omega_1(\tau)\omega_3(\theta)) = \omega_1(\tau)\omega_3(\theta)$ for $(\tau, \theta) \in R \times (0, 2\pi)$. Then we can define an equivalent representation $(U^{\eta,\varepsilon}, L^2(V_{iM}(3), d\bar{y}))$ as follows.

$$(2.1)\qquad \begin{aligned} &U^{\eta,\varepsilon}(x, g)f(y) = e^{i\langle x',\hat{y}\rangle}\pi_{\eta,\varepsilon}(g_0)f(y\cdot g)\,, \\ &(0, s_e(y))(x, g) = (x', g_0)(0, s_e(y\cdot g))\,, \qquad g_0 \in G_0\,. \end{aligned}$$

Clearly $(\tau, \theta) \in R \times (0, 2\pi)$ is a system of coordinates on an open dense set of $V_{iM}(3)$. Simple calculation yields

$$(y_0, y_1, y_2) = M(\operatorname{sh}\tau, \operatorname{ch}\tau\sin\theta, \operatorname{ch}\tau\cos\theta)\,, \qquad d\bar{y} = \operatorname{ch}\tau\, d\tau\, d\theta\,.$$

Therefore, by identifying $L^2(V_{iM}(3), d\bar{y})$ with $\mathfrak{H}^{\eta,\varepsilon} = L^2(R \times (0, 2\pi), \operatorname{ch}\tau\, d\tau d\theta)$ in a trivial manner, we obtain a representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$ equivalent to the one $(U^{\eta,\varepsilon}, \tilde{\mathfrak{H}}^{\eta,\varepsilon})$ above. From now on the former realization will be discussed. By (2.1) it is easy to see that

$$U^{\eta,\varepsilon}(t, 0, 0, e) = e^{iMt\operatorname{sh}\tau}\,.$$

Let $\omega_j$, $1 \leq j \leq 3$, be an infinitesimal operator of the one-parameter unitary group $U^{\eta,\varepsilon}\,(0, \omega_j(t))$, and put

$$\Delta = -\,\omega_1^2 - \omega_2^2 + \omega_3^2\,, \quad F_\pm = -\,\omega_1 \mp i\omega_2\,, \quad H_3 = i\omega_3\,.$$

To be more precise, $\Delta$ stands for the selfadjoint extension of a symmetric operator $-\,\omega_1^2 - \omega_2^2 + \omega_3^2$ whose domain is the Gårding space, while the domains of $F_\pm$ are the intersection of the domains of $\omega_1$ and $\omega_2$. Using (2.1), we can easily get expressions for the restrictions $\omega_j | C_0^\infty(R \times (0, 2\pi))$. That is,

$$\begin{aligned} \omega_1 &= \cos\theta\,\partial_\tau - \operatorname{th}\tau\sin\theta\,\partial_\theta + i\eta\sin\theta/\operatorname{ch}\tau\,, \\ \omega_2 &= -\,\sin\theta\,\partial_\tau - \operatorname{th}\tau\cos\theta\,\partial_\theta + i\eta\cos\theta/\operatorname{ch}\tau\,, \\ \omega_3 &= \partial_\theta\,. \end{aligned}$$

In particular,

$$F_\pm = -\,e^{\mp i\theta}(\partial_\tau \mp \operatorname{th}\tau\,\partial_\theta \mp \eta/\operatorname{ch}\tau)\,.$$

Put $\mathscr{W}_k^{\eta,\varepsilon} = \{f \in \mathfrak{H}^{\eta,\varepsilon};\, H_3 f = kf\}$, $k \in Z/2$. Then $\mathfrak{H}^{\eta,\varepsilon} = \sum_k \oplus \mathscr{W}_k^{\eta,\varepsilon}$, since eigenvalues of $H_3$ lie in $Z/2$ (see Lemma 2.1). Furthermore, it is not hard to show that $\mathscr{W}_k^{\eta,\varepsilon} = \{0\}$, $k \notin Z + \varepsilon$, and

$$\mathscr{W}_k^{\eta,\varepsilon} = \{f(\tau)e^{-ik\theta};\, f \in L^2(R, \operatorname{ch}\tau)\},\ k \in Z + \varepsilon\,.$$

Now put $W_k = L^2(R)$, $k \in Z/2$, and define an onto isometry $J_k^{\eta,\varepsilon} : \mathscr{W}_k^{\eta,\varepsilon} \to W_k$ by $J_k^{\eta,\varepsilon}(f(\tau)e^{-ik\theta}) = f(\tau)\sqrt{\operatorname{ch}\tau/2\pi}$. Then an onto isometry $J^{\eta,\varepsilon} : \mathfrak{H}^{\eta,\varepsilon} \to W^\varepsilon = \sum_{k\in Z+\varepsilon} \oplus W_k$ arises naturally, namely $J^{\eta,\varepsilon} = \sum_{k \in Z+\varepsilon} \oplus J_k^{\eta,\varepsilon}$. It is immediate that

$$(2.2) \qquad J^{\eta,\varepsilon}U^{\eta,\varepsilon}(t/M, 0, 0, e)J^{\eta,\varepsilon-1} = e^{it\,\mathrm{sh}\,\tau}.$$

Using the explicit forms of $\omega_j$, $1 \le j \le 3$, we obtain, for $k \in Z + \varepsilon$,

$$(2.3) \qquad \begin{aligned} J_k^{\eta,\varepsilon}\Delta J_k^{\eta,\varepsilon-1} &= \mathscr{L}_{k,\eta} + 1/4, \\ J_{k\pm1}^{\eta,\varepsilon}F_\pm J_k^{\eta,\varepsilon-1} &= F_{\pm,k,\eta}. \end{aligned}$$

See (1.1) and (1.3) for the definition of $\mathscr{L}_{k,\eta}$ and $F_{\pm,k,\eta}$ respectively. To be more precise, we can verify the equality (2.3) only on $C_0^\infty(R)$. Since $J_k^{\eta,\varepsilon}\Delta J_k^{\eta,\varepsilon-1}$ is selfadjoint, the first equality in (2.3) follows from Theorem 4.3 [6, p. 287]. On the other hand, the second equality is understood to hold on $H_1(R)$. We regard $D_{k,\pm}^\eta$ (see (1.31)) as a subspace of $W_k$ and introduce closed subspaces $\mathscr{D}_\pm^{\eta,\varepsilon} \subset W^\varepsilon$, $\varepsilon = 0$, $1/2$, and $\mathscr{D}_{\pm1}^{0,0} \subset W^0$ as follows.

$$(2.4) \qquad \begin{aligned} \mathscr{D}_\pm^{\eta,\varepsilon} &= \textstyle\sum_{k\in Z+\varepsilon} \oplus J_k^{\eta,\varepsilon-1}D_{k,\pm}^\eta, \\ \mathscr{D}_{\pm1}^{0,0} &= \textstyle\sum_{k\in Z} \oplus J_k^{0,0-1}D_{k,\mathrm{sign}(\pm k+1/2)}^0. \end{aligned}$$

Now we are ready to state main theorems of this paper.

THEOREM 2.1. *Let $\mathscr{D}$ be a closed proper subspace of $\mathfrak{H}^{\eta,\varepsilon}$. Then $\mathscr{D}$ is $P_+(3)$-invariant iff it coincides with one of $\mathscr{D}_\pm^{\eta,\varepsilon}$ (and $\mathscr{D}_{\pm1}^{0,0}$, provided $(\eta, \varepsilon) = (0, 0)$).*

THEOREM 2.2. *The representations of $SU(1, 1)$ realized in $\mathscr{D}_\pm^{\eta,\varepsilon}$, $\mathscr{D}_{-1}^{0,0}$ and $\mathscr{D}_1^{0,0}$ decompose into irreducible ones, respectively, as*

$$\int_{R_+}^{\oplus} T_{(-1/2+i\eta,\varepsilon)}d\eta \oplus \Sigma_{-k\in Z_+ +1+\varepsilon} \oplus T_{(k,\varepsilon)}^{\mp},$$

$$\int_{R_+}^{\oplus} T_{(-1/2+i\eta,0)}d\eta,$$

$$\int_{R_+}^{\oplus} T_{(-1/2+i\eta,0)}d\eta \oplus \Sigma_{-k\in Z_+ +1} \oplus (T_{(k,0)}^{-} \oplus T_{(k,0)}^{+}).$$

See the following passage for the definition of the representation $T_{(-1/2+i\eta,\varepsilon)}$ and $T_{(k,\varepsilon)}^{\pm}$.

*Remark.* It is known [8] that the representation of $SU(1, 1)$ in $\mathfrak{H}^{\eta,\varepsilon}$ decomposes into irreducible ones as

$$[2]\int_{R_+}^{\oplus} T_{(-1/2+i\eta,\varepsilon)}d\eta \oplus \Sigma_{-k\in Z_+ +1+\varepsilon} \oplus (T_{(k,\varepsilon)}^{-} \oplus T_{(k,\varepsilon)}^{+}).$$

The rest of this section will be devoted to the proof of the above theorems. We begin by reviewing some properties of irreducible unitary representations of $G = SU(1, 1)$. We retain the notation due to Vilenkin

[10, Chapter VI]. Thus $T_{(\ell,\varepsilon)}$ with either $(\ell, \varepsilon) = (-1/2 + i\eta, 0)$, $\eta \geq 0$, or $(\ell, \varepsilon) = (-1/2 + i\eta, 1/2)$, $\eta > 0$, stands for a representation belonging to the continuous series, while $T_{(\ell,0)}$ with $-1 < \ell < -1/2$ is a representation belonging to the supplementary series. In this paper the representation $T^{\pm}_{(\ell,\varepsilon)}$ with either $(\ell, \varepsilon) = (\ell, 0)$, $-\ell \in Z_+ + 1$, or $(\ell, \varepsilon) = (\ell, 1/2)$, $-\ell \in Z_+ + 1/2$, is said to belong to the discrete series, even though $T^{\pm}_{(-1/2,1/2)}$ is not a member of the discrete series in the sense that it is not contained in the regular representation of $G$ as a direct sum component. Recall that $C^\infty(T)$ (resp. a subspace of $C^\infty(T)$) is dense in the representation space $H_{\ell,\varepsilon}$ (resp. $H^{\pm}_{\ell,\varepsilon}$) of $T_{(\ell,\varepsilon)}$ (resp. $T^{\pm}_{(\ell,\varepsilon)}$).

LEMMA 2.1. *For the irreducible unitary representation* $T_{(\ell,\varepsilon)}$ *or* $T^{\pm}_{(\ell,\varepsilon)}$ *of* $G = SU(1, 1)$, *define operators* $\omega_j$, $1 \leq j \leq 3$, $F_\pm$, $H_3$, $\Delta$ *and spaces* $\mathscr{W}_k$, $k \in Z/2$, *as for the representation* $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$. *Then* $\mathscr{W}_k = \{\exp\{-i(k - \varepsilon)\theta\}\}$ *if* $k \in Z + \varepsilon$ *and if* $\exp\{-i(k - \varepsilon)\theta\}$ *lies in the representation space, while* $\mathscr{W}_k = \{0\}$ *otherwise. In addition,*

$$F_\pm e^{-i(k-\varepsilon)\theta} = (\pm k - \ell)e^{-i(k-\varepsilon\pm1)\theta}, \qquad \Delta = -\ell(\ell + 1).$$

*Proof.* The function $\exp\{-i(k - \varepsilon)\theta\}$ is known to lie in $\mathscr{W}_k$, if it belongs to the representation space. Since such functions form a complete orthogonal basis of the representation space, $\dim \mathscr{W}_k \leq 1$. Thus $\mathscr{W}_k$ is obtained. The remaining part of the lemma is well-known [10, p. 299 and p. 334]. The sign of $\ell(\ell + 1)$ on p. 334, however, is misprinted. Q.E.D.

A corollary of the next proposition plays an important role in our discussion.

PROPOSITION 2.2. *Let the notation be as in Lemma* 2.1. *Each* $i\omega_j$, $1 \leq j \leq 3$, *restricted to the algebraic* sum $\Sigma_{k \in Z/2} \oplus \mathscr{W}_k$ *is essentially selfadjoint in the representation space.*

*Proof.* Let $H_{\ell,\varepsilon,c}$ be the algebraic sum $\Sigma_k \oplus \mathscr{W}_k$, and denote by $\dot{\omega}_j$ the restriction $\omega_j | H_{\ell,\varepsilon,c}$. Set, further, $C^\infty = C^\infty(T) \cap H_{\ell,\varepsilon}$, where $T$ stands for the unit circle and $H_{\ell,\varepsilon}$ is the representation space. Since a function $T_{(\ell,\varepsilon)}(g)f(e^{i\theta})$ or $T^{\pm}_{(\ell,\varepsilon)}(g)f(e^{i\theta})$ is smooth on $G \times T$ for any $f \in C^\infty$, $C^\infty$ lies in the domain of $\omega_j$ and invariant under $T_{(\ell,\varepsilon)}$ or $T^{\pm}_{(\ell,\varepsilon)}$. Here we used the fact that the uniform convergence in $C^\infty$ implies the convergence in $H_{\ell,\varepsilon}$. Let $\mathring{\omega}_j$ be the restriction $\omega_j | C^\infty$. We shall show that $i\mathring{\omega}_j$ is essentially selfadjoint. Evidently $i\mathring{\omega}_j$ is symmetric, so it remains to show that the image $(\omega_j - \alpha)C^\infty$ is dense in $H_{\ell,\varepsilon}$ for any $\alpha$, $\operatorname{Re}\alpha \neq 0$. For this purpose, assume that an $f$ in $H_{\ell,\varepsilon}$ is orthogonal to the image. Then, since $T_{(\ell,\varepsilon)}(g)$ or $T^{\pm}_{(\ell,\varepsilon)}(g)$ leaves $C^\infty$ invariant, we have

$$\langle T_{(\ell,\varepsilon)}(\omega_j(t))(\omega_j - \alpha)\phi, f\rangle = 0\,, \quad \phi \in C^\infty\,,$$

or a similar relation for $T^{\pm}_{(\ell,\varepsilon)}$. Multiply the both sides by $e^{-\alpha t}$, and integrate on $R_+$ or $-R_+$ according as $\operatorname{Re}\alpha$ is positive or negative. Then it follows that $\langle\phi, f\rangle = 0$, which implies $f = 0$, as desired. Thus $i\mathring{\omega}_j$ is essentially selfadjoint. To complete the proof, it suffices to show that the closure of $\dot{\omega}_j$ is an extension of $\mathring{\omega}_j$, for $\dot{\omega}_j \subset \mathring{\omega}_j$. To this end, we note first that $\dot{\omega}_j$ is a differential operator with smooth coefficients on $T$. Secondly, the partial sum of the Fourier series for any $f \in C^\infty$ lies in $H_{\ell,\varepsilon,c}$ and they and their derivatives uniformly converge to $f$ and its derivative respectively. Now clearly the closure of $\dot{\omega}_j$ is an extension of $\mathring{\omega}_j$. Q.E.D.

COROLLARY 2.3. *For the irreducible unitary representations $T_{(\ell,\varepsilon)}$ belonging to the continuous series and $T^{\pm}_{(\ell,\varepsilon)}$ belonging to the discrete series in our sense, define $\ell^2$-spaces $\ell^2_{\ell,\varepsilon}$ and $\ell^{2\pm}_{\ell,\varepsilon}$ as follows.*

$$\ell^2_{\ell,\varepsilon} = \{(a_k)_{k\in Z+\varepsilon};\ \Sigma_k|a_k|^2 < \infty\}\,,$$
$$\ell^{2\pm}_{\ell,\varepsilon} = \{(a_k)_{k\in Z+\varepsilon,\ \mp k+\ell\geq 0};\ \Sigma_k|a_k|^2 < \infty\}\,.$$

*Put $\ell^2_{\ell,\varepsilon,c} = \{(a_k) \in \ell^2_{\ell,\varepsilon};\ a_k = 0,\ |k| > n,$ for some $n \in Z_+\}$, and define $\ell^{2\pm}_{\ell,\varepsilon,c}$ similarly. Then operators $i\dot{\omega}_j$, $1 \leq j \leq 2$, with domain $\ell^2_{\ell,\varepsilon,c}$ (resp. $\ell^{2\pm}_{\ell,\varepsilon,c}$) are essentially selfadjoint in $\ell^2_{\ell,\varepsilon}$ (resp. $\ell^{2\pm}_{\ell,\varepsilon}$), where $\dot{\omega}_j$ are defined as follows. Let $f_k = (a_{k'})$ be an element of either $\ell^2_{\ell,\varepsilon}$ or $\ell^{2\pm}_{\ell,\varepsilon}$ such that $a_k = 1$ and $a_{k'} = 0$, $k' \neq k$, and set $\dot{F}_{\pm} = -\dot{\omega}_1 \mp i\dot{\omega}_2$. We require*

$$\begin{aligned}\dot{F}_{\pm}f_k &= \mp\sqrt{\eta^2 + (k \pm 1/2)^2}\,f_{k\pm1} && \text{in } \ell^2_{-1/2+i\eta,\varepsilon}\,,\\ &\mp\sqrt{(k\mp\ell)(k\pm\ell\pm1)}\,f_{k\pm1} && \text{in } \ell^{2+}_{\ell,\varepsilon}\,,\\ &\pm\sqrt{(k\mp\ell)(k\pm\ell\pm1)}\,f_{k\pm1} && \text{in } \ell^{2-}_{\ell,\varepsilon}\,.\end{aligned}$$

*Proof.* Let the notation be as in Lemma 2.1, and set

$$e_k = m_k\exp\{-i(k-\varepsilon)\theta\}/\|\exp\{-i(k-\varepsilon)\theta\}\| \in H_{\ell,\varepsilon}, \text{ where } |m_k| = 1\,.$$

In case $(\ell, \varepsilon)$ is a parameter of the continuous series, we can choose $m_k$ so that $m_k/m_{k-1} = -|k+\ell|/(k+\ell)$. In other cases, set $m_k = 1$. Then it can be easily seen that the restriction of $\omega_j$, $j = 1, 2$, in Proposition 2.2 is unitarily equivalent to $\dot{\omega}_j$ in the above lemma. Q.E.D.

The next lemma is concerned with a pair of one-parameter unitary groups.

LEMMA 2.4. *Let $H_j$, $j = 1, 2$, be Hilbert spaces, and $U_j(t)$ be one-parameter continuous unitary groups on $H_j$ with the infinitesimal operators $\Omega_j = dU_j(t)/dt_{t=0}$. If $H_1$ is a closed subspace of $H_2$ and there exists an essentially selfadjoint operator $i\dot{\Omega}$ such that $\dot{\Omega} \subset \Omega_j$, $j = 1, 2$, then $U_1(t) = U_2(t)$ on $H_1$.*

*Proof.* Let $\Omega$ be the closure of $\dot{\Omega}$. Then $i\Omega$ is selfadjoint and clearly $\Omega \subset \Omega_j$. Consequently, for any $n \in Z_+ + 1$ and $h \in H_1$ we have

$$\Omega(1 - n^{-1}\Omega)^{-1}h = \Omega_j(1 - n^{-1}\Omega_j)^{-1}h\,, \qquad j = 1, 2.$$

That is, $\Omega(1 - n^{-1}\Omega)^{-1} = \Omega_j(1 - n^{-1}\Omega_j)^{-1}$ on $H_1$. By the representation theorem for the continuous semigroup [11, p. 248] we get

$$(\exp t\Omega)h = \lim_{n\to\infty}\{\exp t\Omega(1 - n^{-1}\Omega)^{-1}\}h = (\exp t\Omega_j)h\,, \qquad h \in H_1\,.$$

Q.E.D.

We return to the representation $(U^{\eta,\varepsilon}, \mathfrak{H}^{\eta,\varepsilon})$. Recall the definition of the subspaces $\tilde{D}^{\eta}_{k,\pm}$, $\hat{D}^{\eta}_{k,\pm}$ and the isometry $I^{\eta,\varepsilon}_{\pm,k}$ introduced in (1.34). Let us define auxilary Hilbert spaces $D^{\eta,\varepsilon}_{\pm}$, $D^{0,0}_{\pm1}$, $\tilde{D}^{\eta,\varepsilon}_{\pm}$, $\tilde{D}^{0,0}_{\pm1}$, $\hat{D}^{\eta,\varepsilon}_{\pm}$ and $\hat{D}^{0,0}_{\pm1}$ as follows.

$$\begin{aligned} D^{\eta,\varepsilon}_{\pm} &= \Sigma_{k\in Z+\varepsilon} \oplus D^{\eta}_{k,\pm}\,, & D^{0,0}_{\pm1} &= \Sigma_{k\in Z} \oplus D^{0}_{k,\mathrm{sign}(\pm k+1/2)}\,,\\ \tilde{D}^{\eta,\varepsilon}_{\pm} &= \Sigma_{k\in Z+\varepsilon} \oplus \tilde{D}^{\eta}_{k,\pm}\,, & \tilde{D}^{0,0}_{\pm1} &= \Sigma_{k\in Z} \oplus \tilde{D}^{0}_{k,\mathrm{sign}(\pm k+1/2)}\,,\\ \hat{D}^{\eta,\varepsilon}_{\pm} &= \Sigma_{k\in Z+\varepsilon} \oplus \hat{D}^{\eta}_{k,\pm}\,, & \hat{D}^{0,0}_{\pm1} &= \Sigma_{k\in Z} \oplus \hat{D}^{0}_{k,\mathrm{sign}(\pm k+1/2)}\,. \end{aligned}$$

In terms of the isometries $\mathscr{F}_k : D^{\eta}_{k,\pm} \to \tilde{D}^{\eta}_{k,\pm}$ and $I^{\eta,\varepsilon}_{\pm,k} : \tilde{D}^{\eta}_{k,\pm} \to \hat{D}^{\eta}_{k,\pm}$ we can define onto isometries $\mathscr{F}^{\eta,\varepsilon}_{\pm} : D^{\eta,\varepsilon}_{\pm} \to \tilde{D}^{\eta,\varepsilon}_{\pm}$, $\mathscr{F}^{0,0}_{\pm1} : D^{0,0}_{\pm1} \to \tilde{D}^{0,0}_{\pm1}$, $I^{\eta,\varepsilon}_{\pm} : \tilde{D}^{\eta,\varepsilon}_{\pm} \to \hat{D}^{\eta,\varepsilon}_{\pm}$ and $I^{0,0}_{\pm1} : \tilde{D}^{0,0}_{\pm1} \to \hat{D}^{0,0}_{\pm1}$ in an obvious manner. Let $\hat{D}^{\eta,\varepsilon}_{\pm,c}$ be a dense subspace $\{(h_k) \in \hat{D}^{\eta,\varepsilon}_{\pm};\, h_k \in C_0(R_+) \oplus \tilde{E}_{k,\pm},\ h_k = 0 \text{ for large } |k|\}$, and put

$$\mathscr{D}^{\eta,\varepsilon}_{\pm,c} = (I^{\eta,\varepsilon}_{\pm}\mathscr{F}^{\eta,\varepsilon}_{\pm}J^{\eta,\varepsilon})^{-1}\hat{D}^{\eta,\varepsilon}_{\pm,c}\,.$$

Similarly we define $\hat{D}^{0,0}_{\pm1,c}$ and $\mathscr{D}^{0,0}_{\pm1,c}$.

LEMMA 2.5. *Let $\omega_j$, $j = 1, 2$, be the infinitesimal operator of $U^{\eta,\varepsilon}(0, \omega_j(t))$. Then the restriction $i\omega_j|\mathscr{D}^{\eta,\varepsilon}_{\pm,c}$ is essentially selfadjoint in $\mathscr{D}^{\eta,\varepsilon}_{\pm}$. In case $(\ell, \varepsilon) = (0, 0)$, so is the restriction $i\omega_j|\mathscr{D}^{0,0}_{\pm1,c}$ in $\mathscr{D}^{0,0}_{\pm1}$.*

*Proof.* Only the operator $i\omega_j|\mathscr{D}^{0,0}_{1,c}$, $1 \le j \le 2$, is to be discussed. Denote it by $i\dot{\omega}_j$, and set $\hat{\omega}_j = I^{0,0}_1\mathscr{F}^{0,0}_1J^{0,0}\dot{\omega}_j(I^{0,0}_1\mathscr{F}^{0,0}_1J^{0,0})^{-1}$, $\hat{F}_{\pm} = -\hat{\omega}_1 \mp i\hat{\omega}_2$. First, suppose $k$ is a negative integer, we recall the definition of $e_{k,n}$ given after Lemma 1.7. Evidently $\{\mathscr{F}_k e_{k,n}; n = k, k+1, \cdots, -1\}$ is a basis of $\tilde{E}_k$. On account of (1.33) a closed subspace $\hat{E}_n$, $-n \in Z_+ + 1$, of $\hat{D}_1^{0,0}$ spanned by $\{\mathscr{F}_k e_{k,n}; k = n, n-1, \cdots\}$ is invariant under $\hat{F}_\pm$. Moreover, Corollary 2.3, together with (1.33), implies that $i\hat{\omega}_j$ is essentially selfadjoint in $\hat{E}_n$. As one can see easily, this assertion is valid even for $n \in Z_+ + 1$. It remains, therefore, to show the essentially selfadjointness of $i\hat{\omega}_j$ in $\Sigma_{k\in Z} \oplus L^2(R_+) \subset \hat{D}_1^{0,0}$. To this end, let $C_{0,c}$ be the algebraic sum $\Sigma_{k\in Z} \oplus C_0(R_+)$, and we shall prove that the image $(i\hat{\omega}_j - z)C_{0,c}$, $\operatorname{Im} z \neq 0$, is dense in $\Sigma_{k\in Z} \oplus L^2(R_+)$. If $h = (h_{k'})$ is an element of $C_{0,c}$ such that $h_{k'} = 0$ for $k' \neq k$, then we have by (1.35) the following.

$$i\hat{\omega}_j h(\lambda) = (\cdots, 0, a_{jk}(\lambda)h_k(\lambda), 0, b_{jk}(\lambda)h_k(\lambda), 0, \cdots),$$

where $a_{jk}$ and $b_{jk}$ are smooth functions on $R_+$. We consider an operator $i\hat{\omega}_j(\lambda)$ in $\ell^2 = \Sigma_{k\in Z} \oplus C$ with domain $\ell_c^2 = \{(a_k) \in \ell^2; a_k = 0 \text{ for large } |k|\}$ such that

$$i\hat{\omega}_j(\lambda)e_k = (\cdots, 0, a_{jk}(\lambda), 0, b_{jk}(\lambda), 0, \cdots)$$

for $e_k = (\cdots, 0, 0, 1, 0, 0, \cdots)$. It follows from (1.35) and Corollary 2.3 that $i\hat{\omega}_j(\lambda)$ is essentially selfadjoint. Suppose an $h$ in $\Sigma_{k\in Z} \oplus L^2(R_+)$ is orthogonal to $(i\hat{\omega}_j - z)C_{0,c}$, $\operatorname{Im} z \neq 0$. Then we obtain

$$a_{jk}(\lambda)h_{k-1}(\lambda) - z^* h_k(\lambda) + b_{jk}(\lambda)h_{k+1}(\lambda) = 0 \text{ a.e. on } R_+ .$$

Since $i\hat{\omega}_j(\lambda)$ is essentially selfadjoint in $\ell^2$, $(h_k(\lambda))$ is a zero vector in $\ell^2$ a.e. This means $h = 0$ in $\Sigma_{k\in Z} \oplus L^2(R_+)$. We have shown that $i\hat{\omega}_j$ is essentially selfadjoint in $\Sigma_{k\in Z} \oplus L^2(R_+)$, for it is symmetric. Q.E.D.

We are ready for the proof of Theorems 2.1 and 2.2.

*Proof of Theorem* 2.1. We shall prove the sufficiency first. Set $\mathscr{D}_{k,\pm}^{\eta,\varepsilon} = \mathscr{D}_\pm^{\eta,\varepsilon} \cap \mathscr{W}_k^{\eta,\varepsilon}$, $\mathscr{D}_{k,\pm 1}^{0,0} = \mathscr{D}_{\pm 1}^{0,0} \cap \mathscr{W}_k^{0,0}$. It is evident that $U^{\eta,\varepsilon}(0, \omega_3(t))$ leaves $\mathscr{D}_{k,\pm}^{\eta,\varepsilon}$ (and $\mathscr{D}_{k,\pm 1}^{0,0}$ as well, provided $(\eta, \varepsilon) = (0, 0)$) invariant. By (2.2) and Theorem 1.3 $U^{\eta,\varepsilon}(t, 0, 0, e)$, $t \geq 0$, also leaves $\mathscr{D}_\pm^{\eta,\varepsilon}$ invariant. We note that $P_+(3)$ is topologically generated by the subsemigroup $\{(t, 0, 0, e); t \geq 0\}$ and the subgroup $\{(0, g); g \in G\}$, and that so is $G$ by one-parameter groups $\omega_j(t)$, $j = 2, 3$. To complete the proof of sufficiency, it is enough to show that $U^{\eta,\varepsilon}(0, \omega_2(t))$ keeps $\mathscr{D}_\pm^{\eta,\varepsilon}$ (and $\mathscr{D}_{\pm 1}^{0,0}$ as well, if $(\eta, \varepsilon) = (0, 0)$) invariant. But this fact is an immediate consequence of Lemmas 2.4 and 2.5. Secondly, we shall show the necessity of the condition. Assume that $\mathscr{D}$ is a $P_+(3)$-invariant closed proper subspace of $\mathfrak{H}^{\eta,\varepsilon}$. Since $(t, 0, 0, e) \in P(3)$ commutes with $(0, \omega_3(s)) \in P(3)$, $\mathscr{D}_k^{\eta,\varepsilon} = \mathscr{D} \cap \mathscr{W}_k^{\eta,\varepsilon}$ is invariant under $U^{\eta,\varepsilon}(t, 0, 0, e)$, $t \geq 0$. Moreover, $\mathscr{D}$ being $G$-invariant, we have

$$\Delta \mathscr{D}_k^{\eta,\varepsilon} \subset \mathscr{D}_k^{\eta,\varepsilon}, \qquad F_\pm \mathscr{D}_k^{\eta,\varepsilon} \subset \mathscr{D}_{k\pm 1}^{\eta,\varepsilon}, \qquad k \in Z/2 .$$

Thus $\mathscr{D}$ must coincide with one of $\mathscr{D}_\pm^{\eta,\varepsilon}$ (and $\mathscr{D}_{\pm 1}^{0,0}$, provided $(\eta, \varepsilon) = (0, 0)$) in virtue of (2.2), (2.3) and Theorem 1.4. Q.E.D.

*Proof of Theorem* 2.2. Let $\mathscr{D}_{k,\pm}^{\eta,\varepsilon}$ and $\mathscr{D}_{k,\pm 1}^{0,0}$ be the same as in the above proof. First consider the case $\varepsilon = 1/2$. Then $\mathscr{D}_{k,\pm}^{\eta,\varepsilon} = \{0\}$, $k \in Z$ and

$$\begin{aligned} &\dim(\mathscr{D}_{k,-}^{\eta,\varepsilon} \ominus F_+ \mathscr{D}_{k-1,-}^{\eta,\varepsilon}) = 0 , \qquad k \in Z_+ + \varepsilon , \\ &\dim(\mathscr{D}_{k,-}^{\eta,\varepsilon} \ominus F_- \mathscr{D}_{k+1,-}^{\eta,\varepsilon}) = 0 \quad \text{or} \quad 1 \\ &\qquad \text{according as } -k = 1/2 \text{ or } -k \in Z_+ + 3/2 . \end{aligned} \tag{2.5}$$

These relations imply that among the representations belonging to the discrete series only the representations $T_{(k,\varepsilon)}^+$, $-k \in Z_+ + 3/2$, are contained with multiplicity one in $\mathscr{D}_-^{\eta,\varepsilon}$. Since the following unitary equivalences hold

$$(\Delta - 1/4) | \mathscr{D}_{1/2,-}^{\eta,\varepsilon} \simeq \mathscr{L}_{1/2,\eta} | D_{1/2,-}^{\eta} \simeq \int_{R+}^{\oplus} \lambda \, d\lambda ,$$

the representations $T_{(-1/2+i\eta,\varepsilon)}$, $\eta > 0$, are contained in $\mathscr{D}_-^{\eta,\varepsilon}$ as

$$\int_{R+}^{\oplus} T_{(-1/2+i\eta,\varepsilon)} d\eta .$$

Consequently the representation $(U^{\eta,\varepsilon}, \mathscr{D}_-^{\eta,\varepsilon})$ of $G$ admits a decomposition as stated in Theorem 2.1. We can argue similarly for the representation of $G$ in $\mathscr{D}_+^{\eta,\varepsilon}$. Secondly, assume that $\varepsilon = 0$. We shall confine our discussion to the representation $(U^{0,0}, \mathscr{D}_1^{0,0})$. Since $\mathscr{W}_k^{\eta,\varepsilon} = \{0\}$ for $k \notin Z + \varepsilon$, $\mathscr{D}_{k,1}^{0,0} = \{0\}$, $k \in Z + 1/2$. Moreover, $\dim(\mathscr{D}_{k,1}^{0,0} \ominus F_\pm \mathscr{D}_{k\mp 1,1}^{0,0}) = 1$ for $k \in Z \backslash \{0\}$. This means that among the representations in the discrete series only $T_{(k,0)}^{\mp}$, $-k \in Z_+ + 1$, are contained with multiplicity one in $\mathscr{D}_1^{0,0}$. On account of the following unitary equivalences

$$(\Delta - 1/4) | \mathscr{D}_{0,1}^{0,0} \simeq \mathscr{L}_{0,0} | D_{0,+}^{0} \simeq \int_{R+}^{\oplus} \lambda \, d\lambda .$$

We conclude that the representations $T_{(-1/2+i\eta,0)}$, $\eta \geq 0$, are contained as

$$\int_{R+}^{\oplus} T_{(-1/2+i\eta,0)} \, d\eta .$$

We have verified Theorem 2.2 for the representation in $\mathscr{D}_1^{0,0}$. Q.E.D.

## Appendix

The first lemma is concerned with an $n$-th order equation assuming the following form.

$$z^n w^{(n)} + z^{n-1}c_1(z, \lambda)w^{(n-1)} + \cdots + c_n(z, \lambda)w = 0\,, \tag{A.1}$$

where $c_j$, $1 \le j \le n$, are holomorphic in $\{|z| < \delta_1\} \times \{|\lambda| < \delta_2\}$, $c_j(0, \lambda)$ being constant.

LEMMA A.1. (i) *If the above equation has a solution of the form* $z^\alpha(1 + zh(z, \log z))$, *then* $\alpha$ *is an indicial root, that is,*

$$(\alpha-1)\cdots(\alpha-n+1)+c_1(0,\lambda)(\alpha-1)\cdots(\alpha-n+2)+\cdots+c_n(0,\lambda)=0\,. \tag{A.2}$$

(ii) *Suppose* $\alpha_j$, $1 \le j \le k$, *are roots of* (A.2) *such that* $\alpha_j - \alpha_{j+1}$ *is a positive integer and that there are no other roots in* $Z_+ + \alpha_k$. *Assume further that* $\alpha_j$, $1 \le j < k$, *is a simple root while* $\alpha_k$ *is an* $m_k$*-ple root. Then there exists a system of solutions* $w_j(z, \lambda)$, $1 \le j \le k + m_k - 1$, *such that* $w_j$, *being holomorphic in* $\{0 < |z| < \varepsilon;\ \arg z \neq \pi/2\} \times \{|\lambda| < \delta_2\}$ *for some positive* $\varepsilon$ *depending on* $\delta_2$, *takes the following form.*

$$\begin{aligned} &z^{\alpha_1}(1 + zh(z))\,, && j = 1\,,\\ &z^{\alpha_j}(1 + zh(z, \log z))\,, && 2 \le j \le k\,,\\ &z^{\alpha_k}((\log z)^{j-k} + zh(z, \log z))\,, && k < j < k + m_k\,,\end{aligned}$$

*where* $h(z)$ *and* $h(z, \log z)$ *stand for, respectively, a holomorphic function and a polynomial in* $\log z$ *with holomorphic coefficients.*

*Proof.* To verify (i), it suffices to compare the coefficients of $z^\alpha$ on the both sides of (A.1). The Frobenius method yields (ii) [1, p. 133]. Indeed, put $L = z^n d^n/dz^n + z^{n-1}c_1 d/dz^{n-1} + \cdots + c_n$, and denote by $f(\alpha)$ the polynomial on the left side of (A.2). As is well known, we can find a formal series

$$\phi_j(z, \lambda, \alpha) = z^\alpha \sum_{p=0}^{\infty} d_{jp}(\lambda, \alpha) z^p\,, \qquad d_{j0} = (\alpha - \alpha_j)^{j-1}\,,$$

such that $L\phi_j = f(\alpha)z^\alpha(\alpha - \alpha_j)^{j-1}$. Take $\delta$ so small that there is no roots of $f(\alpha)$ in $\{|\alpha - \alpha_j| < \delta\}$ except for $\alpha_j$. Then it can be shown that $d_{jp}(\lambda, \alpha)$ is homomorphic and $|d_{jp}(\lambda, \alpha)| < K^{2p+1}$, $K > 0$, in $\{|\alpha - \alpha_j| < \delta\} \times \{|\lambda| < \delta_2\}$. Setting $\alpha_j = \alpha_k$ for $j > k$, it suffices to put

$$w_j(z, \lambda) = (\partial/\partial\alpha)^{j-1}_{\alpha=\alpha_j}\phi_j(z, \lambda, \alpha), \qquad 1 \le j < k + m_k .$$

By Osgood's lemma [3] $w_j$ is holomorphic in $\{0 < |z| < 1/K;\ \arg z \neq \pi/2\} \times \{|\lambda| < \delta_2\}$. Q.E.D.

Next consider a differential equation

$$d/dz\, w = A(z, \lambda)w\,, \qquad A(z, \lambda) = \sum_{m=-1}^{\infty} A_m(\lambda)z^m\,, \tag{A.3}$$

where $A(z, \lambda)$ is an $M_n$-valued holomorphic function on $\{0 < |z| < \delta_1\} \times \{|\lambda| < \delta_2\}$, $A_{-1}(0, \lambda)$ being constant.

LEMMA A.2. (i) *If the above equation has a solution of the form* $z^\alpha(p + zh(z, \log z))$, *then* $(A_{-1} - \alpha)p = 0$.

(ii) *Assume that* $\alpha_j$, $1 \le j \le k$, *are characteristic roots of* $A_{-1}$ *such that* $\alpha_j - \alpha_{j+1}$ *is a positive integer and that there are no other characteristic roots in* $Z_+ + \alpha_k$. *Assume further that* $\alpha_j$, $1 \le j < k$, *is a simple root. Then there exists a system of solutions* $w_j(z, \lambda)$, $1 \le j \le k$, *such that* $w_j$, *being holomorphic in* $\{0 < |z| < \varepsilon;\ \arg z \neq \pi/2\} \times \{|\lambda| < \delta_2\}$ *for some positive* $\varepsilon$ *depending on* $\delta_2$, *takes the following form.*

$$z^{\alpha_1}(p_1 + zh(z)) \text{ for } j = 1, \quad z^{\alpha_j}(p_j + zh(z, \log z)) \text{ for } 1 < j \le k\,,$$

*where* $(A_{-1} - \alpha_j)p_j = 0$. *The functions* $h(z)$ *and* $h(z, \log z)$ *stand for the same as in Lemma* A.1.

*Proof.* Compare the coefficients of $z^{\alpha-1}$ on the both sides of (A.3). Then (i) follows. The Frobenius method yields (ii) [1, pp. 136–137]. To be more precise, let $\psi(z, \lambda, \alpha, s_0)$ be a formal series $\sum_{m=0}^{\infty} s_m z^{m+\alpha}$ such that $\psi' - A\psi = (\alpha - A_{-1})s_0 z^{\alpha-1}$, where $\psi'$ denotes the formal series $\sum_{m=0}^{\infty}(\alpha + m)z^{\alpha+m-1}$. Then each component of $s_m$ $(m \ge 1)$, is a rational function of $\alpha$. Let $\delta$ be small enough so that only $\alpha_j$ is a characteristic root of $A_{-1}$ in $\{|\alpha - \alpha_j| < \delta\}$. When $s_0 = p_1$, there exists a positive $K$ such that $|s_m(\lambda, \alpha)| < K^{2m+1}$ in $\{|\lambda| < \delta_2\} \times \{|\alpha - \alpha_j| < \delta\}$. We can set $w_1(z, \lambda) = \psi(z, \lambda, \alpha_1, p_1)$. When $s_0 = (\alpha - \alpha_j)^{j-1}p_j$ $(j > 1)$, $s_m(\lambda, \alpha)$ is holomorphic and $|s_m(\lambda, \alpha)| < K^{2m+1}$ in $\{|\lambda| < \delta_2\} \times \{|\alpha - \alpha_j| < \delta\}$ for some positive $K$ depending on $\delta_2$. In this case, set

$$w_j(z, \lambda) = (\partial/\partial\alpha)^{j-1}_{\alpha=\alpha_j}\psi(z, \lambda, \alpha, s_0)\,, \qquad j > 1\,.$$

The desired analyticity follows from Osgood's lemma [3]. Q.E.D.

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*Department of Mathematics*
*Nagoya University*

*Current address:*
*Department of Mathematics*
*Faculty of Education*
*Tokushima University*