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THE INDEX OF ELLIPTIC OPERATORS OVER V-MANIFOLDS

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Introduction

Let M be a compact smooth manifold and let G be a finite group acting smoothly on M. Let E and F be smooth G-equivariant complex vector bundles over M and let $P: \mathscr{C}^\infty(M; E) \to \mathscr{C}^\infty(M; F)$ be a G-invariant elliptic pseudo-differential operator. Then the kernel and the cokernel of the operator P are finite-dimensional representations of G. The difference of the characters of these representations is an element of the representation ring $R(G)$ of G and is called the G-index of the operator P.

$$(1)\qquad \operatorname{ind} P = \operatorname{char}[\operatorname{kernel} P] - \operatorname{char}[\operatorname{cokernel} P]\,.$$

It is well-known that the G-index $\operatorname{ind} P \in R(G)$ depends only on the homotopy class of the elliptic operator and, as Atiyah and Singer showed in [2], $\operatorname{ind} P$ is determined by the stable equivalence class $[\sigma(P)] \in K_G(\tau M)$ of the principal symbol $\sigma(P)$ viewed as the difference bundle over the tangent bundle τM. The Atiyah-Singer index theorem asserts that the value $(\operatorname{ind} P)(g)$ is expressed by the evaluation of a certain characteristic class over the tangent bundle $\tau(M^g)$ of the fixed point set M^g.

$$(2)\qquad (\operatorname{ind} P)(g) = (-1)^{\dim M^g} \langle \operatorname{ch}^g [\sigma(P)] \mathscr{I}^g(M),\ [\tau(M^g)]\rangle\,.$$

Here $\operatorname{ch}^g [\sigma(P)]$ is a class in the compactly supported cohomology group $H_c^*(\tau(M^g); C)$ expressed in the characteristic classes of the complex eigenvector bundles by the action of g on the stable vector bundle $[\sigma(P)|_{\tau(M^g)}]$. $\mathscr{I}^g(M)$ is a class in $H^*(M^g; C)$ expressed in the characteristic classes of the real and complex eigenvector bundles by the action of g on the real vector bundle $\tau M|_{M^g}$. We call these classes over the fixed point set as the residual characteristic classes.

Next we consider the index of the operator $P^G: \mathscr{C}^\infty(M; E)^G \to \mathscr{C}^\infty(M; F)^G$

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between G-invariant sections. By the orthonormality of irreducible characters, we have:

$$\begin{aligned} \operatorname{ind} P^G &= \dim [\text{kernel } P^G] - \dim [\text{cokernel } P^G] \\ &= \frac{1}{|G|} \sum_{g \in G} (\operatorname{ind} P)(g) \qquad (3) \\ &= \frac{1}{|G|} \sum_{g \in G} (-1)^{\dim M^g} \langle \mathrm{ch}^g [\sigma(P)] \mathscr{I}^g(M),\ [\tau(M^g)] \rangle . \end{aligned}$$

The operator P^G can be viewed as an operator over the orbit space $G \backslash M$ in the following sense. The invariant section $s\colon M \to E$ is determined uniquely by the induced section $\bar{s}\colon G \backslash M — G \backslash E$ over the orbit space. So we may consider the invariant sections $\mathscr{C}^\infty(M; E)^G$ as the sections over the orbit space $X = G \backslash M$. The operator P^G operates on these sections and its index $\operatorname{ind} P^G$ depends only on the G-equivariant homotopy class of the principal symbol $[\sigma(P)]$, which is considered to be a section over the orbit space $G \backslash \tau M$. Thus we consider P^G as an operator over $X = G \backslash M$.

We remark that the evaluation in (3) admits a purely local expression over X. Choose G-invariant metrics and connections on manifolds M and M^g, on bundles τM, $\tau(M^g)$ and $\nu(M^g)$ (the normal bundle of M^g in M) and on a stable bundle $\sigma(P)$. Then the evaluations of residual characteristic classes are given by the integrations of the corresponding characteristic forms. For each $x \in M$, we choose a small neighbourhood U_x so that the isotropy subgroup G_x acts on U_x and, for $g \in G$, $U_x \cap gU_x \neq \emptyset$ implies $g \in G_x$. Then the orbit space $G_x \backslash U_x$ is naturally identified with an open subset in X. A family $\{G_x \backslash U_x\}_{x \in M}$ defines an open covering of X. Choose a partition of unity $1 = \sum \phi_x$ subordinate to this coverinig. Then we can rewrite (3) in the following form

$$\operatorname{ind} P^G = \sum_{x \in M} \frac{1}{|G_x|} \sum_{g \in G_x} (-1)^{\dim U_x^g} \int_{\tau(U_x^g)} \phi_x \, \mathrm{ch}^g [\sigma(P)|_{U_x}] \mathscr{I}^g(U_x) . \qquad (4)$$

The orbit space $G \backslash M$ is a typical example of V-manifold, and the above formula (4) can be given an interpretation which still makes sense for general V-manifolds.

The purpose of the present paper is to give an index theorem for elliptic operators over V-manifolds which generalize the formula (4).

Let X be a compact *V-manifold.* (For the precise definitions of V-manifolds and V-bundles, see Kawasaki [6]). For each $x \in X$, there is a neighbourhood U_x and an identification $U_x = G_x \backslash \tilde{U}_x$, where $\tilde{U}_x$ is a neighbourhood of the origin in an effective real representation space of a finite group G_x. For each $y \in U_x$, choose small U_y so that $U_y \subset U_x$, then there is an open embedding $\phi: \tilde{U}_y \to \tilde{U}_x$ that covers the inclusion $U_y \subset U_x$. The choice of such ϕ is unique up to the action of G_x on $\tilde{U}_x$. Each ϕ determines an injective group homomorphism $\lambda_\phi: G_y \to G_x$ that makes ϕ be λ_ϕ-equivariant.

To express our theorem in cohomological terms, we have to assign to each V-manifold X a certain global geometric object over which the residual characteristic classes should be evaluated. If we look at (4), such an object must be a collection of all $\tilde{U}_x^g$'s. Each $\tilde{U}_x^g$ admits the action of the centralizer $Z_{G_x}(g)$ of g in G_x. If g and g' are conjugate in G_x, then U_x^g and $U_x^{g'}$ are diffeomorphic by the action of some element h in G_x $(g' = hgh^{-1})$. So we consider one element g for each conjugacy class (g) in G_x. For each point $x \in X$, let $(1), (h_x^1), \cdots, (h_x^{\rho_x})$ be all the conjugacy classes in G_x. Then we have a natural bijection

$$\begin{aligned}&\{(y, (h_y^j)) \mid y \in U_x,\ j = 1, 2, \cdots, \rho_y\}\\ &\quad \cong \coprod_{i=1}^{\rho_x} Z_{G_x}(h_x^i) \backslash \tilde{U}_x^{h_x^i} .\end{aligned}$$

So we define globally:

$$\Sigma X = \{(x, (h_x^i)) \mid x \in X,\ G_x \neq \{1\},\ i = 1, 2, \cdots, \rho_x\} .$$

Then ΣX has a natural V-manifold structure whose local coordinate coverings are $\tilde{U}_x^h \to Z_{G_x}(h) \backslash \tilde{U}_x^h$ $(h \neq 1)$. The action of $Z_{G_x}(h)$ on $\tilde{U}_x^h$ is not effective. The order of the trivially acting subgroup is called the *multiplicity* of ΣX in X at $(x, (h))$. In general, ΣX has many connected components of varying dimensions. Let $\Sigma_1, \Sigma_2, \cdots, \Sigma_c$ be the connected components of ΣX. Since the multiplicity is locally constant on ΣX, we may assign the multiplicity m_i to each connected component Σ_i.

On each local coordinate $\tilde{U}_x^h$ over ΣX, we have the normal bundle $\nu(\tilde{U}_x^h)$ in $\tilde{U}_x$ and the tangent bundle $\tau(\tilde{U}_x^h)$. On the normal bundle $\nu(\tilde{U}_x^h)$, we have the action of h. Then we have the eigenspace decomposition of $\nu(\tilde{U}_x^h)$

$$\nu(\tilde{U}_x^h) = \bigoplus_{0<\theta\leq\pi} \nu_h^\theta ,$$

$$\begin{cases} hv = e^{i\theta} v & \text{if } v \in \nu_h^\theta \ (0 < \theta < \pi) , \\ hv = -v & \text{if } v \in \nu_h^\pi . \end{cases}$$

The collection of these $Z_{G_x}(h)$-equivariant bundles ν_h^θ $(0 < \theta \leqq \pi)$ and $\tau(\tilde{U}_x^h)$ form a real or complex vector V-bundles over ΣX. By choosing invariant connections, we have a collection of residual characteristic forms

$$\mathscr{I}^h(\tilde{U}_x) \in \Omega^*(\tilde{U}_x^h) \otimes_{\boldsymbol{R}} \boldsymbol{C} .$$

These forms define characteristic classes

$$\mathscr{I}^\Sigma(X) \in H^*(\Sigma X; \boldsymbol{C}) , \quad \text{and} \quad \mathscr{I}(X) \in H^*(X; \boldsymbol{Q}) \quad (h = 1) .$$

By a *V-bundle* E over a V-manifold X, we mean a family $\{(G_x^E, \tilde{E}_x \to \tilde{U}_x)\}$ of equivariant fibre bundles with surjective homomorphisms $G_x^E \to G_x$ and their attaching bundle maps $\{\Phi\}: \tilde{E}_y \to \tilde{E}_x$ for each inclusive pair $U_y \subset U_x$. We call V-bundle E to be *proper* if, for each $x \in X$, $G_x^E = G_x$. The attaching bundle maps $\{\Phi\}$ define a unique induced open embedding $\bar{\Phi}: G_x^E\backslash\tilde{E}_y \to G_x^E\backslash\tilde{E}_x$ of the orbit spaces of total spaces. These induced maps define the total space $E = \bigcup (G_x^E\backslash\tilde{E}_x)$ and the projection $E \to X$. E itself admit a structure of V-manifold.

Let $E \to X$ be a proper V-bundle. A section $s: X \to E$ is called a C^∞ *V-section* if, for each U_x, $s \mid U_x: U_x \to E_x = G_x\backslash\tilde{E}_x$ is covered by a G_x-invariant C^∞ section $\tilde{s}_x: \tilde{U}_x \to \tilde{E}_x$. For a vector V-bundle E, we denote the set of all C^∞ V-sections by $\mathscr{C}_V^\infty(X; E)$, which forms a vector space. On a vector V-bundle E, we can always construct a invariant linear connection, that is, a family of invariant connections on $(G_x^E, \tilde{E}_x \to \tilde{U}_x)$ which are compatible with attaching bundle maps. Then the characteristic forms define a C^∞ V-section of the exterior power of the cotangent vector V-bundle, which represent a cohomology class on X.

Let E and F be proper complex vector V-bundles over X. A linear map $P: \mathscr{C}_V^\infty(X; E) \to \mathscr{C}_V^\infty(X; F)$ is called a *(pseudo-) differential operator* if locally it is covered by invariant (pseudo-) differential operators

$$\tilde{P}_x: \mathscr{C}_c^\infty(\tilde{U}_x; \tilde{E}_x) \text{——} \mathscr{C}^\infty(\tilde{U}_x; \tilde{F}_x)$$

(modulo smoothing operators), which are compatible with attaching maps. We call P to be *elliptic* if each $\tilde{P}_x$ is elliptic. For an elliptic pseudo-differential operator $P: \mathscr{C}_V^\infty(X; E) \to \mathscr{C}_V^\infty(X; F)$, we have the *V-index* defined by:

$$(5) \qquad \operatorname{ind}_V P = \dim [\text{kernel } P] - \dim [\text{cokernel } P] .$$

This index generalize $\operatorname{ind} P^G$ in (3) and (4).

Like G-equivariant case, the V-index depends only on the homotopy class of elliptic operators. The principal symbol $\sigma(P)$ of the operator P is a well-defined C^∞ V-section of the V-bundle $\mathrm{Hom}(E, F)$ over the total space $\tau_V^* X$ of the cotangent vector V-bundle. For P elliptic, the principal symbol $\sigma(P)$ defines a compactly supported difference V-bundle and the index $\mathrm{ind}_V P$ is determined by its stable equivalence class $[\sigma(P)]$. The stable equivalence classes of compactly supported proper difference vector V-bundles over $\tau_V^* X \cong \tau_V X$ form a group $K_V(\tau_V^* X) \cong K_V(\tau_V X)$. ($\tau_V X$ denotes the total space of the tangent vector V-bundle). Then V-index defines a homomorphism

$$\text{(6)} \qquad \mathrm{ind}_V : K_V(\tau_V X) \longrightarrow \boldsymbol{Z} .$$

An element $u \in K_V(\tau_V X)$ is represented by proper complex vector V-bundles E and F over $\tau_V X$ and an isomorphism $\sigma: E \to F$ over $\tau_V X - X$. Then, choosing a suitable invariant connections, we have the residual Chern characters

$$\mathrm{ch}^h(E) - \mathrm{ch}^h(F) \in \Omega^*(\tau(\tilde{U}_x^h)) \otimes_{\boldsymbol{R}} \boldsymbol{C} ,$$

and globally we have the classes

$$\mathrm{ch}^\Sigma(u) \in H_c^*(\tau_V(\Sigma X); \boldsymbol{C}) \quad \text{and} \quad \mathrm{ch}(u) \in H_c^*(\tau_V X; \boldsymbol{Q}) \quad (h = 1) .$$

In this framework, we can state our theorem

THEOREM. *Let X be a compact V-manifold. Then, for $u \in K_V(\tau_V X)$, we have:*

$$\text{(7)} \qquad \begin{aligned} \mathrm{ind}_V(u) = {} & (-1)^{\dim X} \langle \mathrm{ch}(u) \mathscr{I}(X), [\tau_V X] \rangle \\ & + \sum_{i=1}^{c} \frac{(-1)^{\dim \Sigma_i}}{m_i} \langle \mathrm{ch}^\Sigma(u) \mathscr{I}^\Sigma(X), [\tau_V \Sigma_i] \rangle . \end{aligned}$$

As a special case of this theorem, we get the following results:

I) (Kawasaki [6]) Let X be a compact oriented V-manifold of dimension $4k$. As a topological space, X is an oriented rational homology manifold. The signature $\mathrm{Sign}(X)$ of X is defined by the signature of the non-degenerate symmetric bilinear form on the middle dimensional cohomology group $H^{2k}(X; \boldsymbol{Q})$ given by the cup product. Using de Rham cohomology, we can represent $\mathrm{Sign}(X)$ as the V-index of the signature operator $D_+ : \Omega_V^+(X) \to \Omega_V^-(X)$ over V-manifold X. Then we have:

$$\mathrm{Sign}\,(X) = \langle L(X),\ [X]\rangle + \sum_{i=1}^{c} \frac{1}{m_i} \langle L^{\Sigma}(X),\ [\Sigma_i]\rangle\ .$$

The classes $L(X)$ and $L^{\Sigma}(X)$ are defined locally by the residual L-class $L^h(\tilde{U}_x)$, as we have defined $\mathscr{I}(X)$ and $\mathscr{I}^{\Sigma}(X)$.

II) (Kawasaki [7]) Let X be a compact complex V-manifold and let $E \to X$ be a holomorphic vector V-bundle. Then X admits a natural structure of an analytic space and the local holomorphic V-sections of E define a coherent analytic sheaf $\mathcal{O}_V(E)$ over X. The arithmetic genus $\chi(X; E)$ is defined by:

$$\chi(X; E) = \sum_{i=1}^{\dim X} (-1)^i \dim_C H^i(X; \mathcal{O}_V(E))\ .$$

Then $\chi(X; E)$ is represented by the V-index of the Dolbeault complex over the V-manifold X with coefficients in E. We can apply our theorem and we have:

$$\chi(X; E) = \langle \mathscr{T}(X; E),\ [X]\rangle + \sum_{i=1}^{c} \frac{1}{m_i} \langle \mathscr{T}^{\Sigma}(X; E),\ [\Sigma_i]\rangle\ .$$

The classes $\mathscr{T}(X; E)$ and $\mathscr{T}^{\Sigma}(X; E)$ are defined locally by the residual Todd class with coefficients in E.

The proof that we adopt here is completely different from those in the above two reports [6] and [7]. As we have remarked in [6], every V-manifold X is presented as the orbit space of a smooth G-manifold $\tilde{X}$ with only finite isotropy subgroups and with the trivial principal orbit type. We may choose such $(G, \tilde{X})$ with G compact and connected. Let P be an elliptic operator over X. Then we can lift the principal symbol $\sigma(P)$ considered as a difference V-bundle over $\tau_V X$ to a G-equivariant difference bundle over $\tau_G \tilde{X}$, the space of tangent vectors orthogonal to the orbits of G. The lifted symbol determines up to homotopy a transversally elliptic operator $\tilde{P}$ over $\tilde{X}$ relative to G. Then the V-index $\mathrm{ind}_V P$ is equal to the evaluation $(\mathrm{ind}^G \tilde{P})(1_G)$ of the distributional index $\mathrm{ind}^G \tilde{P}$ by the unit function over G.

For the distributional index of transversally elliptic operators, we refer to Atiyah [1]. We use two main results of [1]. One result is an expression of $\mathrm{ind}^T P$, for a transversally elliptic operator P over a manifold M relative to a toral action with only finite isotropy subgroups. The value $(\mathrm{ind}^T P)(1_T)$ is written by the evaluation of the equivariant residual characteristic classes over the orbit spaces $T\backslash\tau_T M^h$ ($h \in T$, $M^h \neq \phi$) (including

$h=1$). By a direct translation, this formula gives the formula (7) in our theorem, when the V-manifold X has the form $X = T\backslash M$. Another result is a reduction formula $(\mathrm{ind}^G P)(1_G) = (\mathrm{ind}^T([\bar\partial]\otimes P))(1_T)$, for a compact connected Lie group G, where T is a maximal torus of G and $[\bar\partial]$ denotes the Dolbeault complex over the flag manifold G/T.

Combining these two results, we get an expression of the V-index using the evaluation of characteristic classes over an auxiliary V-manifold $T\backslash\tilde X$ and its singularities. This new V-manifold $T\backslash\tilde X$ is a fibration (with singularities) over X with generic fibre G/T. We apply the Gysin homomorphism (the integration over the fibre) to these characteristic classes. Then we get classes over the V-manifold X and its singularities. To deduce (7), we need a formula on the equivariant residual Todd classes over the flag manifold G/T. This formula is a generalization of the following result in Borel-Hirzebruch [5].

Let G be a compact connected Lie group and let T be a maximal torus of G. We fix a G-invariant complex structure on the flag manifold G/T. Consider the fibration $\pi: BT \to BG$ of classifying spaces with fibre G/T. Its bundle along the fibre is a complex vector bundle over BT. We denote by $\mathscr{T}_G(G/T)$ the Todd class of this bundle. (This class is the G-equivariant Todd class of the complex G-manifold G/T). Then Borel and Hirzebruch proved the following:

THEOREM (Borel-Hirzebruch [5]). *Let $\pi_!: H^{**}(BT; \mathbf{R}) \to H^{**}(BG; \mathbf{R})$ be the Gysin homomorphism (the integration over the fibre). Then we have:*

$$\pi_! \mathscr{T}_G(G/T) = 1 \in H^{**}(BG; \mathbf{R}) = H_G^{**}(pt; \mathbf{R}) \,, \tag{8}$$

where H_G^{**} denotes the completed equivariant cohomology group for G-spaces.

Let $h \in T$ be an element. The action of h on G/T is holomorphic. So the fixed point set $(G/T)^h$ is a complex submanifold (non-connected) with the holomorphic action of the centralizer $Z_G(h)$. The tangent bundle τ_h and the normal bundle ν_h are the $Z_G(h)$-equivariant complex vector bundles. Let $\nu_h = \bigoplus \nu_h^\theta$ be the eigenspace decomposition by the action of h. Then we define the equivariant residual Todd class by:

$$\begin{aligned}\mathscr{T}_G^h(G/T) &= \mathscr{T}_{Z_G(h)} \prod_{0<\theta<2\pi} \mathscr{T}^\theta_{Z_G(h)}(\nu_h^\theta) \\ &= \mathscr{T}(EZ_G(h) \times_{Z_G(h)} \tau_h) \prod_{0<\theta<2\pi} \mathscr{T}^\theta(EZ_G(h) \times_{Z_G(h)} \nu_h^\theta) \\ &\in H^{**}_{Z_G(h)}((G/T)^h; \mathbf{C}) = H^{**}(EZ_G(h) \times_{Z_G(h)} (G/T)^h; \mathbf{C}) \,.\end{aligned}$$

The base space $EZ_G(h) \times_{Z_G(h)} (G/T)^h$ is a fibration over $BZ_G(h)$ with the fibre $(G/T)^h$. Then we have the Gysin homomorphism $\pi_!: H^{**}_{Z_G(h)}((G/T)^h; \boldsymbol{C}) \to H^{**}_{Z_G(h)}(pt; \boldsymbol{C}) = H^{**}(BZ_G(h; \boldsymbol{C})$.

THEOREM. *The Gysin homomorphism of the equivariant residual Todd class is given by*:

$$(9) \qquad \pi_! \mathcal{T}^h_G(G/T) = 1 \in H^{**}(BZ_G(h); \boldsymbol{C}) = H^{**}_{Z_G(h)}(pt; \boldsymbol{C}) .$$

If we put $h = 1$, we recover (8). The proof of this formula is straightforward. The same technique as in Borel-Hirzebruch [4] is applicable. We can express $\pi_! \mathcal{T}^h_G(G/T) \in H^{**}(BZ_G(h); \boldsymbol{C}) \subset H^{**}(BT; \boldsymbol{C})$ in the power series in the roots of the Lie group G. Then we deduce our formula from the Weyl's relation on the roots of G.

§ 1. Distributional index and V-index

In this section we summarize the results in Atiyah [1] that we need and we shall show the relation between the distributional index of transversally elliptic operators and the V-index of elliptic operators over V-manifolds.

Let G be a compact Lie group and let M be a compact smooth G-manifold without boundary. We choose a G-invariant Riemannian metric on M and we identify the cotangent bundle τ^*M and the tangent bundle τM. We define a subset $\tau_G M$ in τM as the set of all the tangent vectors that are orthogonal to the orbits of G.

Let E and F be G-equivariant smooth complex vector bundles over M and let $P: \mathcal{C}^\infty(M; E) \to \mathcal{C}^\infty(M; F)$ be a G-invariant pseudo-differential operator of order m. By choosing invariant metrics and invariant connections on E and F, we have the space of Sobolev sections $\mathcal{H}^s(M; E)$ and $\mathcal{H}^s(M; F)$ $(s \in \boldsymbol{R})$. Then the operator P extends uniquely to a bounded operator $P: \mathcal{H}^s(M; E) \to \mathcal{H}^{s-m}(M; F)$. Also we have the adjoint operator $P^*: \mathcal{H}^s(M; F) \to \mathcal{H}^{s-m}(M; E)$. The null spaces $\mathcal{N}^s(P)$ and $\mathcal{N}^s(P^*)$ are closed subspaces and admit the structure of Hilbert spaces. We may consider $\mathcal{N}^s(P)$ and $\mathcal{N}^s(P^*)$ as unitary representations of G. We denote by $\hat{G}$ the set of all equivalence classes of irreducible representations of G. For $\alpha \in \hat{G}$, we denote the α-components by $\mathcal{N}^s_\alpha(P)$ and $\mathcal{N}^s_\alpha(P^*)$.

We call a G-invariant pseudo-differential operator $P: \mathcal{C}^\infty(M; E) \to \mathcal{C}^\infty(M; F)$ to be *transversally elliptic* relative to G if the principal symbol $\sigma(P)$ is invertible over $\tau_G M - M$. Then we have:

THEOREM (Atiyah [1]). *Let $P: \mathscr{C}^\infty(M; E) \to \mathscr{C}^\infty(M; F)$ be a transversally elliptic operator. Then for each $\alpha \in \hat{G}$, $\mathscr{N}^s_\alpha(P)$ is finite dimensional and does not depend on s. Furthermore the formal sum*

$$\operatorname{char} \mathscr{N}(P) = \sum_{\alpha \in \hat{G}} \operatorname{char} \mathscr{N}^s_\alpha(P)$$

converges in $\mathscr{H}^{-n-\varepsilon}(G)$ ($n = \dim M$) for any $\varepsilon > 0$.

Now we can define the distributional index:

DEFINITION. Let $P: \mathscr{C}^\infty(M; E) \to \mathscr{C}^\infty(M; F)$ be a transversally elliptic operator relative to G. Then the *distributional index* $\operatorname{ind}^G(P)$ is defined by:

$$\operatorname{ind}^G(P) = \operatorname{char} \mathscr{N}(P) - \operatorname{char} \mathscr{N}(P^*) \in \mathscr{D}'(G)^{\mathrm{inv}}$$

Here we denote by $\mathscr{D}'(G)^{\mathrm{inv}}$ the distributions on G invariant under the inner automorphisms of G.

The distributional index has the following properties:

THEOREM (Atiyah [1]). *The distributional index of a transversally elliptic operator P depends only on the homotopy class of the restriction of the principal symbol $\sigma(P)$ to $\tau_G M - M$*

$$\sigma(P)|_{\tau_G M - M} \in \operatorname{Iso}(\pi^* E, \pi^* F)|_{\tau_G M - M} .$$

COROLLARY. *The distributional index defines a $R(G)$-module homomorphism*

$$\operatorname{ind}^G: K_G(\tau_G M) \longrightarrow \mathscr{D}'(G)^{\mathrm{inv}} .$$

For each $\alpha \in \hat{G}$, the transversally elliptic operator P defines a G-invariant Fredholm operator

$$P_\alpha: \mathscr{H}^s_\alpha(M; E) \longrightarrow \mathscr{H}^{s-m}_\alpha(M; F) .$$

So we may consider $\operatorname{ind}^G(P) = \sum_\alpha \operatorname{ind}(P_\alpha)$. Then by the orthonormality of irreducible characters, we have:

$$(\operatorname{ind}^G P)(1_G) = \operatorname{index}[P^G: \mathscr{C}^\infty(M; E)^G \longrightarrow \mathscr{C}^\infty(M; F)^G] .$$

Now we assume that the action of G on M is of trivial principal orbit type and with only finite isotropy subgroups. Then, by definition, the above number is the V-index of the elliptic operator $P^G: \mathscr{C}^\infty_V(G\backslash M; G\backslash E) \to \mathscr{C}^\infty_V(G\backslash M; G\backslash F)$ over the V-manifold $G\backslash M$. Each G-equivariant bundle

$E \to M$ defines a proper V-bundle $G\backslash E \to G\backslash M$, and vice versa. The V-manifold $G\backslash \tau_G M$ is exactly the total space $\tau_V(G\backslash M)$ of the tangent V-bundle. Then we have the canonical isomorphism $K_G(\tau_G M) \cong K_V(\tau_V(G\backslash M))$ and the following commutative diagram

$$\begin{array}{ccc} K_G(\tau_G M) & \xrightarrow{\mathrm{ind}^G} & \mathscr{D}'(G)^{\mathrm{inv}} \\ \| \wr & & \downarrow \int_G \\ K_V(\tau_V(G\backslash M)) & \xrightarrow{\mathrm{ind}_V} & \boldsymbol{Z} \subset \boldsymbol{C} . \end{array}$$

Conversely, given a V-manifold X, we choose a Riemannian metric on X. Then the total space $O(n)(\tau_V X)$ of the associated tangential orthonormal frame V-bundle is a smooth manifold. The right action of $O(n)$ is of trivial principal orbit type and with only finite isotropy subgroups. Its orbit space is canonically identified with the original V-manifold X. If we choose an injective homomorphism of $O(n)$ into a compact connected Lie group G, then the total space $\tilde{X} = O(n)(\tau_V X) \times_{O(n)} G$ of the associated tangential G-principal V-bundle is a smooth manifold with a right G-action and its orbit space is again a V-manifold X. So we recover the original situation and we also have an identification $K_V(\tau_V X) \cong K_G(\tau_G \tilde{X})$. Thus we reduce the computations of V-index into those of distributional index.

For the computations of distributional index, we write down some of the results in Atiyah [1]. Let G be a compact connected Lie group and let T be its maximal torus. We choose and fix a G-invariant complex structure on the flag manifold G/T. Then we have the Dolbeault complex on G/T and we consider its symbol $[\bar{\partial}]$ as an element of $K_G(\tau(G/T))$. Let M be a smooth G-manifold with only finite isotropy subgroups. We have a G-equivariant diffeomorphism $G \times_T M \cong G/T \times M$ by sending $(g, x) \in G \times_T M$ to (gT, gx). Then we have the equivalences of vector bundles

$$G \times_T \tau_T M \cong \tau_G(G \times_T M) \cong \tau_G(G/T \times M) \cong \tau(G/T) \times \tau_G M .$$

The first equivalence comes from the $(G \times T)$-equivariant bundle map $G \times \tau_T M = \tau_{G\times T}(G \times M)$, where $G \times T$ acts on $G \times M$ by $(g, h)(g', x) = (gg'h^{-1}, hx)$. The third equivalence comes from the natural identification $\tau(G/T \times M) \cong \tau(G/T) \times \tau M$. Then we define a homomorphism $r\colon K_G(\tau_G M) \to K_T(\tau_T M)$ by:

$$r\colon K_G(\tau_G M) \xrightarrow{[\bar{\partial}]\times} K_G(\tau(G/T) \times \tau_G M) \cong K_G(G \times_T \tau_T M) \cong K_T(\tau_T M) .$$

By this homomorphism, we can compute ind^G through ind^T.

THEOREM (Atiyah [1]). *Let M be a compact smooth G-manifold without boundary (with only finite isotropy subgroups)*). Then the following diagram commutes:*

$$\begin{array}{ccc} K_G(\tau_G M) & \xrightarrow{\ r\ } & K_T(\tau_T M) \\ \Big\downarrow \mathrm{ind}^G & & \Big\downarrow \mathrm{ind}^T \\ \mathscr{D}'(G)^{\mathrm{inv}} & \xleftarrow{\ i_*\ } & \mathscr{D}'(T) , \end{array}$$

where $i_*: \mathscr{D}'(T) \to \mathscr{D}'(G)^{\mathrm{inv}}$ is the dual of the restriction $i^*: \mathscr{C}^\infty(G)^{\mathrm{inv}} \to \mathscr{C}^\infty(T)$.

Especially, for $u \in K_G(\tau_G M)$, we have:

$$(\mathrm{ind}^G u)(1_G) = (\mathrm{ind}^T ru)(1_T) . \tag{10}$$

Another result that we need is the following:

THEOREM (Atiyah [1]). *Let M be a compact smooth T-manifold without boundary, with only finite isotropy subgroups. Then for $u \in K_T(\tau_T M)$, we have:*

$$(\mathrm{ind}^T u)(1_T) = \sum_{\substack{h \in T,\, M^h \neq \phi \\ M_i^h \subset M^h}} \frac{(-1)^{\dim (T\backslash M_i^h)}}{m_T(M_i^h)} \{\mathrm{ch}_T^h(u) \mathscr{I}_T^h(M)\}[T\backslash \tau_T M_i^h] , \tag{11}$$

where M_i^h moves over the connected components of M^h and for each M_i^h, we define the multiplicity $m_T(M_i^h)$ by:

$$m_T(M_i^h) = \text{the order of } \{g \in T \mid gx = x, \text{ for any } x \in M_i^h\}^{**)} .$$

We review the definitions of $\mathrm{ch}_T^h(u)$ and $\mathscr{I}_T^h(u)$. Let $i_h: \tau_T M^h \to \tau_T M$ be the inclusion, then $i_h^* u \in K_T(\tau_T M^h)$ admits the eigenspace decomposition $i_h^* u = \oplus_{0 \leqq \theta < 2\pi} u_h^\theta$, where $u_h^\theta \in K_T(\tau_T M^h)$ is the stable eigenvector bundle of eigenvalue $e^{i\theta}$. Then we have an element $\mathrm{ch}_T(u_h^\theta) \in H_{T,c}^*(\tau_T M^h; \boldsymbol{Q}) \cong H_c^*(T\backslash \tau_T M^h; \boldsymbol{Q})$ (the subscript c denotes the cohomology with compact support). We define $\mathrm{ch}_T^h(u) \in H_{T,c}^*(\tau_T M^h; \boldsymbol{C}) \cong H_c^*(T\backslash \tau_T M^h; \boldsymbol{C})$ by:

$$\mathrm{ch}_T^h(u) = \sum_{0 \leqq \theta < 2\pi} e^{i\theta} \mathrm{ch}_T (u_h^\theta) .$$

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*) In Atiyah [1], this theorem is proved without any restriction on isotropy subgroups.

**) The definition of the multiplicity $m(h)$ in Atiyah [1] is incorrect. It depends on the whole group T and the connected component M_i^h in M^h.

Let $\tau_T M|_{M^h} = \tau_T M^h \oplus \nu_{h,T} = \tau_T M^h \oplus (\oplus_{0<\theta\leqq\pi} \nu^\theta_{h,T})$ be the eigenspace decomposition. $\nu^\pi_{h,T}$ is the real eigenvector bundle of eigenvalue -1 and $\nu^\theta_{h,T}$ $(0<\theta<\pi)$ is a complex vector bundle on which the action of h is the multiplication by the scalar $e^{i\theta}$. We denote formally the equivariant Pontrjagin classes of $\tau_T M^h$ and $\nu^\pi_{h,T}$ by $p_T(\tau_T M^h) = \prod (1+x_j^2) \in H^*_T(M^h; \boldsymbol{Q})$ and $p_T(\nu^\pi_{h,T}) = \prod (1+y_j^2) \in H^*_T(M^h; \boldsymbol{Q})$ respectively, and the equivariant Chern classes of $\nu^\theta_{h,T}$ by $c_T(\nu^\theta_{h,T}) = \prod (1+z_j) \in H^*_T(M^h; \boldsymbol{Q})$. Then we define $\mathscr{I}^h_T(M) \in H^*_T(M^h; \boldsymbol{C}) \cong H^*(T\backslash M^h; \boldsymbol{C})$ by:

$$\mathscr{I}^h_T(M) = \det_{\boldsymbol{R}}(1 - h|_{\nu_{h,T}})^{-1} \mathscr{R}_T(\nu^\pi_{h,T}) \Big\{ \prod_{0<\theta<\pi} \mathscr{S}^\theta_T(\nu^\theta_{h,T}) \Big\} \mathscr{I}_T(M^h) ,$$

where

$$\mathscr{I}_T(M^h) = \mathscr{T}_T(\tau_T M^h \otimes_{\boldsymbol{R}} \boldsymbol{C}) = \prod_j \left(\frac{x_j}{1-e^{-x_j}} \frac{-x_j}{1-e^{x_j}} \right) ,$$

$$\mathscr{R}_T(\nu^\pi_{h,T}) = \prod_j \left(\frac{2}{1+e^{y_j}} \frac{2}{1+e^{-y_j}} \right) ,$$

$$\mathscr{S}^\theta_T(\nu^\theta_{h,T}) = \prod_j \left(\frac{1-e^{i\theta}}{1-e^{z_j+i\theta}} \frac{1-e^{-i\theta}}{1-e^{-z_j-i\theta}} \right) .$$

Consider the orbit space $X = T\backslash M$ as a V-manifold. By definition, we can see:

$$X \amalg \Big(\coprod_i \Sigma_i \Big) = T\backslash M \amalg \Big(\coprod_{\substack{h\in T, M^h\neq\emptyset \\ M^h_i \subset M^h}} T\backslash M^h_i \Big) ,$$

$$\tau_V X \amalg \Big(\coprod_i \tau_V \Sigma_i \Big) = T\backslash \tau_T M \amalg \Big(\coprod_{h,i} T\backslash \tau_T M^h_i \Big) .$$

So we may identify $H^*_{T,c}(\tau_T M; \boldsymbol{Q})$ with $H^*_c(\tau_V X; \boldsymbol{Q})$ and $H^*_{T,c}(\tau_T M^h_i; \boldsymbol{C})$ with $H^*_c(\tau_V \Sigma_i; \boldsymbol{C})$. Then, for $u \in K_V(\tau_V X) \cong K_T(\tau_T M)$, we can interpret:

$$\mathrm{ch}(u)\mathscr{I}(X) + \mathrm{ch}^\Sigma(u)\mathscr{I}^\Sigma(X) = \mathrm{ch}_T(u)\mathscr{I}_T(M) + \sum_h \mathrm{ch}^h_T(u)\mathscr{I}^h_T(M) .$$

Thus we have shown that the Atiyah's formula (11) is equivalent to our formula (7), if the V-manifold X is obtained as the orbit space of a toral action.

Now we consider a general V-manifold X. We may assume that X is the orbit space of a G-manifold M. G acts on M with only finite isotropy subgroups and of trivial principal orbit type. Then, for a real or complex G-equivariant vector bundle E, we may identify the G-equivariant characteristic class of E with the characteristic class of the V-bundle $G\backslash E \to X$

(defined by the same polynomial in Pontrjagin classes or Chern classes). We shall rewrite the formula (7) in the word of equivariant characteristic classes.

By the compactness of M and the smoothness of the G-action, the number of orbit types of G-manifold M is finite. Also, all the isotropy subgroups are finite, so the number of conjugacy classes of elements of G with non-empty fixed point set is finite. Let $(1), (h_1), \cdots, (h_\rho)$ be such conjugacy classes. Each fixed point set M^h admits the action of the centralizer $Z_G(h)$. Then the action of h on $\tau M|_{M^h}$ defines the decomposition into eigenvector bundles

$$\tau M|_{M^h} = \tau_{Z_G(h)} M^h \oplus \nu_{h,G} = \tau_{Z_G(h)} M^h \oplus \left(\bigoplus_{0<\theta\leqq\pi} \nu_{h,G}^{\theta} \right) .$$

Since $Z_G(h)$ commutes with h, each summand is $Z_G(h)$-equivariant. Then we define $\mathscr{I}_G^h(M) \in H^*_{Z_G(h)}(M^h; C)$ by;

$$\mathscr{I}_G^h(M) = \det_R (1 - h|_{\nu_{h,G}})^{-1} \mathscr{R}_{Z_G(h)}(\nu_{h,G}^{\pi}) \left\{ \prod_{0<\theta<\pi} \mathscr{S}^{\theta}_{Z_G(h)}(\nu_{h,G}^{\theta}) \right\} \mathscr{I}_{Z_G(h)}(M^h) .$$

We remark that $\nu_{h,G}$ and the normal bundle of M^h in M differ in dimension equal to $\dim G - \dim Z_G(h)$. For $u \in K_V(\tau_V X) \cong K_G(\tau_G M)$, we have $i_h^* u \in K_{Z_G(h)}(\tau_{Z_G(h)} M^h)$ and the eigenspace decomposition $i_h^* u = \oplus_{0\leqq\theta<2\pi} u_h^{\theta}$. Then we define:

$$\mathrm{ch}_G^h(u) = \sum_{0\leqq\theta<2\pi} e^{i\theta} \mathrm{ch}_{Z_G(h)}(u_h^{\theta}) \in H^*_{Z_G(h),c}(\tau_{Z_G(h)} M^h; C) .$$

Let $\Sigma X = \coprod \Sigma_i$ be the singularity V-manifold. Then we have canonical identifications

$$\coprod \Sigma_i = \coprod_{j=1}^{\rho} Z_G(h_j)\backslash M^{h_j} , \qquad \coprod \tau_V \Sigma_i = \coprod_{j=1}^{\rho} Z_G(h_j)\backslash \tau_{Z_G(h_j)} M^{h_j} .$$

Let $Z_G(h)\backslash M^h = \coprod Z_G(h)\backslash M_i^h$ be the decomposition into connected components. Each M_i^h is $Z_G(h)$-invariant but not connected in general. We define the multiplicity $m_G(M_i^h)$ by:

$$m_G(M_i^h) = \text{the order of } \{g \in Z_G(h) \mid gx = x, \text{ for any } x \in M_i^h\} .$$

Now we can rewrite the formula (7) into:

$$\text{(12)} \qquad (\mathrm{ind}^G u)(1_G) = \sum_{\substack{(h)\in(G) \\ M_i^h \subset M^h}} \frac{(-1)^{\dim (Z_G(h)\backslash M_i^h)}}{m_G(M_i^h)} \{\mathrm{ch}_G^h(u) \mathscr{I}_G^h(M)\}[Z_G(h)\backslash \tau_{Z_G(h)} M_i^h] ,$$

where we denote by (G) the set of conjugacy classes of G. We shall deduce this formula from (11) and a computation in the equivariant Chern classes on the flag manifold G/T.

§ 2. Gysin homomorphisms (integrations over the fibre)

Let G be a compact connected Lie group and let M be a compact G-manifold without boundary. We assume that G acts on M with only finite isotropy subgroups. Let T be a maximal torus of G. We choose and fix a G-invariant complex structure on the flag manifold G/T. Then, by (10) and (11), we have, for $u \in K_G(\tau_G M)$:

$$\begin{aligned}(\mathrm{ind}^G u)(1_G) &= (\mathrm{ind}^T ru)(1_T) \\ &= \sum_{\substack{h\in T \\ M_i^h \subset M^h}} \frac{(-1)^{\dim (T\backslash M_i^h)}}{m_T(M_i^h)} \{\mathrm{ch}_T^h (ru)\mathscr{I}_T^h(M)\}[T\backslash\tau_T M_i^h] \,. \end{aligned} \tag{13}$$

Here $M_i^h \subset M^h$ denotes a connected component. In the sequel we omit i's since all the arguments are parallel.

To deduce (12), we need to reform (13) into the evaluation over $[Z_G(h)\backslash\tau_{Z_G(h)}M^h]$'s. We use the Gysin homomorphisms. Consider the commutative diagram

$$\begin{array}{ccc} ET \times_T M^h & \xrightarrow{\pi} & EZ_G(h) \times_{Z_G(h)} M^h \\ \downarrow & & \downarrow \\ T\backslash M^h & \xrightarrow{\pi} & Z_G(h)\backslash M^h \,. \end{array}$$

The vertical maps induce the identifications $H^*(T\backslash M^h; \boldsymbol{Q}) \cong H_T^*(M^h; \boldsymbol{Q})$ and $H^*(Z_G(h)\backslash M^h; \boldsymbol{Q}) \cong H_{Z_G(h)}^*(M^h; \boldsymbol{Q})$. The upper π is a fibration with fibre $Z_G(X)/T$. We orient $Z_G(h)/T$ by the induced complex structure from G/T. We denote the orientation sheaf on M^h by $o(M^h)$. Then we have the Gysin homomorphism $\pi_!: H_T^*(M^h; o(M^h) \otimes \boldsymbol{Q}) \to H_{Z_G(h)}^*(M^h; o(M^h) \otimes \boldsymbol{Q})$. We may reconstruct $\pi_!$ by using the Leray-Serre spectral sequence of the map $\pi: T\backslash M^h — Z_K(h)\backslash M$. Then we have the following proposition:

PROPOSITION. *The Gysin homomorphism* $\pi_!$:

$$H_T^*(M^h; o(M^h) \otimes \boldsymbol{Q}) \longrightarrow H_{Z_G(h)}^*(M^h; o(M^h) \otimes \boldsymbol{Q})$$

is a $H_{Z_G(h)}^*(M^h; \boldsymbol{Q})$*-module homomorphism. For* $x \in H_T^*(M^h; o(M^h) \otimes \boldsymbol{Q})$, *we have the following formula*:

$$\frac{1}{m_T(M^h)}\langle x, [T\backslash M^h]\rangle = \frac{1}{m_G(M^h)}\langle \pi_! x, [Z_G(h)\backslash M^h]\rangle .$$

Also we have the Thom isomorphisms

$$\psi_T : H_T^*(M^h; o(M^h)\otimes \boldsymbol{Q}) \longrightarrow H_{T,c}^*(\tau_T M^h; \boldsymbol{Q})$$

and

$$\psi_{Z_G(h)} : H_{Z_G(h)}^*(M^h; o(M^h)\otimes \boldsymbol{Q}) \longrightarrow H_{Z_G(h),c}^*(\tau_{Z_G(h)} M^h; \boldsymbol{Q}) .$$

Then we define $\tau\pi_! = \psi_{Z_G(h)} \circ \pi_! \circ (\psi_T)^{-1} : H_{T,c}^*(\tau_T M^h; \boldsymbol{Q}) \to H_{Z_G(h),c}^*(\tau_{Z_G(h)} M^h; \boldsymbol{Q})$. It is also a $H_{Z_G(h)}^*(M^h; \boldsymbol{Q})$-homomorphism. Looking carefully at the orientations of $T\backslash\tau_T M^h$ and $Z_G(h)\backslash\tau_{Z_G(h)}M^h$, we have, for $y \in H_{T,c}^*(\tau_T M^h; \boldsymbol{Q})$:

$$\frac{1}{m_T(M^h)}\langle y, [T\backslash\tau_T M^h]\rangle = \frac{(-1)^{m_h}}{m_T(M^h)}\langle \tau\pi_! y, [Z_G(h)\backslash\tau_{Z_G(h)} M^h]\rangle ,$$
$$(m_h = \tfrac{1}{2}\dim_{\boldsymbol{R}}(Z_G(h)/T) = \dim_{\boldsymbol{C}}(Z_G(h)/T)) .$$

We apply $\tau\pi_!$ to $\{\mathrm{ch}_T^h(ru)\mathscr{I}_T^h(M)\}$ in (13). Then we get:

$$(14)\qquad \begin{aligned}&(\mathrm{ind}^G u)(1_G) = \sum_{h\in T}\frac{\varepsilon_G(M^h)}{m_G(M^h)}(-1)^{m_h}\tau\pi_!\{\mathrm{ch}_T^h(ru)\mathscr{I}_T^h(M)\}[Z_G(h)/\tau_{Z_G(h)}M^h] ,\\ &(\varepsilon_G(M^h) = (-1)^{\dim(Z_G(h)\backslash M^h)}) .\end{aligned}$$

We compute each term $(-1)^{m_h}\tau\pi_!\{\mathrm{ch}_T^h(ru)\mathscr{I}_T^h(M)\}$ independently. First we consider $\mathscr{I}_T^h(M) \in H_T^*(M^h; \boldsymbol{C})$. We have isomorphisms:

$$H_T^*(M^h; \boldsymbol{C}) \cong H_{Z_G(h)}^*(Z_G(h)\times_T M^h; \boldsymbol{C}) \cong H_{Z_G(h)}^*(Z_G(h)/T \times M^h) .$$

Recall the definition:

$$\mathscr{I}_T^h(M) = \det_{\boldsymbol{R}}(1 - h|_{\nu_{h,T}})^{-1}\mathscr{R}_T(\nu_{h,T}^\pi)\Big\{\prod_{0<\theta<\pi}\mathscr{S}_T^\theta(\nu_{h,T}^\theta)\Big\}\mathscr{I}_T(M^h) .$$

$\nu_{h,T}$ is a $Z_G(h)$-equivariant bundle and decomposes equivariantly into:

$$\nu_{h,T} = \nu_{h,G} \oplus \tau_0(G/Z_G(h)) ,$$

where $\tau_0(G/Z_G(h))$ denotes the tangent space of $G/Z_G(h)$ at the identity coset. (We denote by the same symbol the vector space and the trivial vector bundle). So, if we lift the T-equivariant bundle $\nu_{h,T}$ to a $Z_G(h)$-equivariant bundle over $Z_G(h)\times_T M^h \cong Z_G(h)/T \times M^h$, we may consider it as the pull-back of a $Z_G(h)$-equivariant bundle $\nu_{h,T}$ over M^h. Since $Z_G(h)/T$ is a complex submanifold of G/T, $\tau_0(G/Z_G(h))$ is a complex vector space with a linear action of h. h does not have any non-zero fixed vector on $\tau_0(G/Z_G(h))$. Let $\oplus_{0<\theta<2\pi}\tau_0^\theta(G/Z_G(h))$ be the eigenspace decomposition. We define:

$$\mathscr{I}_G^h(G/Z_G(h))_0 = \det_R(1-h|_{\tau_0(G/Z_G(h))})^{-1}\Big\{\prod_{0<\theta<2\pi}\mathscr{S}^\theta_{Z_G(h)}(\tau_0^\theta(G/Z_G(h)))\Big\}$$
$$\in H^{**}_{Z_G(h)}(pt; \boldsymbol{C}) .$$

Then we have:

$$\det_R(1-h|_{\nu_{h,T}})^{-1}\mathscr{R}_T(\nu^\pi_{h,T})\Big\{\prod_{0<\theta<\pi}\mathscr{S}^\theta_T(\nu^\theta_{h,T})\Big\}$$
$$= \mathscr{I}_G^h(G/Z_G(h))_0 \times \det(1-h|_{\nu_{h,G}})^{-1}\mathscr{R}_{Z_G(h)}(\nu^\pi_{h,G})\Big\{\prod_{0<\theta<\pi}\mathscr{S}^\theta_{Z_G(h)}(\nu^\theta_{h,G})\Big\} ,$$

where the first factor is in $H^{**}_{Z_G(h)}(pt; \boldsymbol{C})$ and the second factor is in $H^*_{Z_G(h)}(M^h; \boldsymbol{C})$. Also we have a T-equivariant decomposition:

$$\tau_T M^h = \tau_{Z_G(h)} M^h \oplus \tau_0(Z_G(h)/T) .$$

If we lift $\tau_T M^h$ over $Z_G(h)/T \times M^h$, then $\tau_{Z_G(h)}M^h$ is a $Z_G(h)$-equivariant bundle over M^h and $\tau_0(Z_G(h)/T)$ is the tangent bundle of $Z_G(h)/T$. Hence we have:

$$\mathscr{I}_T(M^h) = \mathscr{I}_{Z_G(h)}(Z_G(h)/T) \times \mathscr{I}_{Z_G(h)}(M^h) .$$

As a whole, we have:

$$\mathscr{I}_T^h(M) = \mathscr{I}_{Z_G(h)}(Z_G(h)/T) \times \mathscr{I}_G^h(G/Z_G(h))_0 \times \mathscr{I}_G^h(M)$$
$$\in H^*_{Z_G(h)}(Z_G(h)/T \times M^h; \boldsymbol{C}) ,$$

where the first factor is in $H^{**}_{Z_G(h)}(Z_G(h)/T; \boldsymbol{Q})$, the second factor is in $H^{**}_{Z_G(h)}(pt; \boldsymbol{C})$ and the third factor is in $H^*_{Z_G(h)}(M^h; \boldsymbol{C})$.

Next we compute $\mathrm{ch}_T^h(ru) \in H^*_{T,c}(\tau_T M^h; \boldsymbol{C})$. By definition, we have ru $[\bar\partial] \times u \in K_G(\tau(G/T) \times \tau_G M) \cong K_T(\tau_T M)$. So

$$i_h^* ru = [\bar\partial_{G/T}|_{Z_G(h)/T}] \times i_h^* u \in K_{Z_G(h)}(\tau(Z_G(h)/T) \times \tau_{Z_G(h)} M^h) .$$

Since $Z_G(h)/T$ is a complex submanifold of G/T, we have $[\bar\partial_{G/T}|_{Z_G(h)/T}] = [\bar\partial_{Z_G(h)/T}]\lambda_{-1}(\tau_0(G/Z_G(h))) \in K_{Z_G(h)}(\tau(Z_G(h)/T))$. Hence we have:

$$i_h^* ru = [\bar\partial_{Z_G(h)/T}] \times \lambda_{-1}(\tau_0(G/Z_G(h))) \times i_h^* u ,$$

where the first factor is in $K_{Z_G(h)}(\tau(Z_G(h)/T))$, the second factor is in $R(Z_G(h))$ and the third factor is in $K_{Z_G(h)}(\tau_{Z_G(h)}M^h)$. Consider the eigenspace decomposition by the action of h. The action is trivial on $Z_G(h)/T$. So we have:

$$\sum e^{i\theta}(i_h^* ru)^\theta = [\bar{\partial}_{Z_G(h)/T}] \times (\sum e^{i\theta}\lambda_{-1}(\tau_0^\theta(G/Z_G(h)))) \times (\sum e^{i\theta}(i_h^* u)^\theta)\,.$$

Applying the Chern character on both sides, we have:

$$\begin{aligned}\mathrm{ch}_T^h(ru) &= \mathrm{ch}_{Z_G(h)}[\bar{\partial}_{Z_G(h)/T}] \times \mathrm{ch}_G^h(\lambda_{-1}(\tau_0(G/Z_G(h)))) \times \mathrm{ch}_G^h(u)\\ &\in H_{T,c}^*(\tau_T M^h; C) \cong H_{Z_G(h),c}^*(\tau(Z_G(h)/T) \times \tau_{Z_G(h)} M^h; C)\,,\end{aligned}$$

where the first factor is in $H_{Z_G(h),c}^{**}(\tau(Z_G(h)/T); C)$, the second factor is in $H_{Z_G(h)}^{**}(pt; C)$ and the third factor is in $H_{Z_G(h),c}^*(\tau_{Z_G(h)} M^h; C)$. Combining this with the computation on $\mathscr{I}_T^h(M)$, we have:

$$\begin{aligned}\mathrm{ch}_T^h(ru)\mathscr{I}_T^h(M) &= \mathrm{ch}_{Z_G(h)}[\bar{\partial}_{Z_G(h)/T}]\mathscr{I}_{Z_G(h)}(Z_G(h)/T)\\ &\quad\times \mathrm{ch}_G^h(\lambda_{-1}(\tau_0(G/Z_G(h))))\mathscr{I}_G^h(G/Z_G(h))_0\\ &\quad\times \mathrm{ch}_G^h(u)\mathscr{I}_G^h(M)\\ &\in H_{T,c}^*(\tau_T M^h; C) \cong H_{Z_G(h),c}^*(\tau(Z_G(h)/T) \times \tau_{Z_G(h)} M^h; C)\,,\end{aligned}$$

where the first factor is in $H_{Z_G(h),c}^{**}(\tau(Z_G(h)/T; Q)$, the second factor is in $H_{Z_G(h)}^{**}(pt; C)$ and the third factor is in $H_{Z_G(h),c}^*(\tau_{Z_G(h)} M^h; C)$. We have also:

$$\mathrm{ch}_{Z_G(h)}[\bar{\partial}_{Z_G(h)/T}]\mathscr{I}_{Z_G(h)}(Z_G(h)/T) = (-1)^{m_h}\psi(\mathscr{T}_{Z_G(h)}(Z_G(h)/T))\,,$$

$$(\psi: H_{Z_G(h)}^{**}(Z_G(h)/T; Q) \longrightarrow H_{Z_G(h),c}^{**}(\tau(Z_G(h)/T); Q),\ \text{Thom isomorphism})\,,$$

$$\mathrm{ch}_G^h(\lambda_{-1}(\tau_0(G/Z_G(h))))\mathscr{I}_G^h(G/Z_G(h))_0 = \mathscr{T}_G^h(G/Z_G(h))_0\,,$$

(the residual Todd class restricted at the identity component).

By the identification $H_{T,c}^*(\tau_T M^h; C) \cong H_{Z_G(h),c}^*(\tau(Z_G(h)/T) \times \tau_{Z_G(h)} M^h; C)$, $\tau\pi_!$ is given by the composite:

$$\begin{aligned}\tau\pi_!: H_{Z_G(h),c}^*&(\tau(Z_G(h)/T) \times \tau_{Z_G(h)} M^h; C)\\ &\xrightarrow{\psi^{-1}} H_{Z_G(h)}^*(Z_G(h)/T \times M^h; o(M^h) \otimes C)\\ &\xrightarrow{\pi_!} H_{Z_G(h)}^*(M^h; o(M^h) \otimes C)\\ &\xrightarrow{\psi} H_{Z_G(h),c}^*(\tau_{Z_G(h)} M^h; C)\,.\end{aligned}$$

Then we can see:

$$\begin{aligned}&(-1)^{m_h}\tau\pi_!\{\mathrm{ch}_T^h(ru)\mathscr{I}_T^h(M)\}\\ &\qquad= \pi_!\{\mathscr{T}_{Z_G(h)}(Z_G(h)/T)\mathscr{T}_G^h(G/Z_G(h))_0\}\,\mathrm{ch}_G^h(u)\mathscr{I}_G^h(M)\,,\\ &(\pi_!: H_{Z_G(h)}^{**}(Z_G(h)/T; C) \longrightarrow H_{Z_G(h)}^{**}(pt; C))\,.\end{aligned}$$

Thus we have proved:

$$(\mathrm{ind}^G u)(1_G) = \sum_{\substack{h\in T\\ M^h\neq\emptyset}} \frac{\varepsilon_G(M^h)}{m_G(M^h)}\{\pi_!\{\mathscr{T}_{Z_G(h)}(Z_G(h)/T)\mathscr{T}_G^h(G/Z_G(h))_0\} \times \mathrm{ch}_G^h(u)\mathscr{I}_G^h(M)\}[Z_G(h)\backslash\tau_{Z_G(h)} M^h]\,. \tag{15}$$

We compare this formula with the final form (12). In (12), the summation moves over the conjugacy classes (h) in G such that $M^h \neq \emptyset$, but in (15), the summation moves over all the elements in T such that $M^h \neq \emptyset$. We recall that every conjugacy class (h) in G meets T by finite (non-zero) times. So in (15), we sum up first the terms corresponding to the elements that belong to the same conjugacy class in G.

Let h and h' be elements in T conjugate in G. Choose $g \in G$ such that $ghg^{-1} = h'$. We denote by ϕ_g the action of g on M and by ι_g the inner automorphism induced by g. Then $\phi_g: M \to M$ is ι_g-equivariant and maps M^h onto $M^{h'}$. It induces bundle equivalences $\tau_h\phi_g: \tau_{Z_G(h)}M^h \to \tau_{Z_G(h')}M^{h'}$ and $\nu_h\phi_g: \nu_{h,G} \to \nu_{h',G}$. These equivalences are $[\iota_g: Z_G(h) \to Z_G(h')]$-equivariant. This shows $\phi_g^* \mathscr{I}_G^{h'}(M) = \mathscr{I}_G^h(M)$ and $(\tau_h\phi_g)^* \operatorname{ch}_G^{h'}(u) = \operatorname{ch}_G^h(u)$. Hence we have:

$$\begin{aligned}(\tau_h\phi_g)^*\{\pi_!\{\mathscr{T}_{Z_G(h')}(Z_G(h')/T)\mathscr{T}_G^{h'}(G/Z_G(h'))_0\} \operatorname{ch}_G^{h'}(u)\mathscr{I}_G^{h'}(M)\} \\ = \{\iota_g^*\pi_!\{\mathscr{T}_{Z_G(h')}(Z_G(h')/T)\mathscr{T}_G^{h'}(G/Z_G(h'))_0\}\} \operatorname{ch}_G^h(u)\mathscr{I}_G^h(M)\,.\end{aligned}$$

For each conjugacy class (h) in G, we put $(h) \cap T = \{h_1, h_2, \cdots, h_{w(h)}\}$. For each j, we choose $g_j \in G$ such that $h_j = g_j h g_j^{-1}$. Then we have:

$$\begin{aligned}(\operatorname{ind}^G u)(1_G) = \sum_{\substack{(h)\in(G)\\ M^h\neq\emptyset}} \frac{\varepsilon_G(M^h)}{m_G(M^h)} \Bigg\{\Bigg\{\sum_{j=1}^{w(h)} \iota_{g_j}^*\pi_!\{\mathscr{T}_{Z_G(h_j)}(Z_G(h_j)/T) \\ \times \mathscr{T}_G^{h_j}(G/Z_G(h_j))_0\}\Bigg\} \operatorname{ch}_G^h(u)\mathscr{I}_G^h(M)\Bigg\}[Z_G(h)\backslash\tau_{Z_G(h)}M^h]\,.\end{aligned} \tag{16}$$

Now we consider the class

$$\sum_{j=1}^{w(h)} \iota_{g_j}^*\pi_!\{\mathscr{T}_{Z_G(h_j)}(Z_G(h_j)/T)\mathscr{T}_G^{h_j}(G/Z_G(h_j))_0\} \in H_{Z_G(h)}^{**}(pt; C)\,.$$

The action of h on $\tau_0(G/Z_G(h))$ has no fixed non-zero vector. By an elementary consideration, we have:

$$(G/T)^h = \coprod_{j=1}^{w(h)} g_j^{-1}Z_G(h_j)/T\,.$$

Recall the definition of $\mathscr{T}_G^h(G/T) \in H_{Z_G(h)}^{**}((G/T)^h; C)$. We can see that $\mathscr{T}_{Z_G(h)}(Z_G(h)/T)\mathscr{T}_G^h(G/Z_G(h))_0$ is the restriction of $\mathscr{T}_G^h(G/T)$ onto the component $Z_G(h)/T$. The holomorphic action of g_j on G/T defines a map $\psi_{g_j}: g_j^{-1}Z_G(h_j)/T \to Z_G(h_j)/T$. It is ι_{g_j}-equivariant. Hence we have:

$$\begin{aligned}&\iota_{g_j}^*\pi_!\{\mathscr{T}_{Z_G(h_j)}(Z_G(h_j)/T)\mathscr{T}_G^{h_j}(G/Z_G(h_j))_0\} \\ &\quad = \pi_!\{\psi_{g_j}^*\mathscr{T}_{Z_G(h_j)}(Z_G(h_i)/T)\mathscr{T}_G^{h_j}(G/Z_G(h_j))_0\} \\ &\quad = \pi_!(\mathscr{T}_G^h(G/T)|_{g_j^{-1}Z_G(h_j)/T})\,.\end{aligned}$$

Thus we have proved:

$$(\mathrm{ind}^G u)(1_G)$$

(17) $$= \sum_{\substack{(h)\in(G)\\ M^h\neq\emptyset}} \frac{\varepsilon_G(M^h)}{m_G(M^h)}\{\pi_!(\mathscr{T}_G^h(G/T))\,\mathrm{ch}_G^h(u)\mathscr{I}_G^h(M)\}[Z_G(h)\backslash\tau_{Z_G(h)}M^h]\,,$$

$$(\pi_!\colon H^{**}_{Z_G(h)}((G/T)^h; C) \longrightarrow H^{**}_{Z_G(h)}(pt; C))\,.$$

To complete the proof it will suffice to show:

$$\pi_!(\mathscr{T}_G^h(G/T)) = 1 \in H^{**}_{Z_G(h)}(pt; C)\,.$$

This will be done in the next section.

§ 3. Equivariant residual Todd classes over flag manifolds

Let G be a compact connected Lie group and let T be a maximal torus of G. Choose and fix a G-invariant complex structure on the flag manifold G/T. Let $h\in T$ be an element. Then the fixed point set $(G/T)^h$ is a complex submanifold (closed but not connected in general). It admits the holomorphic action of the centralizer $Z_G(h)$. Let $E\,(=EG)\to E/G\,(=BG)$ be the universal G-principal bundle. Then we have an associated bundle: $E\times_{Z_G(h)}(G/T)^h \to E/Z_G(h)$ $(=BZ_G(h))$. Over its total space $E\times_{Z_G(h)}(G/T)^h$, we have vector bundles

$$\tau((G/T)^h)_{Z_G(h)} = E\times_{Z_G(h)}\tau((G/T)^h)\,,$$
$$\nu^\theta((G/T)^h)_{Z_G(h)} = E\times_{Z_G(h)}\nu^\theta((G/T)^h) \qquad (0<\theta<2\pi)\,,$$

(ν^θ denotes the eigenvector bundle by the action of h). Then we define:

$$\mathscr{T}_G^h(G/T) = \mathscr{T}(\tau((G/T)^h)_{Z_G(h)})\prod_{0<\theta<2\pi}\mathscr{T}^\theta(\nu^\theta((G/T)^h)_{Z_G(h)})$$
$$\in H^{**}(E\times_{Z_G(h)}(G/T)^h; C) = H^{**}_{Z_G(h)}((G/T)^h; C)\,.$$

$\pi\colon E\times_{Z_G(h)}(G/T)^h \to E/Z_G(h)$ defines the Gysin homomorphism

$$\pi_!\colon H^{**}(E\times_{Z_G(h)}(G/T)^h; C) \longrightarrow H^{**}(E/Z_G(h); C)\,.$$

The purpose of this section is to prove the following formula

(18) $$\pi_!\mathscr{T}_G^h(G/T) = 1\in H^{**}(E/Z_G(h); C) = H^{**}_{Z_G(h)}(pt; C)\,.$$

This is the last formula in the previous section.

Let $Z_G(h)_0\subset Z_G(h)$ denote the identity component. Then the projection $E/Z_G(h)_0\to E/Z_G(h)$ is a finite regular covering. The induced map

$H^{**}(E/Z_G(h); \boldsymbol{C}) \to H^{**}(E/Z_G(h)_0; \boldsymbol{C})$ is injective. So we may reduce the structure group $Z_G(h)$ to $Z_G(h)_0$. We denote by π' the projection

$$\pi' \colon E \times_{Z_G(h)_0} (G/T)^h \longrightarrow E/Z_G(h)_0 .$$

Then it will suffice to show:

$$\pi'_! \mathscr{T}^h_G(G/T) = 1 \in H^{**}(E/Z_G(h)_0; \boldsymbol{C}) .$$

Let $W(G) = N_G(T)/T$ and $W(Z_G(h)_0) = N_{Z_G(h)_0}(T)/T$ be the Weyl group of G and $Z_G(h)_0$ respectively. For each right coset $[w_j]$ in $W(G)/W(Z_G(h)_0)$, choose one representative $g_j \in N_G(T)$. Then, as a $Z_G(h)_0$-manifold, $(G/T)^h$ decomposes into a disjoint union

$$(G/T)^h = \coprod_{[w_j]\in W(G)/W(Z_G(h)_0)} (Z_G(h)_0 g_j^{-1})/T .$$

Put $h_j = g_j h g_j^{-1}$, then the holomorphic action of g_j maps $(Z_G(h)_0 g_j^{-1})/T$ onto $Z_G(h_j)_0/T$. This map is $[\iota_{g_j} \colon Z_G(h)_0 \to Z_G(h_j)_0]$-equivariant. Over each component $(Z_G(h)_0 g_j^{-1})/T$ in $(G/T)^h$, we may translate everything onto $Z_G(h_j)_0/T$ by the action of g_j. Then the bundles

$$E \times_{Z_G(h)_0} \tau((G/T)^h) \quad \text{and} \quad E \times_{Z_G(h)_0} \nu^\theta((G/T)^h)$$

are translated to:

$$E \times_{Z_G(h_j)_0} \tau(Z_G(h_j)_0/T) \cong E \times_T \tau_0(Z_G(h_j)_0/T) ,$$
$$E \times_{Z_G(h_j)_0} \nu^\theta(Z_G(h_j)_0/T) \cong E \times_T \tau_0^\theta(G/Z_G(h_j)_0) .$$

Then we have:

$$\pi'_!\{\mathscr{T}^h_G(G/T)|_{(Z_G(h)_0 g_j^{-1})/T}\}$$
$$= \iota^*_{g_j}(\pi_j)_! \left\{\mathscr{T}_T(\tau_0(Z_G(h_j)_0/T)) \prod_{0<\theta<2\pi} \mathscr{T}^\theta_T(\tau^\theta_0(G/Z_G(h_j)_0))\right\} ,$$

$$(\pi_j \colon E/T \longrightarrow E/Z_G(h_j)_0,\ \iota_{g_j} \colon E/Z_G(h)_0 \longrightarrow E/Z_G(h_j)_0) .$$

We can describe these classes in terms of the roots of G. Let $a_1, a_2, \cdots, a_m$ be the positive roots of G, corresponding to the invariant complex structure on G/T (see Borel-Hirzegruch [4]). Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_m$ be the root space decomposition. That is: $\mathfrak{g} = \tau_0(G)$ and $\mathfrak{h} = \tau_0(T)$. T acts on $\mathfrak{g}$ by the conjugacy. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_m$ is the irreducible decomposition of this T-action. $\mathfrak{h}$ is the trivial summand. a_k $(k = 1, 2, \cdots, m)$ is a linear functional on $\mathfrak{h}$ such that, on $\mathfrak{a}_k \cong \boldsymbol{C}$, the action of T is given by:

$$hz = e^{2\pi i a_k(H)} z\ ,$$
$$(h \in T,\ z \in \mathfrak{a}_k \cong \boldsymbol{C},\ H \in \mathfrak{h} \text{ such that } \exp H = h)\ .$$

For the fixed $h \in T$, we choose $H \in \mathfrak{h}$ such that $\exp H = h$ and we put $H_j = w_j H = \mathrm{Ad}(g_j)H$. Then the T-invariant subspaces $\tau_0(Z_G(h_j)_0/T)$ and $\tau_0^\theta(G/Z_G(h_j)_0)$ in $\tau_0(G/T) = \mathfrak{g}/\mathfrak{h} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_m$ are given by:

$$\tau_0(Z_G(h_j)_0/T) = \bigoplus_{\substack{k;\ a_k(H_j)\equiv 0 \\ \bmod \boldsymbol{Z}}} \mathfrak{a}_k\ ,$$
$$\tau_0^\theta(G/Z_G(h_j)_0) = \bigoplus_{\substack{k;\ a_k(H_j)\equiv \theta/2\pi \\ \bmod \boldsymbol{Z}}} \mathfrak{a}_k\ .$$

By Borel-Hirzebruch [4], we may identify $H^{**}(BT; \boldsymbol{R}) = H^{**}(E/T; \boldsymbol{R})$ with the completion of the symmetric tensor algebra $S^{**}(\mathfrak{h}^*)$. We denote by $[a_k] \in H^2(E/T; \boldsymbol{R})$ the corresponding class to $a_k \in \mathfrak{h}^*$. Then the equivariant total Chern classes are written by:

$$c_T(\tau_0(Z_G(h_j)_0/T)) = \prod_{k;\ a_k(H_j)\equiv 0} (1 + [a_k]) \in H^{**}(E/T; \boldsymbol{R})\ ,$$
$$c_T(\tau_0^\theta(G/Z_G(h_j)_0)) = \prod_{k;\ a_k(H_j)\equiv \theta/2\pi} (1 + [a_k]) \in H^{**}(E/T; \boldsymbol{R})\ .$$

Hence we have:

$$\begin{aligned}
&\mathscr{T}_T(\tau_0(Z_G(h_j)_0/T)) \prod_{0<\theta<2\pi} \mathscr{T}_T^\theta(\tau_0^\theta(G/Z_G(h_j)_0)) \\
&= \prod_{k;\ a_k(H_j)\equiv 0} \frac{[a_k]}{1 - e^{-[a_k]}} \prod_{0<\theta<2\pi} \prod_{k;\ a_k(H_j)\equiv\theta/2\pi} \frac{1}{1 - e^{-[a_k]-i\theta}} \\
&= \left\{\prod_{k;\ a_k(H_j)\equiv 0} [a_k]\right\} \left\{\prod_{k=1}^m \frac{1}{1 - e^{-[a_k]-2\pi i a_k(H_j)}}\right\}\ .
\end{aligned}$$

By Borel-Hirzebruch [5], we can compute the Gysin homomorphism $(\pi_j)_!$. We remark that $\{a_k \mid a_k(H_j) \equiv 0 \bmod \boldsymbol{Z}\}$ are the positive roots of $Z_G(h_j)_0$. Then we have:

$$\begin{aligned}
&\left\{\prod_{k;\ a_k(H_j)\equiv 0} [a_k]\right\} (\pi_j)_! \left\{\mathscr{T}_T(\tau_0(Z_G(h_j)_0/T)) \prod_{0<\theta<2\pi} \mathscr{T}_T^\theta(\tau_0^\theta(G/Z_G(h_j)_0))\right\} \\
&= \sum_{w\in W(Z_G(h_j)_0)} \mathrm{sgn}\,(w) \left\{\prod_{k;\ a_k(H_j)\equiv 0} [wa_k]\right\} \left\{\prod_{k=1}^m \frac{1}{1 - e^{-[wa_k]-2\pi i a_k(H_j)}}\right\}\ .
\end{aligned}$$

For $w \in W(Z_G(h_j)_0)$, we have:

$$\mathrm{sgn}\,(w) \left\{\prod_{k;\ a_k(H_j)\equiv 0} [wa_k]\right\} = \prod_{k;\ a_k(H_j)\equiv 0} [a_k]\ ,$$
$$wa_k(H_j) = a_k(w^{-1}H_j) = a_k(H_j) \qquad (k = 1, 2, \cdots, m)\ .$$

Hence we have:

$$(\pi_j)_! \left\{ \mathcal{T}_T(\tau_0(Z_G(h_j)_0/T)) \prod_{0<\theta<2\pi} \mathcal{T}_T^\theta(\tau_0^\theta(G/Z_G(h_i)_0)) \right\}$$
$$= \sum_{w \in W(Z_G(h_j)_0)} \prod_{k=1}^m \frac{1}{1 - e^{-[wa_k] - 2\pi i w a_k(H_j)}} .$$

The conjugation $\iota_{g_j}: E/Z_G(h)_0 \to E/Z_G(h_j)_0$ is covered by the map $\iota_{g_j}: E/T \to E/T$. So, in cohomology, $\iota_{g_j}^*$ is given by the action of the element $w_j^{-1} \in W(G)$. Then we have:

$$\pi'_! \mathcal{T}_G^h(G/T) = \sum_j \iota_{g_j}^*(\pi_j)_! \left\{ \mathcal{T}_T(\tau_0(Z_G(h_j)_0/T)) \prod_\theta \mathcal{T}_T^\theta(\tau_0^\theta(G/Z_G(h_j)_0)) \right\}$$
$$= \sum_{\substack{[w_j] \in W(G)/W(Z_G(h)_0) \\ w \in W(Z_G(h_j)_0)}} \prod_{k=1}^m \frac{1}{1 - e^{-[w_j^{-1} w a_k] - 2\pi i w a_k(H_j)}} .$$

Here, $wa_k(H_j) = wa_k(w_j H) = w_j^{-1} w a_k(H)$ and in summation $w_j^{-1} w$ move just all over $W(G)$. Hence:

$$\pi'_! \mathcal{T}_G^h(G/T) = \sum_{w \in W(G)} \prod_{k=1}^m \frac{1}{1 - e^{-[wa_k] - 2\pi i w a_k(H)}} .$$

Recall the Weyl's relation that was used in Borel-Hirzebruch [4]. That is, as a function in $X \in \mathfrak{h}$, we have:

$$\sum_{w \in W(G)} \prod_{k=1}^m \frac{1}{1 - e^{-wa_k(X)}} \equiv 1 .$$

Replace X by $X + 2\pi i H$ and we get:

$$\sum_{w \in W(G)} \prod_{k=1}^m \frac{1}{1 - e^{-wa_k(X) - 2\pi i w a_k(H)}} \equiv 1 .$$

The formal power series expansion of this expression gives a relation in $S^{**}(\mathfrak{h}^*) \otimes \boldsymbol{C} = H^{**}(E/T; \boldsymbol{C})$. This shows:

$$\pi_! \mathcal{T}_G^h(G/T) = 1 \in H^{**}(E/Z_G(h); \boldsymbol{C}) \subset H^{**}(E/T; \boldsymbol{C}) .$$

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