SOME REMARKS CONCERNING DEMAZURE'S CONSTRUCTION OF NORMAL GRADED RINGS

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Introduction

In [1], Demazure showed a new way of constructing normal graded rings using the concept of "rational coefficient Weil divisors" of normal projective varieties and he showed, among other things, the following

THEOREM ([1], 3.5). *If* $R = \bigoplus_{n \geq 0} R_n$ *is a normal graded ring of finite type over a field k and if T is a homogeneous element of degree 1 in the quotient field of R, then there exists unique divisor $D \in \mathrm{Div}\,(X, Q)$ ($X = \mathrm{Proj}\,(R)$), such that $R_n = H^0(X, \mathcal{O}_x(nD)) \cdot T^n$ for every $n \geq 0$. (See (1.1) for the definition of $\mathrm{Div}\,(X, Q)$ and $\mathcal{O}_x(nD)$.)*

Let us denote the ring R above by $R = R(X, D)$. In this note we want to consider the following problems concerning $R = R(X, D)$.

(1) What is the depth of R? In particular, when is R a Macaulay ring or a Gorenstein ring?

(2) When is R a rational singularity?

The paper is divided into three sections. In § 1, we calculate the divisor class group of R. Although the contents of this section are included implicitly in [1], we need to state the results explicitly to define the canonical class cl (K_R) of R in § 2.

In § 2, we seek the condition for R to be a Macaulay ring or a Gorenstein ring. First, we express the local cohomology groups of R by the cohomology groups of $\mathcal{O}_x(nD)$ ($n \in Z$). Then, using Grothendieck duality, we calculate the canonical class cl (K_R) of R and, in particular, we can find the condition for R to be a Gorenstein ring.

In § 3, we establish a criterion for R to be a rational singularity when X is smooth and $\mathrm{Supp}\,(D - \lfloor D \rfloor)$ has only normal crossings as its singularity. (See (1.1) for the definition of $\lfloor D \rfloor$.) This criterion gives us very

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abundant examples of rational singularities and that will be of some in-
terest as there are not many examples of rational singularities known to
us, yet.

§1. The divisor class group of $R(X, D)$

Throughout this paper, we shall use the following notations.

NOTATION (1.1). k is a fixed field.

X is a normal irreducible projective scheme over k. We assume dim
$X \geqq 1$.

$k(X)$ is the rational function field of X.

$\mathrm{Irr}^1(X)$ is the set of irreducible subvarieties of codimension 1 of X.

$\mathrm{Div}(X)$ is the group of Weil divisors of X.

$\mathrm{Div}(X, \mathbf{Q}) = \mathrm{Div}(X) \otimes_Z \mathbf{Q}$ is the group of "rational coefficient Weil
divisors on X". If $E = \sum r_V \cdot V$ and $E' = \sum r'_V \cdot V$ are elements of $\mathrm{Div}(X, \mathbf{Q})$,
$E \geqq E'$ means that $r_V \geqq r'_V$ for every $V \in \mathrm{Irr}^1(X)$. We write

$$\lfloor E \rfloor = \sup \{Z \in \mathrm{Div}(X) \,|\, Z \leqq E\} = \sum [r_V] \cdot V$$

(where $[r]$ is the largest integer not larger than r for $r \in \mathbf{Q}$).

$\mathcal{O}_X(E) = \mathcal{O}(\lfloor E \rfloor)$ for $E \in \mathrm{Div}(X, \mathbf{Q})$. We consider $\mathcal{O}_X(E)$ as a subsheaf of
the constant sheaf $k(X)$.

$D = \sum p_V/q_V \cdot V$ is a fixed element of $\mathrm{Div}(X, \mathbf{Q})$ (where $p_V, q_V \in \mathbf{Z}$, $q_V > 0$
and $(p_V, q_V) = 1$ for every $V \in \mathrm{Irr}^1(X)$) satisfying the condition;

(A) There is a positive integer N such that ND is an ample Cartier
divisor.

Throughout this paper, we shall use the letters D, N, p_V and q_V in
this sense.

$R = R(X, D) = \bigoplus_{n \geqq 0} H^0(X, \mathcal{O}_X(nD)) \cdot T^n \subset k(X)[T]$ (T is an indeterminate).

$\mathfrak{m} = R_+ = \bigoplus_{n > 0} H^0(X, \mathcal{O}_X(nD)) \cdot T^n$.

$C = C(X, D) = \mathrm{Spec}_X(\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(nD) \cdot T^n)$ and $C^+ = \mathrm{Spec}_X(\bigoplus_{n \geqq 0} \mathcal{O}_X(nD)$
$\cdot T^n)$. Note that C is an open subscheme of C^+. We put $S^+ = C^+ - C$.
C and C^+ have the natural G_m-actions induced by the gradings. We have
the natural homomorphism $\Psi: C^+ \to \mathrm{Spec}(R)$ which maps C isomorphically
onto $\mathrm{Spec}(R) - \{\mathfrak{m}\}$ and contracts S^+ to the point $\mathfrak{m}$. Note that Ψ is a
projective morphism. In fact, C^+ is isomorphic to $\mathrm{Proj}(R^\natural)$ in the notation
of E.G.A. Chapter II, 8.2.

$\pi: C \to X$ and $\pi^+: C^+ \to X$ be the canonical projections.

$F_V = \pi^{-1}(V)_{\mathrm{red}} \in \mathrm{Irr}^1(C)$ for $V \in \mathrm{Irr}^1(X)$.

$\mathrm{Cl}(Y)$ (resp. $\mathrm{Cl}(R)$) is the divisor class group of a normal variety Y (resp. of R).

$P(Y)$ (resp. $P(R)$) is the group of principal divisors of Y (resp. of R).

Let us put the letter H to show that something on R or on C is stable under the G_m-action. For example, $H\mathrm{Div}(R)$ is the subgroup of $\mathrm{Div}(R)$ consisting of the homogeneous divisors of R and $H\mathrm{Irr}^1(C)$ is the set of irreducible subsets of codimension 1 on C which are stable under the G_m-action on C.

Now, we will recall some facts to calculate $\mathrm{Cl}(R)$.

(1.2) As $C \simeq \mathrm{Spec}(R) - \{\mathfrak{m}\}$ and as $\dim R \geqq 2$ ($\dim R = \dim X + 1$), $\mathrm{Cl}(R) \simeq \mathrm{Cl}(C)$. Also, we have $\mathrm{Cl}(R) \simeq H\mathrm{Div}(R)/HP(R)$ ([7], Proposition 7.1).

(1.3) ([1], 2.6 and 2.8) There is a natural bijection between $\mathrm{Irr}^1(X)$ and $H\mathrm{Irr}^1(C)$ given by $V \to F_V$. The mapping $\pi^* : \mathrm{Div}(X) \to \mathrm{Div}(C)$ is given by $\pi^*(V) = q_V \cdot F_V$ for $V \in \mathrm{Irr}^1(X)$. In particular, $\pi^*(D) = \sum p_V \cdot F_V \in \mathrm{Div}(C)$.

By π^*, we can identify the group $H\mathrm{Div}(C)$ with the subgroup

$\mathrm{Div}(X, D) = \{\sum r_V \cdot V \in \mathrm{Div}(X, Q) \,|\, q_V r_V \in Z \text{ for every } V \in \mathrm{Irr}^1(X)\}$ of $\mathrm{Div}(X, Q)$. Note that the bijection $\mathrm{Div}(X, D) \to H\mathrm{Div}(R)$ is given by $E \to \bigoplus_{n \in Z} H^0(X, \mathcal{O}_X(E + nD)) \cdot T^n$.

(1.4) If we denote the quotient field of R by $Q(R)$, $Q(R) = k(C) = k(X)(T)$. Every homogeneous element of $Q(R)$ can be written in the form $f \cdot T^n$, where $f \in k(X)$ and $n \in Z$.

(1.5) ([1], 2.9) $\mathrm{div}(T) = \pi^*(D) = \sum p_V \cdot V$ in $\mathrm{Div}(C)$.

After these observations, we can easily get the following

THEOREM (1.6). *There is an exact sequence*

$$0 \longrightarrow Z \overset{\theta}{\longrightarrow} \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(R) \longrightarrow \mathrm{Coker}(\alpha) \longrightarrow 0$$

where θ is given by $\theta(1) = LD$ ($L = LCM\{q_V \,|\, V \in \mathrm{Irr}^1(X)\}$) and α is the homomorphism $Z \to \bigoplus_V Z/q_V Z$ given by $\alpha(1) = (p_V \,(\mathrm{mod}\, q_V))_V$.

Proof. Look at the following commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 \longrightarrow & P(X) & \longrightarrow & \mathrm{Div}\,(X) & \longrightarrow & \mathrm{Cl}\,(X) & \longrightarrow 0 \\
& \downarrow & & \downarrow{\scriptstyle \pi^*} & & \downarrow & & \\
0 \longrightarrow & HP(R) \longrightarrow & H\,\mathrm{Div}\,(R) = \mathrm{Div}\,(X, D) & \longrightarrow & \mathrm{Cl}\,(R) & \longrightarrow 0 \\
& \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\
& I & \xrightarrow{\ \alpha\ } & \oplus_V Z/q_V Z & \longrightarrow & \mathrm{Coker}\,(\alpha) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
& 0 & & 0 & & 0 & &
\end{array}
$$

where the rows and columns are exact. The group $I \simeq Z$ is generated by div (T). As $\alpha\,(\mathrm{div}\,(T)) = \beta(\pi^*(D))$, $\mathrm{Ker}\,(\alpha) \simeq Z$ and is generated by LD.

COROLLARY (1.7). *R is factorial if and only if* $\mathrm{Cl}\,(X)$ *is generated by* LD *and* q_V's *are pairwise coprime*.

Remark. If k is algebraically closed, we can classify all factorial graded rings of dimension 2 by this method since the only normal projective curve X with $\mathrm{Cl}\,(X) \simeq Z$ is P^1. As the result, we rediscover Theorem 5.1 of Mori [5].

§2. The local cohomology groups and the canonical module of $R(X, D)$

First, we note the following fact.

LEMMA (2.1). *There is a canonical isomorphism* $\widetilde{R(n)} \simeq \mathcal{O}_X(nD)$ *on* $X = \mathrm{Proj}\,(R)$ *for every* $n \in Z$.

Proof. By the assumption (A), we can assume that $\mathcal{O}_X(mD)$ is generated by its global sections for every sufficiently large m. Let us take $f \in R_m$ and $g \in R_{m+n}$. If m is a multiple of N, mD is an ample Cartier divisor and g/f is a section of $\mathcal{O}_X(nD)$ on the open set $D_+(f)$ of $X = \mathrm{Proj}\,(R)$. ($D_+(f)$ is the standard notation used in E.G.A. Chapter 2.) If m is not a multiple of N, we can take some $f' \in R_{m'}$ such that $m + m'$ is a multiple of N. As $g/f = gf'/ff'$ on $D_+(ff')$ and as $D_+(ff')$ covers $D_+(f)$ when f' varies, we can reduce to the case when m is a multiple of N. Thus we have the natural homomorphism $\widetilde{R(n)} \to \mathcal{O}_X(nD)$. As $\mathcal{O}_X(mD)$ and $\mathcal{O}_X((m + n)D)$ are

generated by global sections for sufficiently large m, this map is surjective and the injectivity is obvious.

PROPOSITION (2.2). *There is a canonical isomorphism of graded R-modules*

$$H_{\mathfrak{m}}^p(R) \cong \bigoplus_{n \in Z} H^{p-1}(X, \mathcal{O}_X(nD)) \quad (p \geqq 2) .$$

Proof. It is shown in [3], § 5 that there is a canonical isomorphism

$$H_{\mathfrak{m}}^p(R) \cong \bigoplus_{n \in Z} H^{p-1}(X, \widetilde{R(n)}) \quad (p \geqq 2) .$$

And by (2.1), $\widetilde{R(n)} \cong \mathcal{O}_X(nD)$ for every $n \in Z$.

COROLLARY (2.3). depth $R = p + 1$, *where p is the minimal positive integer such that $H^p(X, \mathcal{O}_X(nD)) \neq 0$ for some $n \in Z$.*

COROLLARY (2.4). *$R(X, D)$ is a Macaulay ring if and only if $H^p(X, \mathcal{O}_X(nD))$ $= 0$ for $1 \leqq p < \dim X$ and for every $n \in Z$.*

EXAMPLE (2.5). In the following cases, $R(X, D)$ is a Macaulay ring for every $D \in \mathrm{Div}(X, Q)$ satisfying the condition (A).

(a) X is a curve.

(b) X is a projective space or a Grassmann variety.

(c) X is a smooth complete intersection in a projective space and $\dim X \geqq 3$.

EXAMPLE (2.6). If X is a rational ruled surface and if D is an ample divisor on X, it is known that $R(X, D)$ is a Macaulay ring. But for $D \in$ $\mathrm{Div}(X, Q)$ satisfying the condition (A), this is no longer true. For example, if $X = P^1 \times P^1$ and $D = 1/2.\Delta - 1/5.F_1 - 1/5.F_2$, where Δ is the diagonal and F_1 and F_2 are fibres of the first projection, then $H^1(X, \mathcal{O}_X(D))$ $= H^1(X, \mathcal{O}_X(-F_1 - F_2)) \neq 0$, while $10.D = 5\Delta - 2F_1 - 2F_2$ is an ample Cartier divisor.

Now we will calculate the canonical class $\mathrm{cl}(K_R)$ of R. Recall that K_R is defined by $K_R = (H_{\mathfrak{m}}^d(R))^*$, where $d = \dim R = \dim X + 1$. See [2], (2.1.2) and (1.2) for the definition of the functor $(\ )^*$ and K_R.

First, we recall the following fact.

LEMMA (2.7). *If Y is a normal irreducible projective variety over k and if $E \in \mathrm{Div}(Y)$, then we have the nonsingular pairing*

$$H^n(Y, \mathcal{O}_Y(E)) \times H^0(Y, \mathcal{O}_Y(K_Y - E)) \to k$$

where $d = \dim Y$ and $\mathcal{O}_Y(K_Y) = \omega_Y$ is the dualizing sheaf on Y.

Proof. By Grothendieck duality (cf. [8], Chapter I, (1.3)), $H^d(Y, \mathcal{O}_Y(E))$ is dual to $\mathrm{Ext}^0_{\mathcal{O}_Y}(\mathcal{O}_Y(E), \mathcal{O}_Y(K_Y)) = H^0(Y, \mathscr{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y(E), \mathcal{O}_Y(K_Y)))$ and it is easy to see that $\mathscr{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y(E), \mathcal{O}_Y(K_Y)) \cong \mathcal{O}_Y(K_Y - E)$.

THEOREM (2.8). *The canonical module K_R of R is given by*

$$K_R = \bigoplus_{n \in \mathbf{Z}} H^0(X, \mathcal{O}_X(K_X + D' + nD)),$$

where K_X is the canonical divisor of X and $D' = \sum_V (q_V - 1)/q_V$.

Proof. We have $(K_R)_n = (H^d_{\mathfrak{m}}(R))^*_n = (H^{d-1}(X, \mathcal{O}_X(-nD)))^* \cong H^0(X, \mathcal{O}_X(K_X - \lfloor -nD \rfloor)) = H^0(X, \mathcal{O}_X(K_X + D' + nD))$ by (2.2) and (2.7) (where ( )* means the dual vector space. Note that $-\lfloor -nD \rfloor$ is not equal to $\lfloor nD \rfloor$ but to $\lfloor nD + D' \rfloor$). As these isomorphisms of all degrees are compatible with the multiplication of homogeneous elements of R, we get the desired result.

COROLLARY (2.9). *If R is a Macaulay ring, R is a Gorenstein ring if and only if there is an integer $a = a(R)$ such that $K_X + D' - aD = \mathrm{div}(f)$ for some $f \in k(X)$.*

Proof. As R is a Macaulay normal domain, R is a Gorenstein ring if and only if $\mathrm{cl}(K_R) = 0$ in $\mathrm{Cl}(R)$. In the notation of (1.3) and (1.6), $K_X + D' \in \mathrm{Div}(X, D)$ and $\mathrm{cl}(K_R) = \mathrm{cl}(K_X + D')$ by (2.8). So, $\mathrm{cl}(K_R) = 0$ if and only if $K_X + D' \in \mathrm{Div}(X, D)$ is in the image of $HP(R)$. But if fT^a is a homogeneous element of $Q(R)$ ($f \in k(X)$), the image of $\mathrm{div}(fT^a)$ in $\mathrm{Div}(X, D)$ is $\mathrm{div}_X(f) + aD$ by (1.5).

Remark (2.10). In [2], (3.1.4), we have defined the invariant $a(R)$ of R by

$$a(R) = -\min\{m \mid (K_R)_m \neq 0\} = \max\{m \mid (H^d_{\mathfrak{m}}(R))_m \neq 0\}$$

and showed that if R is a Gorenstein ring, then $K_R = R(a(R))$. It is easy to show that the integer $a(R)$ in (2.9) coincides with this definition.

Remark (2.11). If R is a Macaulay ring, then X is a Macaulay scheme. In fact, if R is a Macaulay ring, so is the N-th Veronese subring $R^{(N)} = R(X, ND)$. As ND is an ample Cartier divisor, we can say that X is a Macaulay scheme by [2], (5.1.10). But even if R is a Gorenstein ring, X

need not be a Gorenstein scheme. For example, if $R = k[U, V, W]$, where we put $\deg(U) = \deg(V) = 1$ and $\deg(W) = n$, then $X = \text{Proj}(R)$ is a Gorenstein scheme if and only if $n = 1$ or 2.

§ 3. A criterion for rational singularities

In this section, we assume $\text{char}(k) = 0$ since the definition of rational singularities is not founded yet in positive characteristics. Recall that R is a rational singularity if for some (or, equivalently, every) resolution $\Phi: Y \to \text{Spec}(R)$, the higher direct images $R^p \Phi_*(\mathcal{O}_Y) = 0$ for all $p > 0$. We will say that a scheme has only rational singularities if every singularity of it is a rational singularity.

The key lemma of this section is the following.

LEMMA (3.1). *If X and D satisfy the following conditions, then the scheme $C^+ = C^+(X, D)$ has only rational singularities.*

(1) *X has only rational singularities.*

(2) *If $x \in X$ is a singular point, then D is a Cartier divisor on some neighborhood of x.*

(3) *If $x \in X$ is a smooth point, then there is a regular parameter system $(z, \cdots, z_d)$ of the local ring $\mathcal{O}_{X,x}$ such that $\text{Supp}(D - \lfloor D \rfloor)$ is defined by $z_1 \cdots z_t$ $(t \leq d)$ at x.*

Proof. First, notice that if E is a Cartier divisor on X, $C^+(X, D)$ has only rational singularities if and only if so does $C^+(X, D + E)$ as the problem is local with respect to X. If $\mathcal{O}_X|_U \cong \mathcal{O}_X(D)|_U$ for some open set U of X, then $C^+(X, D)|_U \cong U \times A^1$. Thus $C^+(X, D)$ has only rational singularities over some neighborhood of a singular point x of X by (1) and (2). If x is a smooth point of X, we may assume that $D = \sum_{i=1}^{t} p_i/q_i \cdot V_i$ near x, where $(p_i, q_i) = 1$, $0 < p_i < q_i$, $0 \leq t \leq d$ and the defining equation of V_i is z_i at x by (3). Then,

$$C^+(X, D) \times_x \text{Spec}(\mathcal{O}_{X,x}) = \text{Spec}(\mathcal{O}_{X,x}[z_1^{-a_1} \cdots z_t^{-a_t} \cdot T^n \mid np_i \geq a_i q_i$$
$$\text{for } i = 1, \cdots, t]) .$$

As the completion of this ring is isomorphic to that of a normal semigroup ring, this ring is known to be a rational singularity ([4], Chapter I, § 3).

Remark (3.2). If $\text{Supp}(D - \lfloor D \rfloor)$ has "bad" singularities, C^+ does have non-rational singularities even if X is smooth. For example, let $U = A^2(u, v)$ and $D = 1/q \cdot Z$, where Z is defined by $f(u, v) = 0$. Then $C^+ =$

$\mathrm{Spec}\,(k[u, v, T, f^{-1}T^q])$ is a hypersurface in $A^4(u, v, T, w)$ defined by $T^q - w \cdot f(u, v) = 0$, which is not a rational singularity if q and $\deg(f)$ are large.

THEOREM (3.3). *If X and D satisfy the conditions of (3.1) and if $\Phi: Y \to \mathrm{Spec}\,(R)$ is a resolution of $\mathrm{Spec}\,(R)$, then we have canonical isomorphisms*

$$R^p\Phi_*(\mathcal{O}_Y) \cong \bigoplus_{n \geq 0} H^p(X, \mathcal{O}_X(nD))$$

for all $p \geq 0$. In particular, R is a rational singularity if and only if R is a Macaulay ring and the invariant $a(R)$ is negative.

Proof. As $R^p\Phi_*(\mathcal{O}_Y)$ does not depend on the choice of the resolution Φ, we may assume that Φ is the composition of $\Theta: Y \to C^+$ and $\Psi: C^+ \to \mathrm{Spec}\,(R)$. As C^+ has only rational singularities, $R^p\Theta_*(\mathcal{O}_Y) = 0$ $(p > 0)$ and we have isomorphisms

$$R^p\Phi_*(\mathcal{O}_Y) \cong R^p\Psi_*(\mathcal{O}_+) \cong H^p(C^+, \mathcal{O}_{C^+}) \cong H^p(X, \bigoplus_{n \geq 0} \mathcal{O}_X(nD)) \cong \bigoplus_{n \geq 0} H^p(X, \mathcal{O}_X(nD))$$

by the definition of C^+.

EXAMPLE (3.4). If X is a curve, the conditions of (3.1) are always satisfied. So, R is a rational singularity if and only if $a(R) < 0$. This has been proved by Pinkham [6].

EXAMPLE (3.5). Let $X = P^d$ and H_i $(i = 1, \cdots, s)$ be hyperplanes of X in general position. If D is a rational coefficient linear combination of H_i's and satisfies the condition (A), X and D satisfy the conditions of (3.1) and $R(X, D)$ is a rational singularity if and only if $\deg(\lfloor nD \rfloor) \geq -d$ for every $n \geq 0$. For example, let p, q be positive integers with $(p, q) = 1$, $H, H_1, \cdots, H_p$ be hyperplanes in general position and $D = bH + \sum_{i=1}^p a/q \cdot H_i$, where a, b are integers which satisfy $ap + bq = 1$. Then $R = R(X, D) \cong k[S, T_0, \cdots, T_d]/(S^q - h_1 \cdots h_p)$, where $T_0, \cdots, T_d$ are homogeneous coordinates of P^d and $h_i = h_i(T_0, \cdots, T_d)$ is the equation of H_i $(i = 1, \cdots, p)$. As $\deg(S) = p$ and $\deg(T_i) = q$ $(i = 0, \cdots, d)$, $a(R) = pq - p - (d + 1)q$. So, R is a rational singularity if and only if $pq - p - (d + 1)q < 0$.

EXAMPLE (3.6). Let (X_1, D_1) and (X_2, D_2) be pairs of a variety and a divisor satisfying the conditions of (3.1). Then, if we put $(X, D) = (X_1 \times X_2, p_1^*(D_1) + p_2^*(D_2))$ (where $p_i: X \to X_i$ $(i = 1, 2)$ are the projections), then it is easy to see that $C^+(X, D)$ has only rational singularities and the resulting ring $R(X, D)$ is the Segre product $R(X_1, D_1) \sharp R(X_2, D_2)$. As the local

cohomology groups of Segre products are computed in [2], (4.1.5), we can easily check if $R(X, D)$ is a rational singularity or not. For example, though $R = k[x, y, z]/(x^2 + y^3 + z^7)$ (deg $(x) = 21$, deg $(y) = 14$ and deg $(z) = 6$) is not a rational singularity, the Segre product $R \sharp R'$ is a rational singularity for any rational double point R'.

CONJECTURE (3.7). If R is a Macaulay graded ring with isolated singularity and if $a(R) < 0$, then is R a rational singularity?

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