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SMOOTHNESS OF SOLUTIONS OF PARABOLIC EQUATIONS IN REGIONS WITH EDGES

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§ 1. Introduction

We consider the mixed initial-boundary value problem for the parabolic equation

$$(1.1)\qquad Lu = \sum_{i,j=1}^{2} a_{ij}(x, t)u_{x_i x_j} + \sum_{j=1}^{2} a_j(x, t)u_{x_j} + b(x, t)u - u_t = f(x, t)$$

in a region $\Omega \times (0, T]$, where $x = (x_1, x_2)$ and $\Omega \subset \boldsymbol{R}^2$ is a simply-connected bounded domain having corners.

Our main objective will be the study of smoothness properties of solutions of that problem. Early investigations of that type concern *elliptic* equations in domains with a *smooth* boundary, starting with the Dirichlet problem for the Laplace and Poisson equations and proceeding to general second order elliptic equations as well as general boundary conditions; see S. Agmon, A. Douglis and L. Nirenberg [1]. A more recent survey of the elliptic case and further references are given by D. Gilbarg and N. S. Trudinger [6].

Similar work on *parabolic* equations appeared later; we mention in particular investigations by A. Friedman [5] on the first boundary value problem, by Z. Itô [8] and L. I. Kamynin and V. N. Maslennikova [9] on the second boundary value problem and by N. V. Zitarašu [15] on general boundary value problems. Further references are given in [12].

The case of a *nonsmooth boundary* was treated by E. A. Volkov [14], V. A. Kondrat'ev [10] and others whose work is discussed or mentioned in [7]; all these papers concern *elliptic* equations, whereas we shall deal with *parabolic* equations.

We want to mention that those problems in regions with edges and

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corners are also of practical importance in applications, for instance in heat flow (cf. [4]), mechanics of continua and numerical analysis (difference methods, subtraction of singularities, acceleration of convergence; cf. [11], which treats elliptic equations but involves ideas which are also relevant in connection with parabolic equations).

It is known that in the case of a *smooth boundary* $\partial\Omega$ of Ω, the smoothness of solutions increases with that of the coefficients of (1.1) and the boundary data. Indeed, if $\partial\Omega$ is of class $C^{2+\alpha}$, $0<\alpha<1$, the coefficients of (1.1) are of class $C^{\alpha}(\bar{G})$, $G=\Omega\times(0,T]$, and u is a solution of (1.1) in G satisfying

$$u(x,0)=0 \qquad \text{on } \bar{\Omega}$$
$$\beta(x,t)u+\eta(x,t)u_n=0 \qquad \text{on } \partial\Omega\times(0,T]\,,$$

where u_n is the outer normal derivative and $\beta\in C^{2+\alpha}(\partial\Omega\times(0,T])$, $\eta\in C^{1+\alpha}(\partial\Omega\times(0,T])$, then $u\in C^{2+\alpha}(\bar{G})$.

In this paper we show that the increase of the smoothness of solutions with that of the coefficients of (1.1) and the boundary data is no longer true if $\partial\Omega$ has corners, and obtain a smoothness theorem for this case.

§ 2. Main result

Let $\Omega\subset\boldsymbol{R}^2$ be a simply connected bounded domain with boundary $\partial\Omega$. For simplicity and without loss of generality we assume that $\partial\Omega$ has a single corner at the origin with interior angle $\gamma>0$. We assume that the two arcs Γ_1 and Γ_2 forming the corner have the representation

$$\Gamma_1\colon\ x_1=g_2(x_2)$$
$$\Gamma_2\colon\ x_2=g_1(x_1)$$

with g_1 and g_2 of class $C^{2+\alpha}$ and $g_1(0)=0$, $g_2(0)=0$, $g_1'(0)=0$, and $g_2'(0)=\cot\gamma$.

We consider the problem [cf. (1.1)]

$$\text{(2.1)}\qquad Lu=\sum_{i,j=1}^{2}a_{ij}(x,t)u_{x_ix_j}+\sum_{j=1}^{2}a_j(x,t)u_{x_j}+b(x,t)u-u_t=f(x,t)\qquad \text{in } G\,,$$

$$\text{(2.2a)}\qquad u(x,0)=0\qquad \text{on } \bar{\Omega}\,,$$

$$\text{(2.2b)}\qquad \beta(x,t)u+\eta(x,t)u_n=0\qquad \text{on } S=(\partial\Omega\backslash\{0\})\times(0,T]\,,$$

where $\beta\in C^{2+\alpha}(S)$, $\eta\in C^{1+\alpha}(S)$, and

$$\beta = 1,\ \eta = 0 \text{ on } \Gamma_1, \qquad \beta = 0,\ \eta = 1 \text{ on } \Gamma_2 .$$

If a_{ij}, a_j, b, $f \in C^\alpha(\bar{G})$, then for any bounded solution of (2.1)–(2.2) we have

$$u \in C^{2+\alpha}(G_1) \cap C^0(\bar{G}) \tag{2.3}$$

where $G_1 = \Omega_1 \times (0, T]$ and Ω_1 is any compact subregion of $\bar{\Omega}$ having positive distance from 0.

To investigate the smoothness of solutions near the edge, we take any fixed $t_0 \in [0, T]$ and transform the equation

$$\sum_{i,j=1}^{2} a_{ij}(0, t_0) u_{x_i x_j} = 0 \tag{2.4}$$

to canonical form. Then γ at $(0, t_0)$ is transformed to

$$\omega(t_0) = \arctan \frac{[a_{11}(0, t_0)a_{22}(0, t_0) - a_{12}^2(0, t_0)]^{1/2}}{a_{22}(0, t_0) \cot \gamma - a_{12}(0, t_0)} .$$

Clearly, the value $\omega(t_0)$ does not depend on the particular choice of that transformation.

We now state our main result. From (2.3) it follows that it suffices to consider the smoothness of solutions near the edge.

THEOREM 1. *Let u be a bounded solution of* (2.1), (2.2) *in* $\mathscr{G}_c = \Omega_c^* \times (0, T]$, *where*

$$\Omega_c^* = \{x | x \in \Omega,\ 0 < |x| \leqq c\} .$$

Assume a_{ij}, a_j, b, $f \in C^\alpha(\mathscr{G}_c)$ in (2.1), *where $0 < \alpha < 1$, and, furthermore, $\omega(t) < \frac{1}{2}\pi$ for every $t \in [0, T]$. Then u, as a function of x, satisfies*

$$u \in C^\nu(\mathscr{G}_c) , \tag{2.5}$$

for some $\nu \in (1, 2]$.

The proof of this theorem will result from Theorem 2 (below) and will be given at the end.

§ 3. The case of a cylindrical sector

Let $t_0 \in [0, T]$ be fixed and define

$$G_\sigma = \Omega_\sigma \times I_\sigma ,$$

where

$$\Omega_\sigma = \{(r, \theta)|, 0 < r \leqq \sigma,\ \tfrac{1}{2}\pi - \omega < \theta < \tfrac{1}{2}\pi\}$$

with $\omega = \omega(t_0) < \frac{1}{2}\pi$ and r, θ given by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and

$$I_\sigma = \{t|t \in (0, T],\ |t - t_0| \leqq \sigma\}\ .$$

Furthermore, let

$$\Gamma_1 = \{(r, \theta)|r \leqq \sigma,\ \theta = \tfrac{1}{2}\pi - \omega\}\ ,$$
$$\Gamma_2 = \{(r, \theta)|r \leqq \sigma,\ \theta = \tfrac{1}{2}\pi\}\ .$$

As the mixed initial-boundary conditions we take

(3.1a) $$u(x, 0) = 0 \qquad \text{on } \Omega_\sigma\ ,$$

(3.1b) $$u|\Gamma_1 = u_n|\Gamma_2 = 0, \qquad 0 < t \leqq T\ .$$

Then we have the following result.

THEOREM 2. *Let u be a bounded solution of the problem* (2.1), (3.1) *in G_σ with $\omega < \frac{1}{2}\pi$. Suppose that a_{ij}, a_j, b, $f \in C^\alpha(\overline{G}_\sigma)$, $0 < \alpha < 1$, and $a_{ij}(0, t_0) = \delta_{ij}$. Then u, as a function of x, satisfies*

(3.2) $$u \in C^\nu(\overline{G}_a)\ ,$$

where $a < \sigma, \nu = \min(2, \pi/2\omega - \varepsilon)$ and $\varepsilon > 0$ is arbitrarily small.

We shall obtain a proof of this theorem by first proving two lemmas in the next two sections.

§ 4. Estimation of solutions

LEMMA 1. *Under the assumptions of Theorem* 2 *there exists a positive $a < \sigma$ such that in $\overline{G}_a$ (cf. Sec.* 3)

(4.1) $$|u(x, t)| \leqq Mr^\nu\ ,$$

where $\nu = \min(2, \pi/2\omega - \varepsilon)$ and $\varepsilon > 0$ is arbitrarily small.

Proof. In G_σ we consider the function

$$v(x) = -Mr^\nu \cos\lambda(\tfrac{1}{2}\pi - \theta)\ ,$$

$$\nu = \min\left(2, \frac{\pi}{2\omega} - \varepsilon\right) < \frac{\pi - 2\delta}{2\omega} = \lambda, \qquad 0 < \delta < \tfrac{1}{2}\pi\ .$$

We shall prove that $v(x)$ serves as a barrier for the solution u of (2.1), (3.1). We have

$$Lv = M[(\lambda^2 - \nu^2) \cos \lambda(\tfrac{1}{2}\pi - \theta) - h(x, t)]r^{\nu-2} + Mh_1(x, t)r^{\nu-1} + Mh_2(x, t)r^{\nu} ,$$

where $h(x, t)$ is continuous, $h(0, t_0) = 0$, and h_1 and h_2 are bounded. Thus for every $\varepsilon_1 > 0$ we can find an $a < \sigma$ such that in G_a,

$$Lv \geqq M[(\lambda^2 - \nu^2) \cos \lambda(\tfrac{1}{2}\pi - \theta) - \varepsilon_1]r^{\nu-2} .$$

Since for $\frac{1}{2}\pi - \omega \leqq \theta \leqq \frac{1}{2}\pi$ we have

$$\sin \delta \leqq \cos \lambda(\tfrac{1}{2}\pi - \theta) \leqq 1 ,$$

in G_a we thus obtain

$$Lv \geqq M[(\lambda^2 - \nu^2) \sin \delta - \varepsilon_1]r^{\nu-2}$$

with arbitrarily small $\varepsilon_1 > 0$ and sufficiently small $a < \sigma$. We now take $\varepsilon_1 < (\lambda^2 - \nu^2) \sin \delta$ and use the fact that $f(x, t)$ is bounded in G_σ, say, $|f(x, t)| \leqq K$. Consequently, by taking M sufficiently large (if $\nu = 2$) and a sufficiently small (if $\nu < 2$), we obtain in G_a

$$Lv \geqq K \geqq f(x, t) .$$

We now want to apply in that region the maximum principle to $w = u - v$. We have $Lw \leqq 0$ in G_a. Furthermore, $w_n = 0$ on $\Gamma_2 \times I_a$ and $w \geqq 0$ on $\Gamma_1 \times I_a$. Since u is bounded in G_σ, by taking M sufficiently large we can make $w \geqq 0$ on

$$\{(r, \theta) | r = a, \tfrac{1}{2}\pi - \omega < \theta < \tfrac{1}{2}\pi\} \times I_a .$$

Hence, by the maximum principle (cf. [13]) we have $w \geqq 0$ in $\overline{G}_a$ with sufficiently small a; that is,

$$u \geqq -Mr^{\nu} \cos \lambda(\tfrac{1}{2}\pi - \theta) \geqq -Mr^{\nu} .$$

The other part of (4.1) can be obtained similarly. This proves Lemma 1.

§ 5. Proof of Theorem 2

To prove Theorem 2, we need another lemma, as follows.

LEMMA 2. *Let u be a bounded solution of* (2.1), (2.2) *in* G_σ. *Suppose that* a_{ij}, a_j, b, $f \in C^\alpha(\overline{G}_\sigma)$. *Then if for some* μ, $0 < \mu \leqq 2$, *and* $a < \sigma$ *we have*

$$|u(x, t)| \leqq Mr^{\mu} \qquad \text{in } \overline{G}_a ,$$

it follows that

(5.1) $$\left|\frac{\partial^k u(x,t)}{\partial x_1^{k_1}\partial x_2^{k-k_1}}\right| \leqq M_k r^{\mu-k} \qquad in\ \bar{G}_a$$

where $k_1 = 0$, 1 *or* 2, $k_1 \leqq k$, $k = 1, 2$.

Proof. It suffices to indicate the basic idea since similar proofs were used in [2] and [3]. In Ω_a we define

$$D_s = \{(r,\theta) | 2^{-s-2}a \leqq r \leqq 2^{-s-1}a,\ \tfrac{1}{2}\pi - \omega < \theta < \tfrac{1}{2}\pi\}\,,$$
$$D'_s = D_{s-1} \cup D_s \cup D_{s+1}, \qquad s = 0, 1, \cdots,$$

and set

$$R_s = D_s \times I_a \qquad R'_s = D'_s \times I_a\,.$$

The transformation

$$x_i = 2^{-s}y_i, \qquad i = 1, 2\,,$$

transforms (2.1), (2.2) into

$$\sum \tilde{a}_{ij}\tilde{u}_{y_i y_j} + 2^{-s}\sum \tilde{a}_j \tilde{u}_{y_j} + 2^{-2s}\tilde{b}\tilde{u} - 2^{-2s}\tilde{u}_t = 2^{-2s}\tilde{f}, \qquad \tilde{\beta}\tilde{u} + \tilde{\eta}\tilde{u}_n = 0\,.$$

Furthermore, it maps R_s onto R_0 and R'_s onto R'_0. In R_0 and R'_0 we apply Schauder estimates for the solution of the transformed problem to obtain

$$\|\tilde{u}\|_{2+\alpha}^{R_0} \leqq C[\|\tilde{u}\|_0^{R'_0} + 2^{-2s}\|\tilde{f}\|_\alpha^{R'_0}]\,.$$

This yields in R_s

$$\left|\frac{\partial^k u(x,t)}{\partial x_1^{k_1}\partial x_2^{k-k_1}}\right| \leqq c_0 2^{-(\mu-k)s}\,,$$

which entails the assertion of Lemma 2.

From this lemma (with $\mu = \nu$) we shall now obtain Theorem 2.

Proof of Theorem 2. It is sufficient to show that

(5.2) $$\frac{|u_x(P) - u_x(Q)|}{d(P,Q)^{\nu-1}} \leqq H$$

for any two points P: (r_1, θ_1, t_1) and Q: (r_2, θ_2, t_2) in $\bar{G}_a$. Here

$$d(P,Q)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + |t_1 - t_2|\,,$$

as usual. Without restriction, let $0 \leqq r_2 \leqq r_1$. If $r_2 \leqq r_1/2$ or $|t_1 - t_2|^{1/2} \geqq r_1/2$, then $d(P,Q) \geqq r_1/2$ and (5.2) can be obtained using the bound (5.1) of u_x. Now let $r_2 > r_1/2$ and consider the region

$$S_P = \{(x, t) | (x, t) \in G_a,\ \tfrac{1}{2} r_1 \leqq r \leqq r_1,\ |t - t_1| \leqq \tfrac{1}{4} r_1^2\} .$$

The transformation

$$x_i = 2r_1 z_i / a, \qquad i = 1, 2 ,$$

maps S_P onto

$$G'_P = \{(z, t) | \tfrac{1}{4} a \leqq \rho \leqq \tfrac{1}{2} a,\ |t - t_1| \leqq \tfrac{1}{4} r_1^2\} ,$$

where $\rho^2 = z_1^2 + z_2^2$. It transforms equation (2.1) to

$$\sum a_{ij}^* u_{z_i z_j}^* + \frac{2r_1}{a} \sum a_j^* u_j^* + \left(\frac{2r_1}{a}\right)^2 b^* u^* - \left(\frac{2r_1}{a}\right)^2 u_t^* = \left(\frac{2r_1}{a}\right)^2 f^*$$

In G'_P and

$$G''_P = \{(z, t) | \tfrac{1}{8} a \leqq \rho \leqq a, |t - t_1| \leqq \tfrac{1}{4} r_1^2\}$$

we apply a Schauder estimate to get

$$\| u^* \|_{2+\alpha}^{G'_P} \leqq C\left[\| u^* \|_0^{G''_P} + \left(\frac{2r_1}{a}\right)^2 \| f^* \|_\alpha^{G''_P}\right] .$$

As in the proof of Lemma 2 we obtain

(5.3) $$\| u^* \|_{2+\alpha}^{G'_P} \leqq M_2 r_1^\nu .$$

Now

$$H_{\nu-1}^{G_P}(u_x) = \left(\frac{2r_1}{a}\right)^\nu H_{\nu-1}^{G'_P}(u_z^*) ,$$

where $H_{\nu-1}^{G_P}(u_x)$ is the Hölder coefficient of the Hölder condition for u_x in G_P (with exponent $\nu - 1$). From (5.3) we thus obtain

$$H_{\nu-1}^{G_P}(u_x) \leqq M_3 .$$

This proves Theorem 2.

§ 6. Proof of Theorem 1. Concluding remarks

Consider any point $(0, t_0)$, $t_0 \in [0, T]$. We straighten the boundary around $(0, t_0)$ by the transformation

(6.1) $$y_1 = x_1 - g_2(x_2), \qquad y_2 = x_2 - g_1(x_1) .$$

Equation (2.1) is then transformed to a parabolic equation with principal part

$$\sum_{i,j=1}^{2} A_{ij}(y, t)U_{y_i y_j} .$$

Clearly, (6.1) is of class $C^{2+\alpha}$ and the value of the Jacobian at $(0, t_0)$ is 1. By another linear transformation we cast

$$\sum_{i,j=1}^{2} A_{ij}(0, t_0)U_{y_i y_j} = 0$$

to canonical form. These two transformations map Ω_c^* in Theorem 1 onto a sector Ω_σ of angle $\omega(t_0)$. In $G_\sigma = \Omega_\sigma \times (0, T]$ with a suitable $\sigma > 0$ the transformed functions satisfy all the conditions of Theorem 2. Hence the conclusion of Theorem 2 applies to the transformed solution. Since the composite of those transformations is of class $C^{2+\alpha}$ and locally injective, Theorem 1 follows.

Remark 1. In [3] we were concerned with the initial Dirichlet problem for a parabolic equation with coefficients a_{ij} independent of t. Our present results now permit us to extend our considerations in [3] for the case $n = 2$ as follows.

THEOREM 3. *Let u be a bounded solution of the initial-Dirichlet problem*

$$\begin{aligned} Lu &= f && \text{in } G_c \\ u(x, 0) &= 0 && \text{on } \bar{\Omega} \\ u &= 0 && \text{on } S \end{aligned}$$

with L, Ω and S as in Sec. 2. Suppose that $a_{ij}, a_j, b, f \in C^\alpha(\bar{G}_c)$ and $\omega(t) < \pi$ for every $t \in [0, T]$. Then

$$u \in C^\nu(\bar{G}_c) \tag{6.2}$$

for some $\nu \in (1, 2]$.

To prove this, find a bound for u, using the idea of the proof of the above Lemma 1 and then continue as in [3].

Remark 2. If u and the coefficients of the equation in Theorem 2 are extended by symmetry across $\theta = \frac{1}{2}\pi$, then in $\bar{\tilde{G}}_c \backslash J$ with $\tilde{G}_c = \tilde{\Omega}_c \times I_c$ and

$$\begin{aligned} \tilde{\Omega}_c &= \{(r, \theta) | 0 < r \leqq c, \tfrac{1}{2}\pi - \omega < \theta < \tfrac{1}{2}\pi + \omega\} , \\ J &= \{(\theta, t) | \theta = \tfrac{1}{2}\pi, \ t \in I_c\} \end{aligned}$$

the extended function satisfies an initial *Dirichlet* problem (instead of a *mixed* problem), and we may still apply the maximum principle for generalized solutions to obtain a bound for the solution. However, in Lemma 2 it is required that we have a bound for the solution *everywhere*. Hence the approach just mentioned would not be of help in the present case.

Remark 3. Combining Theorems 1 and 3 we obtain

THEOREM 4. *Let $\Omega \subset \boldsymbol{R}^2$ be a bounded domain whose boundary $\partial\Omega$ is a simple polygon, with sides $\Gamma_1 \cdots, \Gamma_p$ of class $C^{2+\alpha}$ and vertices $P_j = \Gamma_j \cap \Gamma_{j+1}$, $j = 1, \cdots, p$, $(\Gamma_{p+1} = \Gamma_1)$ with angles γ_j such that for the corresponding $\omega_j = \omega(P_j, t)$ we have*

$$\omega_j = \tfrac{1}{2}\xi_j(t)(\kappa_j + \kappa_{j+1}), \qquad 0 < \xi_j < \pi,$$
$$\kappa_j = 0 \text{ or } 1, \quad \kappa_j + \kappa_{j+1} \neq 0, \quad j = 1, \cdots, p \ (\kappa_{p+1} = \kappa_1).$$

Let u be a bounded solution of the problem

$$Lu = f \quad \text{in } \Omega \times I, \quad I = (0, T] ,$$
$$u(x, 0) = 0 \qquad \text{in } \bar{\Omega} ,$$
$$\kappa_j u + (1 - \kappa_j)u_n = 0 \qquad \text{on } \Gamma_j \times I ,$$

with L as in (2.1) *and a_{ij}, a_j, b, $f \in C^\alpha(\bar{\Omega} \times \bar{I})$, $0 < \alpha < 1$. Then*

$$u \in C^{2+\alpha}(\bar{\Omega}_1 \times \bar{I}) , \tag{6.3}$$

where $\bar{\Omega}_1$ is any compact subregion of $\bar{\Omega}$ with positive distance from the corners, and

$$u \in C^{\nu_j}([\bar{\Omega} \cap N_j] \times \bar{I}) \qquad \nu_j = \min_{t \in I} (2, \pi/\xi_j(t) - \varepsilon) , \tag{6.4}$$

where $\varepsilon > 0$ is arbitrarily small and N_j is a sufficiently small closed disk centered at P_j.

The proof is obvious; indeed, for $\kappa_j + \kappa_{j+1} = 1$ (mixed data) it follows from Theorem 1 and for $\kappa_j + \kappa_{j+1} = 2$ (Dirichlet data) from Theorem 3.

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