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BOUNDARY BEHAVIOR OF POSITIVE HARMONIC FUNCTIONS IN BALLS OF R^n

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§ 1. Introduction

Let R^n be the real n-dimensional euclidean space. Elements of R^n are denoted by $x = (x_1, \cdots, x_n)$, and $\|x\|$ denotes the euclidean norm of x. The open ball $B(x, r)$ with center x and radius r is defined by

$$B(x, r) := \{y : \|y - x\| < r\}$$

and the sphere $S(x, r)$ is defined by

$$S(x, r) := \{y : \|y - x\| = r\} .$$

In particular $B := B(0, 1)$ is the unit ball and $S := S(0, 1)$ is the unit sphere.

Let u be a positive harmonic function in B. Then by the Herglotz Theorem ([2], p. 29) there exists a positive Borel measure μ on the unit sphere S such that

$$u(z) = \frac{1}{\sigma_n} \int_S P(z, x) d\mu(x) \qquad (x \in S)$$

holds for all $z \in B$. $P(z, x)$ is the Poisson kernel for B defined by

$$P(z, x) := \frac{1 - \|z\|^2}{\|z - x\|^n}$$

and σ_n is the surface area of S.

Now the question arises of the relationship between the limiting behavior of $u(z)$ as z approaches a boundary point and the measure μ on the boundary. To study this question we define the open polar cap $J(a, r)$ having center $a \in S$ and radius r by

$$J(a, r) := \{x \in S : \|x - a\| < r\}$$

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and the symmetric derivative $(D\mu)(a)$ of the measure μ at a surface point a by

$$(D\mu)(a) := \lim_{r\to 0} \frac{\mu}{m}[J(a, r)]$$

where

$$\frac{\mu}{m}[J(a, r)] := \frac{\mu[J(a, r)]}{m[J(a, r)]}$$

and m is Lebesgue measure on the sphere. We note that $\gamma_n' r^{n-1} \leqq m[J(a, r)] \leqq \gamma_n r^{n-1}$ for $0 \leqq r < 1$, where γ_n, γ_n' are constants depending only on the dimension n. The first main result regarding the limiting behavior of u is the following, which is due to Fatou (1906) in case $n = 2$ and to Bray and Evans (1927) for $n = 3$.

FATOU'S THEOREM. *Let u be a positive harmonic function in the open unit ball of $\boldsymbol{R}^n$ having measure μ in its Herglotz representation. If $(D\mu)(a)$ exists, then the radial limit $\lim_{r\to 1} u(ra)$ exists and is equal to $(D\mu)(a)$.*

Remark. The Theorem of Fatou also holds in the case $(D\mu)(a) = +\infty$. We indicate a short proof of this fact. Let $x \in J(a, 1 - r)$. Using the triangle inequality we get

$$\|ra - x\| \leqq \|ra - a\| + \|a - x\| \leqq 2(1 - r) .$$

From this it follows that

$$\begin{aligned} u(ra) &= \frac{1}{\sigma_n}\int_S \frac{1 - r^2}{\|ra - x\|^n} d\mu(x) \geqq \frac{1}{\sigma_n}\int_{J(a,1-r)} \frac{1 - r}{2^n(1 - r)^n} d\mu(x) \\ &= \frac{1}{2^n\sigma_n}\frac{\mu[J(a, 1 - r)]}{(1 - r)^{n-1}} \geqq \frac{\gamma_n'}{2^n\sigma_n}\frac{\mu}{m}[J(a, 1 - r)] \to +\infty \end{aligned}$$

for $r \to 1$, since $(D\mu)(a) = +\infty$.

Fatou's Theorem also holds, if z approaches the point a in a non-tangential manner (in a Stolz domain at a), see e.g. [2], p. 55. But if we replace Stolz domains by more general regions the situation changes dramatically. In this case the boundary behavior of u can be very erratic. To see this, let us introduce regions $R(a, \delta, \gamma)$ in B touching the unit sphere at a.

DEFINITION. For $a \in S$, $\delta > 0$, $\gamma \geqq 1$ let

$$R(a, \delta, \gamma) := \{z \in B: 1 - \|z\| \geqq \delta\, \|z' - a\|^\gamma\}\backslash\{a\} ,$$

where z' is the radial projection of z on the sphere, i.e. $z' = z/\|z\|$ if $z \neq 0$ and we define $z' = a$ if $z = 0$.

Evidently the radius $\{ra: 0 \leqq r < 1\}$ lies in $R(a, \delta, \gamma)$ for any $\delta > 0$, $\gamma \geqq 1$. If $\gamma = 1$ the region $R(a, \delta, \gamma)$ is essentially a Stolz domain at a, i.e. a cone with vertex at a and aperture depending on δ. As γ increases, $R(a, \delta, \gamma)$ touches the unit sphere with an increasing degree of tangency, e.g. for $\gamma = 2$ and $n = 2$, $R(a, \delta, \gamma)$ is essentially the interior of an oricycle at a. In two dimensions the regions $R(a, \delta, \gamma)$ were introduced by Cargo [1] to study tangential limits of Blaschke products.

§ 2. The distinction between angular and tangential boundary behavior

We will now make clear the difference between $\gamma = 1$ and $\gamma > 1$ regarding the boundary behavior of u. For simplicity we choose the dimension $n = 2$, the boundary point $a = 1$ and $\delta = 1$. Let $\gamma > 1$ be given. We construct a positive harmonic function u, such that $u(z_n) \to 0$ for every sequence $z_n \in R(a, \delta, 1)$, $z_n \to a$, but for any number $c \geqq 0$ (including $c = +\infty$) there exists a sequence $z_n \in R(a, \delta, \gamma)$, $z_n \to a$ with $u(z_n) \to c$. In other words, the partial cluster sets of u on $R(a, \delta, 1)$ or $R(a, \delta, \gamma)$ respectively consist of only one point 0 or is the whole interval $[0, \infty]$ respectively. One should note that the simple example of a harmonic function h given in Helms ([2], p. 54) does not work here, since the partial cluster set of h on $R(a, \delta, \gamma)$ consists of exactly one point in all cases $1 < \gamma < 2$.

Construction of the measure. We choose the discrete singular measure

$$\mu = \sum_{k=1}^{\infty} s_k \delta_{(a_k)}$$

with $a_k = \exp(it_k)$, $t_k = 2^{-k}$, $s_k = t_k^{\beta}$, $1 < \beta < \gamma$. $\delta_{(a_k)}$ is the Dirac measure associated with the point a_k. Let u be the positive harmonic function with this measure μ in its Herglotz representation.

1. *Case.* Let $0 < r < \frac{1}{2}$, then there exists an index $n \in N$ with

$$t_{n+1} < r \leqq t_n .$$

This implies

$$0 \leqq \frac{\mu}{m}[J(a, r)] \leqq \frac{1}{2t_{n+1}} \mu[J(a, t_n)] .$$

A short calculation yields

$$\frac{1}{2t_{n+1}}\mu[J(a, t_n)] = \frac{1}{2t_{n+1}}\sum_{k=n}^{\infty} s_k < 2t_n^{\beta-1} \to 0 \qquad (n \to \infty) .$$

Therefore $(D\mu)(a) = 0$ and it follows by Fatou's Theorem $u(z_n) \to 0$ for any sequence $z_n \in R(a, \delta, 1)$, $z_n \to a$.

2. *Case.* We choose points $z_n = r_n e^{it_n}$ with $1 - r_n = t_n^\gamma$. Thus $z_n \in R(a, \delta, \gamma)$ for all $n \in N$. Let I_n be the interval around t_n defined by

$$I_n := \{e^{it} : |t - t_n| < t_n^\gamma\} .$$

For n sufficiently large, i.e. $n > (\gamma - 1)^{-1}$, the only point t_k belonging to I_n is the point t_n. Therefore $\mu(I_n) = s_n$. By standard estimations we obtain

$$\begin{aligned} u(z_n) &\geqq \int_S \frac{1 - r_n}{(1 - r_n)^2 + (t_n - t)^2} d\mu \geqq \int_{I_n} \frac{t_n^\gamma}{t_n^{2\gamma} + (t_n - t)^2} d\mu \\ &\geqq \frac{1}{2t_n^\gamma}\int_{I_n} d\mu = \frac{1}{2}t_n^{\beta-\gamma} \to \infty \qquad (n \to \infty) . \end{aligned}$$

Thus $u(z_n) \to \infty$ $(n \to \infty)$.

For $n \in N$ let $A_n := B(a, 1/n) \cap R(a, \delta, \gamma)$. Since u is continuous on A_n and A_n is a connected set, $u(A_n)$ is also connected. Since $u(A_n) \subset \boldsymbol{R}$ and the only connected subsets of $\boldsymbol{R}$ are the intervals, $u(A_n)$ must be an interval. From the construction above we see that $u(A_n) = (0, \infty)$ for every $n \in N$. Therefore the partial cluster set of u on $R(a, \delta, \gamma)$ is $\bigcap_{n=1}^{\infty} \overline{u(A_n)} = [0, \infty]$.

§ 3. The problem

Let u be a positive harmonic function in B represented by the measure μ on S and let $a \in S$. We are interested in the behavior of u in the region $R(a, \delta, \gamma)$. Of course, this depends on the measure μ on S and especially on the behavior of μ in a neighbourhood of a. In view of the possibly erratic limiting behavior, our main problem is to give a condition on the measure μ such that u is bounded in $R(a, \delta, \gamma)$. An obvious necessary condition is that μ be continuous at the point a, i.e. $\mu(\{a\}) = 0$, and an obvious sufficient condition is that there exist $r > 0$ such that $\mu[J(a, r)] = 0$. But there are much weaker conditions, e.g. one can show that the condition

$$\int_S \frac{d\mu(x)}{\|a - x\|^{\gamma(n-1)}} < \infty$$

is sufficient. Let us give an interpretation of this condition. Take the neighbourhood $J(a, r)$ of the point a. Then the condition says that $\mu[J(a, r)]$ tends to zero with some speed as $r \to 0$. It is not hard to show that the condition above is not necessary. In order to get a necessary and sufficient condition we introduce the following maximal function.

DEFINITION. For a finite positive Borel measure μ on S and a region $R = R(a, \delta, \gamma)$, we define the real function $M = M(R, \mu, x)$ on S and the number $N = N(R, \mu)$ by

$$M(R, \mu, x) := \sup \left\{ \frac{\mu}{m}[J(x, r)] : r > \delta \|x - a\|^{\gamma} \right\} ,$$

$$N(R, \mu) := \sup \{ M(R, \mu, z') : z \in R(a, \delta, \gamma) \} ,$$

where z' is the radial projection of z onto the sphere S (with the agreement $z' = a$ if $z = 0$).

Remark. In the special case $n = 2$, $\gamma = 1$ and μ absolutely continuous, $M(R, \mu, x)$ is the Hardy-Littlewood maximal function.

Some auxiliary results will be given in the next section.

§ 4. Auxiliary results

LEMMA 1. *There exist positive constants C_1 and C_2, depending only on the dimension n, such that for all $z \in B$, $x \in S$ the Poisson kernel can be estimated as follows:*

$$C_1 \frac{1 - \|z\|}{(1 - \|z\|)^n + \|x - z'\|^n} \leqq P(z, x) \leqq C_2 \frac{1 - \|z\|}{(1 - \|z\|)^n + \|x - z'\|^n} ,$$

where z' is the radial projection of z onto S. In case of $z = 0$ the inequality holds for any $z' \in S$. One can choose $C_1 = 1/n$ and $C_2 = 2 \cdot 3^n$.

Proof. We note that $\|z - z'\| = 1 - \|z\| = \operatorname{dist}(z, S)$ and

$$(1) \qquad 1 - \|z\| \leqq \|x - z\| .$$

The triangle inequality implies

$$(2) \qquad \|x - z\| \leqq \|x - z'\| + (1 - \|z\|)$$

and

$$(3) \qquad \|x - z'\| \leqq \|x - z\| + (1 - \|z\|) \leqq 2\|x - z\| .$$

Now (1), (2), (3) imply

$$\tfrac{1}{3}\{\|x - z'\| + (1 - \|z\|)\} \leqq \|x - z\| \leqq \{\|x - z'\| + (1 - \|z\|)\} .$$

After exponentiating and further estimating we obtain

$$(\tfrac{1}{3})^n\{\|x - z'\|^n + (1 - \|z\|)^n\} \leqq \|x - z\|^n \leqq n\{\|x - z'\|^n + (1 - \|z\|)^n\} ,$$

and from this the result follows.

LEMMA 2. *If u is a positive harmonic function in B with associated measure μ in the Herglotz representation, then*

$$u(z) \leqq CM(R, \mu, z')$$

holds for all $z \in R(a, \delta, \gamma)$. C is a constant depending only on the dimension n.

Remark. In the special case of a Stolz domain, i.e. $\gamma = 1$, and for μ absolutely continuous, the estimate in Lemma 2 is essentially known. See E. Stein ([3], p. 62, Theorem 1a) for an analogous n-dimensional statement.

Proof. Fix a point $z \in R(a, \delta, \gamma)$. We decompose the sphere S in a union of subsets S_k depending on z. Let $S_k = S_k(z)$ be defined as follows:

$$S_0 := \{x \in S : \|x - z'\| < 1 - \|z\|\} ,$$
$$S_k := \{x \in S : 2^{k-1}(1 - \|z\|) \leqq \|x - z'\| < 2^k(1 - \|z\|)\} ,$$

where $k = 1, 2, \cdots$. For $k = 0, 1, 2, \cdots$ let $I_k := \bigcup_{1=0}^{k} S_1$. We note that I_k is the open polar cap $J[z', 2^k(1 - \|z\|)$ of radius $2^k(1 - \|z\|)$. Let p be the smallest integer k with $2^k(1 - \|z\|) \geqq 1$. We put $I_p = S$. Therefore,

$$m(I_k) \leqq \gamma_n 2^{k(n-1)}(1 - \|z\|)^{n-1} \qquad (k < p) .$$

Using Lemmas 1 and 2 we can estimate as follows:

$$\begin{aligned}
\int_S P(z, x)\, d\mu(x) &\leqq C_2 \int_S \frac{1 - \|z\|}{(1 - \|z\|)^n + \|x - z'\|^n} d\mu(x) \\
&\leqq C_2 \left\{ \int_{S_0} \frac{1 - \|z\|}{(1 - \|z\|)^n} d\mu(x) + \sum_{k=1}^{p} \int_{S_k} \frac{1 - \|z\|}{\|x - z'\|^n} d\mu(x) \right\} \\
&\leqq C_2 \left\{ \frac{\mu(I_0)}{(1 - \|z\|)^{n-1}} + \sum_{k=1}^{p} \frac{\mu(I_k)}{2^{n(k-1)}(1 - \|z\|)^{n-1}} \right\} \\
&\leqq C_2 \gamma_n \left\{ \frac{\mu(I_0)}{m(I_0)} + \sum_{k=1}^{p} \frac{2^n \mu(I_k)}{2^k m(I_k)} \right\} \\
&\leqq C_2 \gamma_n \left(1 + 2^n \sum_{k=1}^{p} \frac{1}{2^k}\right) \sup \left\{ \frac{\mu}{m}[J(z', r)] : r > (1 - \|z\|) \right\}
\end{aligned}$$

$$\leqq C_2\gamma_n(1+2^n)M(R, \mu, z') .$$

This yields $u(z) \leqq CM(R, \mu, z')$ with $C = C_2\gamma_n(1+2^n)/\sigma_n$. Since the constant C is independent of the special point $z \in R(a, \delta, \gamma)$, the inequality holds for all $z \in R(a, \delta, \gamma)$.

§ 5. The main result

THEOREM. *A positive harmonic function u in B with measure μ in its Herglotz representation is bounded in the region $R(a, \delta, \gamma)$ if and only if $N(R, \mu) < \infty$.*

Proof. One direction follows from Lemma 1. To prove the converse, assume that u is bounded in $R(a, \delta, \gamma)$. Then there exists an absolute constant M, such that

$$\int_S P(z, x)d\mu(x) \leqq M \tag{1}$$

holds for all $z \in R(a, \delta, \gamma)$. To prove $N(R, \mu) < \infty$ we have to show that there exists an absolute constant C such that

$$\frac{\mu}{m}[J(z', r)] \leqq C \tag{2}$$

holds for all $z \in R(a, \delta, \gamma)$ and all $r > \delta \|z' - a\|^\gamma$. It is clear that we may assume $r < 1$. Let such a pair z, r be given. We choose a special point $z_r = (1-r)z'$. It follows that $\|z_r\| = 1 - r < 1$, i.e. $z \in B$ and $1 - \|z_r\| = r$. Since $r > \delta \|z' - a\|^\gamma$ we have $z_r \in R(a, \delta, \gamma)$, i.e. $1 - \|z_r\| \geqq \delta \|z' - a\|^\gamma$. Using our assumption (1) for the point z_r and Lemma 1 we obtain

$$\begin{aligned} M &\geqq \int_S P(z_r, x)d\mu(x) \geqq C_1 \int_S \frac{1 - \|z_r\|}{(1 - \|z_r\|)^n + \|x - z_r'\|^n} d\mu(x) \\ &\geqq C_1 \int_{\|x - z_r'\| < r} \frac{r}{r^n + r^n} d\mu(x) = \frac{1}{2}\frac{C_1}{r^{n-1}}\mu[J(z_r', r)] \\ &\geqq \frac{1}{2} C_1\gamma_n' \frac{\mu}{m}[J(z', r)] . \end{aligned}$$

Note that $z' = z_r'$. Thus we have

$$\frac{\mu}{m}[J(z', r)] \leqq C$$

with the constant $C = 2M/C_1\gamma_n'$, which is independent of z and r. Therefore we have established (2).

References

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